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THE FUCHSIAN CAUCHY PROBLEM

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Abstract. This article presents a global version of the main theorem by Baouendi and Goulaouic [1], in the space of differentiable functions with respect to Fuchsian variable, and holomorphic with respect to other variables. We have no assumptions on the characteristic exponents, and no hyperbolicity conditions.

1. Introduction

Baouendi and Goulaouic [1] generalized the Cauchy-Kowalevsky and Holmgren theorems for the Cauchy problem

$$\begin{aligned} \mathcal{P}u(t,x) &= f(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^q \\ D_t^l u(0,x) &= w_l(x), \quad 0\le l\le m-k-1 \end{aligned} \tag{1.1}$$

with

$$\mathcal{P} = t^k D_t^m + \sum_{j=m-k}^{m-1} a_j(x) t^{j-m+k} D_t^j + \sum_{j=0}^{m-1}\sum_{|\beta|\le m-j} a_{j,\beta}(t,x) t^{\max\{0,j-m+k+1\}} D_t^j D_x^\beta \tag{1.2}$$

called Fuchsian operators with weight $m-k$ with respect to t. Its associated characteristic polynomial (or indicial polynomial) is defined by

$$\begin{aligned} \mathcal{C}(\lambda,x) &= \lambda(\lambda-1)\dots(\lambda-m+1) + a_{m-1}(x)\lambda(\lambda-1)\dots(\lambda-(m-1)+1) \\ &\quad + \dots + a_{m-k}(x)\lambda(\lambda-1)\dots(\lambda-(m-k)+1) \end{aligned}$$

and its m roots $\lambda_1(x),\dots,\lambda_k(x)$, $\lambda_{k+1}=0,\ \dots,\lambda_m=m-k-1$ are called characteristic exponents (or characteristic index) of $\mathcal{P}$ at x. Under the condition

(C1) For any integer $\lambda\ge m-k$, $\mathcal{C}(\lambda,0)\ne 0$,

they solve (1.1) in the space of real-analytic functions in a neighborhood of the origin. When the problem (1.1) is considered in the spaces of partial-analytic functions (C_k^{m-k+h} class on the variable t which will be defined in Section 2.), the condition (C1) is not sufficient to solve this problem. In order to show the well-posedness in these functional spaces, they imposed an additional assumption of the form

(C2) $\Re e\lambda_l(x) < m-k+h$, for $x\in\mathcal{V}$ and $1\le l\le m$,

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where $\mathcal{V}$ is a connected neighborhood of 0 in $\mathbb{C}^q$.

Yamane [12] gave a global version of the main result of [1] for Fuchsian operators with polynomial coefficients, by using the condition (C1) and the method of successive approximations. Mandai [6] has been interested in the assumptions of characteristic exponents. Applying the method of Frobenius, he could omit them, in particular for hyperbolic equations considered by Tahara in different functional spaces.

When $m = k$, (case of weight 0), Lope [5] multiplied the coefficients $a_{j,\beta}(t,x)$ by $\mu(t)^{|\beta|}$ where $\mu(t)$ is a weight function introduced by Tahara [10], in order to construct another class of Fuchsian operators slightly more general than the ones given in [1]. After specifying some properties of this function, without the condition (C1), he showed the very important role of the condition (C2) to invert some operators for getting a unique local solution of$\mathcal{C}_m^0$ class in t and analytic in x for the above problem. On the other hand, Tahara [9] used the condition (C1) for any $x \in \mathbb{R}^q$ as an essential argument under suitable hyperbolicity conditions on $\mathcal{P}$ to prove $\mathcal{C}^\infty$ well-posedness of the Cauchy problem (1.1).

The goal of our work is to establish the existence and uniqueness of a solution of (1.1) in spaces of partially holomorphic functions for Fuchsian operators with the principal part whose coefficients are polynomial in non-Fuchsian variables with a particular degree. The consideration of this type of operators is classic and natural for global solution of the Cauchy problem. They have been considered by several authors, in characteristic and non characteristic case, as [7, 12, 8]. Hamada [4] gave an example of operators with polynomial coefficients for which the associated Cauchy problem, with some polynomial data, does not admit any global solution.

The main result of this paper is obtained without the hyperbolicity condition and other conditions related to the characteristic exponents. We use the same techniques as in [3]. We construct some Banach spaces, following the idea developed in [11], where we reduce our differential problem into inversion of some operator.

Contrary to [5], our techniques allow us to give result of Nagumo's type (see Theorem 3.4) for Fuchsian operators without the conditions (C1) and (C2). On other hand, we use the condition (C1) in our main result.

2. Notations and Main result

Let $q \in \mathbb{N}^*$ and $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{N}^q$, we denote $|\beta| = \sum_{i=1}^q \beta_i$. In what follows, $x = (x_1, \dots, x_q) \in \mathbb{C}^q$, for $1 \le i \le q$, $D_{x_i} = \frac{\partial}{\partial x_i}$ is the partial differentiation with respect to x_i, $D_x = (D_{x_1}, \dots, D_{x_q})$ and we set

$$x^\beta = \prod_{i=1}^q x_i^{\beta_i}, \quad D_x^\beta = D_{x_1}^{\beta_1} \dots D_{x_q}^{\beta_q},$$
$$|x| = \max_{1 \le i \le q} |x_i|, \quad B_R^x = \{x \in \mathbb{C}^q : |x| < R\}.$$

For fixed positive integers $s > m + 1$ and $s' = s - 1$, we set:

$$\Delta_{\rho,R} = \left\{(t,x) \in \mathbb{R} \times \mathbb{C}^q : (\rho R)^{s'/s} (\rho|t|)^{1/s} + |x_1| + \dots + |x_q| < \rho R\right\}$$
$$|\Delta_{\rho,R}| = \{(|t|, x) : (t,x) \in \Delta_{\rho,R}\}.$$

Let $\mathbb{C}[[X]]$ be the space of formal series on X whose coefficients belong to $\mathbb{C}$ and $\mathcal{H}(\mathbb{C}^q)$ be the space of entire functions in $\mathbb{C}^q$. For $h \in \mathbb{N}$, $\mathcal{U} \subset \mathbb{R}$ and $\Omega \subset \mathbb{C}^q$ open sets, $\mathcal{C}^{h,\omega}(\mathcal{U}\times\Omega)$ denote the algebra of functions $f(t,x)$ of class $\mathcal{C}^h$ in t on $\mathcal{U}$ and holomorphic in x on Ω.

For $m\in\mathbb{N}$, we consider $\mathcal{C}^{h,\omega}_m(\mathcal{U}\times\Omega)$ the set of functions $f\in\mathcal{C}^{h,\omega}(\mathcal{U}\times\Omega)$ such that for every $0\le\gamma\le m$, $t^\gamma f\in\mathcal{C}^{h+\gamma,\omega}(\mathcal{U}\times\Omega)$.

The expression $f=O(t^\nu)$ in $\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$ means

$$\exists g\in\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q): f(t,x)=t^\nu g(t,x)\,.$$

Consider problem (1.1) for a Fuchsian operator $\mathcal{P}$ given in (1.2). Assume that the coefficients $a_j(x)=a_j$ is a constant in $\mathbb{C}$ and $a_{j,\beta}(t,x)\in\mathcal{C}^{\infty,\,\omega}(\mathbb{R}\times\mathbb{C}^q)$ satisfy the following assumption

(H0) For any (j,β) such that $j+|\beta|=m$, the functions $a_{j,\beta}(t,x)$ are polynomial in x with $\operatorname{ord}_x a_{j,\beta}(t,x)<|\beta|$ and their coefficients are of $\mathcal{C}^\infty$ class in t on $\mathbb{R}$.

Remark 2.1. The Fuchsian characteristic polynomial associated with operator $\mathcal{P}$ given in (1.2), is defined by the identity:

$$\forall\lambda\in\mathbb{N},\ t^{-\lambda+m-k}\mathcal{P}t^\lambda\big|_{t=0}=\mathcal{C}(\lambda,x)\,.$$

Remark 2.2. From the hypothesis $a_j(x)$ constant in $\mathbb{C}$, we can write $\mathcal{C}(\lambda,x)=\mathcal{C}(\lambda)$.

According to this remark, the condition (C1) becomes:

(C1) For any integer$\lambda\ge m-k$, $\quad\mathcal{C}(\lambda)\neq0$.

Under the above hypotheses on coefficients of $\mathcal{P}$, our main result is as follows.

Theorem 2.3. *Let $\mathcal{P}$ be a Fuchsian operator defined by* (1.2). *If condition* (C1) *holds, then for any functions $f\in\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$ and $w_1,\dots,w_{m-k-1}\in\mathcal{H}(\mathbb{C}^q)$, there exists a unique solution $u\in\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$ of Cauchy problem* (1.1).

Remark 2.4. Take the expression of operator $\mathcal{P}$ given in (1.2) and look for solution u of our Fuchsian Cauchy problem (1.1) in the form

$$u(t,x)=\sum_{l=0}^{m-k-1}\frac{w_l(x)}{l!}t^l+t^{m-k}v(t,x)$$

which satisfies the initial conditions, then problem (1.1) is equivalent to

$$\mathcal{P}t^{m-k}v(t,x)=f(t,x)-\mathcal{P}\Big[\sum_{l=0}^{m-k-1}\frac{w_l(x)}{l!}t^l\Big]\,. \tag{2.1}$$

Note that the right hand side is a known function belongs to $\mathcal{C}^{\infty,\,\omega}(\mathbb{R}\times\mathbb{C}^q)$, and the operator $\mathcal{P}_1$ defined by $\mathcal{P}_1=\mathcal{P}t^{m-k}$is also Fuchsian with weight 0 and its Fuchsian characteristic polynomial $\mathcal{C}_1$ satisfies

$$\forall\lambda\in\mathbb{N},\quad\mathcal{C}_1(\lambda)=t^{-\lambda}\mathcal{P}_1t^\lambda\Big|_{t=0}.$$

Since

$$\forall\lambda\in\mathbb{N},\quad t^{-\lambda}\mathcal{P}_1t^\lambda\Big|_{t=0}=t^{-\lambda-(m-k)}t^{m-k}\mathcal{P}t^{m-k}t^\lambda\Big|_{t=0},$$

from remark 2.1 we have

$$\forall\lambda\in\mathbb{N},\quad\mathcal{C}_1(\lambda)=\mathcal{C}(\lambda+m-k)\,.$$

Then the condition (C1) implies

$$\forall\lambda \in \mathbb{N}, \quad \mathcal{C}_1(\lambda) \neq 0\,. \tag{2.2}$$

Hence, the transformation of problem (1.1) to (2.1), allows us to limit our studies for the case weight 0, with the condition (2.2), in the functional space $\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$.

3. Fuchsian Cauchy problem with weight 0

Let $\mathcal{P}$ be a Fuchsian operator with weight 0, expressed in the form

$$\mathcal{P} = t^m D_t^m + \sum_{j=0}^{m-1} a_j(x) t^j D_t^j + \sum_{j=0}^{m-1} \sum_{0<|\beta|\leq m-j} t a_{j,\beta}(t,x) t^j D_t^j D_x^\beta\ . \tag{3.1}$$

We assume that $a_j(x) = a_j$ are constants in $\mathbb{C}$ and $a_{j,\beta} \in \mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$ satisfying (H0).

Theorem 3.1. *Let $\mathcal{P}$ a Fuchsian operator with weight* 0*defined in* (3.1)*. Under above hypotheses, if its Fuchsian characteristic polynomial $\mathcal{C}(\lambda) \neq 0$ for all $\lambda \in \mathbb{N}$, then for any functions f in $\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$, the following equation*

$$\mathcal{P}u(t,x) = f(t,x) \tag{3.2}$$

admits a unique solution $u \in \mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$.

The sketch of the proof is follows:
- We first give two decompositions of operator $\mathcal{P}$.
- We solve (3.2) in the spaces $\mathcal{C}_m^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$. For that, we transform our problem to inversion of an operator $(I+\mathcal{B})$.
- We introduce Banach spaces which cover $\mathcal{C}_m^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$ and prove that $\|\mathcal{B}\| < 1$.
- We complete the proof in the subsection 3.2.3.

3.1. **Decomposition of operator $\mathcal{P}$.** We have the following properties:

Lemma 3.2 ([1])**.** *Let $j \in \mathbb{N}$. we have*

(1) $t^j D_t^j = \Big(D_t t\Big)^j + \sum_{k=0}^{j-1} c_k \Big(D_t t\Big)^k$, *where $c_k \in \mathbb{Z}$.*

(2) *For every $u \in \mathcal{C}^\infty(\mathbb{R})$, there exists a set of functions $\{u_l\}_{0\leq l\leq j} \subset \mathcal{C}^\infty(\mathbb{R})$ such that*

$$\forall\, v \in \mathcal{C}^\infty(\mathbb{R}), \quad t^j u D_t^j v = \sum_{l=0}^{j} t^l D_t^l[u_l v] \tag{3.3}$$

From this lemma, we can rewrite $\mathcal{P}$ in the following forms

$$\mathcal{P} = (D_t t)^m + \sum_{j=0}^{m-1} b_j (D_t t)^j + \sum_{j=0}^{m-1} \sum_{0<|\beta|\leq m-j} t\widetilde{b}_{j,\beta}(t,x)(D_t t)^j D_x^\beta \tag{3.4}$$

where $b_j \in \mathbb{C}$ and $\widetilde{b}_{j,\beta} \in \mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$, or

$$\mathcal{P} = t^m D_t^m + \sum_{j=0}^{m-1} a_j t^j D_t^j - \sum_{j=0}^{m-1} t^{j+1} D_t^j \mathcal{B}_{m-j} \tag{3.5}$$

where $\mathcal{B}_{m-j} = \sum_{0<|\beta|\leq m-j} b_{j,\beta}(t,x) D_x^\beta$ and $b_{j,\beta} \in \mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$.

Remark 3.3. According to (H0), for all (j,β) such that $j+|\beta|=m$, the coefficients $\widetilde{b}_{j,\beta}(t,x)$ and $b_{j,\beta}(t,x)$ are also polynomials in x with $\mathrm{ord}_x\,\widetilde{b}_{j,\beta}(t,x)<|\beta|$ and $\mathrm{ord}_x\,b_{j,\beta}(t,x)<|\beta|$ whose coefficients are of class $\mathcal{C}^\infty$ in t on $\mathbb{R}$.

3.2. **Resolution of equation** (3.2) **in** $\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$**.** Let $\mathcal{P}$ a Fuchsian operator defined in (3.1) with the same assumptions. Take a positive integer ν enough large such that

$$\sum_{j<m}|b_j|\frac{1}{\nu^{m-j}}<\frac{1}{3}\,. \tag{3.6}$$

Theorem 3.4. *For any function $f=O(t^\nu)$ in $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$,there exists a unique solution $u=O(t^\nu)$ in $\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$ of the equation* (3.2)

Proof. We start by seeking an equivalent problem. It is known that

$$\forall u\in\mathcal{C}^{0,\omega}_1(\mathbb{R}\times\mathbb{C}^q),\quad D_tt[t^\nu u(t,x)]=t^\nu(D_tt+\nu)u(t,x)$$

then using the expression (3.4) of the operator $\mathcal{P}$ and a change of unknown $u(t,x)=t^\nu v(t,x)$ with $v\in\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$, we transform (3.2) to

$$\mathcal{P}_1v(t,x)=g(t,x), \tag{3.7}$$

where

$$\mathcal{P}_1=(D_tt+\nu)^m+\sum_{j=0}^{m-1}b_j(D_tt+\nu)^j+\sum_{j=0}^{m-1}\sum_{0<|\beta|\le m-j}t\widetilde{b}_{j,\beta}(t,x)D_x^\beta(D_tt+\nu)^j$$

and $g\in\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$ such that $f(t,x)=t^\nu g(t,x)$.

We also show the following proposition.

Proposition 3.5. *For all $j\in\mathbb{N}^*$, the operator $(D_tt+\nu)^j$ is an isomorphism from $\mathcal{C}^{0,\omega}_j(\mathbb{R}\times\mathbb{C}^q)$ to $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$, and its inverse H^j_ν is defined by*

$$H^j_\nu=\underbrace{H_\nu\dots H_\nu}_{j\ factors},\quad \text{where } H_\nu u(t,x)=\int_0^1\sigma^\nu u(\sigma t,x)d\sigma\,.$$

Then we can look for a solution of (3.7) in the form

$$v(t,x)=H^m_\nu\psi(t,x)\quad\text{and}\quad\psi\in\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$$

which establishes the equivalence of (3.2) with

$$(I+\mathcal{B})\psi=g \tag{3.8}$$

where

$$\mathcal{B}=\sum_{j=0}^{m-1}b_jH^{m-j}_\nu+\sum_{j=0}^{m-1}\sum_{0<|\beta|\le m-j}t\widetilde{b}_{j,\beta}(t,x)D_x^\beta H^{m-j}_\nu\,.$$

In the next subsection we construct a suitable Banach space, on which we solve problem (3.8).

3.2.1. *The Banach spaces* $E^{0,\omega}_{\rho,R,a}$. Let

$$\Phi(t,x)=\sum_{\delta\in\mathbb{N}^q}\varphi_\delta(t)\frac{x^\delta}{\delta!},\quad \Psi(t,x)=\sum_{\delta\in\mathbb{N}^q}\psi_\delta(t)\frac{x^\delta}{\delta!}$$

be two formal series in $\mathbb{R}_+[[x]]$. We denote $\Psi(t,x)\ll\Phi(t,x)$ if for all $\delta\in\mathbb{N}^q$, $\psi_\delta(t)\le\varphi_\delta(t)$.

For $u\in\mathcal{C}^{0,\omega}(\Delta_{\rho,R})$, $u(t,x)\lll\Phi(t,x)$ means

$$\forall|t|<R,\quad\forall\delta\in\mathbb{N}^q,\quad\Big|D^\delta_x u(t,0)\Big|\le\varphi_\delta(t)\,.$$

Proposition 3.6. *For $u,v\in\mathcal{C}^{0,\omega}(\Delta_{\rho,R})$, we have the following:*

(1) $\Big(u(t,x)\lll\Phi(t,x)$ *and* $\Phi(t,x)\ll\Psi(t,x)\Big)$ *imply* $u(t,x)\lll\Psi(t,x)$;
(2) *If* $\lambda,\mu\in\mathbb{C}$, *we have* $\Big(u(t,x)\lll\Phi(t,x)$ *and* $v(t,x)\lll\Psi(t,x)\Big)$ *implies* $\lambda u(t,x)+\mu v(t,x)\lll|\lambda|\Phi(t,x)+|\mu|\Psi(t,x)$;
(3) $\Big(u(t,x)\lll\Phi(t,x)$ *and* $v(t,x)\lll\Psi(t,x)\Big)$ *implies* $u(t,x)v(t,x)\lll\Phi(t,x)\Psi(t,x)$;
(4) $u(t,x)\lll\Phi(t,x)$ *if and only if* $\forall\gamma\in\mathbb{N}^q$, $D^\gamma_x u(t,x)\lll D^\gamma_x\Phi(t,x)$.

To construct our Banach spaces, we use the same formal series given in [8]. Let τ and ξ be one-dimensional variables. For all $R,a\in\mathbb{R}^*_+$, we set

$$\varphi^a_R(\xi)=\frac{e^{a\xi}}{R-\xi}\in\mathbb{R}_+[[\xi]]$$

which converges in the open set $\{\xi\in\mathbb{C}:|\xi|<R\}$ and satisfies the following properties.

Lemma 3.7 ([8, Lemma 1.4])**.** *For all $p,l\in\mathbb{N}$, we have*

(1) $D^p\varphi^a_R\ll a^{-l}D^{p+l}\varphi^a_R$;
(2) $D^p\varphi^a_R\ll\dfrac{p!}{(p+l)!}R^lD^{p+l}\varphi^a_R$.

We also put

$$\Phi^a_R(\tau,\xi)=\sum_{p\in\mathbb{N}}\tau^pR^{s'p}\frac{D^{sp}\varphi^a_R(\xi)}{(sp)!}\in\mathbb{R}_+[[\tau,\xi]],$$

which converges in the set $\{(\tau,\xi)\in\mathbb{R}\times\mathbb{C}:R^{s'/s}|\tau|^{1/s}+|\xi|<R\}$ and satisfies the following estimates

$$\frac{1}{R-(\tau+\xi)}\ll\Phi^a_R(\tau,\xi),\tag{3.9}$$

$$\forall\,\eta>1,\quad\frac{\eta R}{\eta R-(\tau+\xi)}\Phi^a_R(\tau,\xi)\ll\frac{\eta}{\eta-1}\Phi^a_R(\tau,\xi)\,.\tag{3.10}$$

For parameters $\rho,a>1$, for all $t\in\mathbb{R}$ and $x=(x_1,\dots,x_q)\in\mathbb{C}^q$, we set:

- $\tau=\rho|t|$, $\xi=x_1+\dots+x_q$;
- $\Phi^a_{\rho,R}(t,x)=\Phi^a_{\rho R}(\rho|t|,\xi)=\sum_{p\in\mathbb{N}}\tau^p(\rho R)^{s'p}\dfrac{D^{sp}\varphi^a_{\rho R}(\xi)}{(sp)!}$;
- $E^{0,\omega}_{\rho,R,a}=\{u\in\mathcal{C}^{0,\omega}(\Delta_{\rho,R}),\ \exists C\ge0:u(t,x)\lll C\Phi^a_{\rho,R}(t,x)\}$;
- $\|u\|=\min\{C\ge0:u(t,x)\lll C\Phi^a_{\rho,R}(t,x)\}$

Proposition 3.8. *$\big(E^{0,\omega}_{\rho,R,a}, \|\cdot\|\big)$ is a Banach space.*

The proof of the above proposition follow the steps in [11, Proposition 6.1].

Remark 3.9 ([2])**.** Let $\rho_1 \geq \rho_2$ and $R_1 \geq R_2$, for all $a_1, a_2 \in \mathbb{R}_+$. Then we have $E^{0,\omega}_{\rho_1,R_1,a_1} \subset E^{0,\omega}_{\rho_2,R_2,a_2}$ with continuous injection.

Proposition 3.10. *Let $R_0, \rho \in \mathbb{R}^*_+$ and $f \in \mathcal{C}^{0,\omega}([-R_0,+R_0] \times \overline{B^x_{\rho R_0}})$, then for all $R \in]0, R_0[$,*

$$f(t,x) \lll C\frac{\rho R}{\rho R-(\tau+\xi)}\,,$$

where $C = \sup_{[-R_0,+R_0]\times\overline{B^x_{\rho R_0}}} |f(t,x)|$.

Proof. For $R \in]0, R_0[$, applying the Cauchy's Estimate to holomorphic function $x \to f(t,x)$ in $B^x_{\rho R}$ and bounded by C, we obtain

$$\forall |t|<R,\ \forall \delta\in\mathbb{N}^q, \quad |D^\delta_x f(t,0)| \leq C\frac{|\delta|!}{(\rho R)^{|\delta|}}\,. \tag{3.11}$$

Since $\Big(\dfrac{\rho R}{\rho R-\tau}\Big)^{|\delta|+1} \geq 1$ and

$$D^\delta_x\Big(\frac{\rho R}{\rho R-(\tau+\xi)}\Big)\Big|_{x=0} = D^{|\delta|}_\tau\Big(\frac{\rho R}{\rho R-\tau}\Big) = \frac{|\delta|!}{(\rho R)^{|\delta|}}\Big(\frac{\rho R}{\rho R-\tau}\Big)^{|\delta|+1},$$

from (3.11), we have

$$\forall |t|<R,\ \forall \delta\in\mathbb{N}^q, \quad |D^\delta_x f(t,0)| \leq C\, D^\delta_x\Big(\frac{\rho R}{\rho R-(\tau+\xi)}\Big)\big|_{x=0}$$

which implies

$$f(t,x) \lll C\frac{\rho R}{\rho R-(\tau+\xi)}\,. \tag{3.12}$$

□

According to (3.9), we have the following result.

Remark 3.11. If $f \in \mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$, then for all R, ρ and a, $f \in E^{0,\omega}_{\rho,R,a}$ and $\|f\|$ is independent of parameter a. See [2, Proposition 3.6].

Corollary 3.12. *Let $p \in \mathbb{N}$ and let a polynomial $F(t,x) = \sum_{|\gamma|\leq p} f_\gamma(t)x^\gamma$ where $f_\gamma \in \mathcal{C}^0(\mathbb{R})$, then for all $\rho, R > 0$ such that $\rho R > 1$,we have*

$$F(t,x) \ll M(R)(\rho R)^p\frac{\rho R}{\rho R-(\tau+\xi)},$$

where $M(R) = \mathrm{card}\{\gamma\in\mathbb{N}^q : |\gamma|\leq p\}\max_{|\gamma|\leq p}\big\{\sup_{|t|<R}|f_\gamma(t)|\big\}$, with card A *denoting the cardinality of the set A.*

3.2.2. *Resolution of equation* (3.8) *in* $E^{0,\omega}_{\rho,R,a}$.

Proposition 3.13. *Let* $\Phi(t,x)=\sum_\delta \varphi_\delta(t)\frac{x^\delta}{\delta!}\in\mathbb{R}_+[[x]]$ *and* $u\in\mathcal{C}^{0,\omega}(\Delta_{\rho,R})$ *such that* $u(t,x)\lll\Phi(t,x)$, *then*

$$\forall j\in\mathbb{N},\quad \mathcal{H}^j_\nu u(t,x)\lll\mathcal{H}^j_\nu\Phi(t,x)$$

Proof. Let $u\in\mathcal{C}^{0,\omega}(\Delta_{\rho,R})$ such that $u(t,x)\lll\Phi(t,x)$, then

$$\forall|t|<R,\ \forall\delta\in\mathbb{N}^q,\quad \Big|D^\delta_x u(t,0)\Big|\le\varphi_\delta(t)\,. \tag{3.13}$$

By applying operator $\mathcal{H}^j_\nu$ to the function u, we obtain

$$\mathcal{H}^j_\nu u(t,x)=\int_{[0,1]^j}\prod_{l=1}^j(\sigma_l)^\nu u(\prod_{l=1}^j\sigma_l\ t,x)\prod_{l=1}^j d\sigma_l\,. \tag{3.14}$$

which implies

$$\forall\delta\in\mathbb{N}^q,\quad \Big|D^\delta_x\mathcal{H}^j_\nu u(t,0)\Big|\le\int_{[0,1]^j}\prod_{l=1}^j(\sigma_l)^\nu\Big|D^\delta_x u(\prod_{l=1}^j\sigma_l\ t,0)\Big|\prod_{l=1}^j d\sigma_l\,. \tag{3.15}$$

From (3.13) we obtain: $\forall|t|<R,\ \forall\delta\in\mathbb{N}^q$,

$$\Big|D^\delta_x\mathcal{H}^j_\nu u(t,0)\Big|\le\int_{[0,1]^j}\prod_{l=1}^j(\sigma_l)^\nu\varphi_\delta(\prod_{l=1}^j\sigma_l t)\prod_{l=1}^j d\sigma_l=\mathcal{H}^j_\nu\varphi_\delta(t)\,.$$

Hence, we have

$$\mathcal{H}^j_\nu u(t,x)\lll\mathcal{H}^j_\nu\Phi(t,x)\,.$$

□

Proposition 3.14. *For all* $R>0$, *there exists* ρ_0 *such that, for any* $\rho>\rho_0$, *there exists* $a_\rho>0$ *such that for any* $a>a_\rho$, *equation* (3.8) *admits a unique solution* $\psi\in E^{0,\omega}_{\rho,R,a}$.

To prove this proposition, we show that $\|\mathcal{B}\|<1$ in $E^{0,\omega}_{\rho,R,a}$. For that we establish the intermediate results.

Lemma 3.15. *For all* $u\in E^{0,\omega}_{\rho,R,a}$, *we have*

$$D^\beta_x\mathcal{H}^{m-j}_\nu u(t,x)\lll\|u\|\sum_{p\in\mathbb{N}}\frac{1}{(\nu+p+1)^{m-j}}\tau^p(\rho R)^{s'p}\frac{D^{sp+|\beta|}\varphi^a_{\rho R}(\xi)}{(sp)!}\,. \tag{3.16}$$

Proof. Let $u\in E^{0,\omega}_{\rho,R,a}$, then $u(t,x)\lll\|u\|\Phi^a_{\rho,R}(t,x)$. From Proposition 3.13 and the fourth majoration of Proposition 3.6, we obtain

$$D^\beta_x\mathcal{H}^{m-j}_\nu u(t,x)\lll\|u\|D^\beta_x\mathcal{H}^{m-j}_\nu\Phi^a_{\rho,R}(t,x)\,. \tag{3.17}$$

Using definitions of the formal series $\Phi^a_{\rho,R}$ and the operator $\mathcal{H}^{m-j}_\nu$, we get

$$\begin{aligned}D^\beta_x\mathcal{H}^{m-j}_\nu\Phi^a_{\rho,R}(t,x)&=\sum_{p\in\mathbb{N}}\prod_{l=1}^{m-j}\int_0^1(\sigma_l)^{\nu+p}d\,\sigma_l(\rho|t|)^p(\rho R)^{s'p}\frac{D^{sp+|\beta|}\varphi^a_{\rho R}(\xi)}{(sp)!}\\&=\sum_{p\in\mathbb{N}}\frac{1}{(\nu+p+1)^{m-j}}\rho^p|t|^p(\rho R)^{s'p}\frac{D^{sp+|\beta|}\varphi^a_{\rho R}(\xi)}{(sp)!}\,.\end{aligned}$$

By substituting this right hand side in (3.17), we complete the proof. □

Lemma 3.16. *Let $u \in E^{0,\omega}_{\rho,R,a}$, then*

(1) *If $j+|\beta|<m$, then*
$$t\ D^{\beta}_x \mathcal{H}^{m-j}_{\nu} u(t,x) \lll \|u\| C_{j,\beta}(R,\rho) a^{-1} \Phi^a_{\rho,R}(t,x)$$
(2) *For allj $+|\beta|=m$, we have*
$$t\ D^{\beta}_x \mathcal{H}^{m-j}_{\nu} u(t,x) \lll C_0 \|u\| \rho^{-1} (\rho R)^{1-|\beta|} \Phi^a_{\rho,R}(t,x),$$
where $C_0 \in \mathbb{R}^*_+$ and $C_{j,\beta}(R,\rho)$ is a positive constant independent of parameter a.

Proof. From Lemma 3.7, we have
$$D^{sp+|\beta|} \varphi^a_{\rho R}(\xi) \ll a^{-1} (\rho R)^{s-(|\beta|+1)} \frac{(sp+|\beta|+1)!}{(s(p+1))!} D^{s(p+1)} \varphi^a_{\rho R}(\xi)\,.$$
When we substitute this result in (3.15), we get
$$\begin{aligned} t\ D^{\beta}_x \mathcal{H}^{m-j}_{\nu} u(t,x) &\lll |t|\ \|u\| a^{-1} (\rho R)^{-|\beta|} \sum_{p\in\mathbb{N}} \frac{1}{(\nu+p+1)^{m-j}} \tau^p (\rho R)^{s'p} (\rho R)^{(s-1)} \\ &\quad \times \frac{(sp+|\beta|+1)!}{(sp)!} \frac{D^{s(p+1)} \varphi^a_{\rho R}(\xi)}{(s(p+1))!} \end{aligned} \tag{3.18}$$
Since $\frac{(sp+|\beta|+1)!}{(sp)!} \le (sp+|\beta|+1)^{|\beta|+1}$ and the sequence
$$\Big(\frac{(sp+|\beta|+1)^{|\beta|+1}}{(\nu+p+1)^{m-j}} \Big)_p$$
converges for all $(j,\beta) : j+|\beta|<m$, then there exists $C_0>0$ such that
$$\forall p \in \mathbb{N}, \quad \frac{(sp+|\beta|+1)!}{(sp)!} \frac{1}{(\nu+p+1)^{m-j}} \le C_0$$
From (3.18), we deduce
$$\begin{aligned} t\ D^{\beta}_x \mathcal{H}^{m-j}_{\nu} u(t,x) &\lll \|u\| C_0\ a^{-1} \rho^{-1} (\rho R)^{-|\beta|} \sum_{p\in\mathbb{N}} (\rho|t|) \tau^p (\rho R)^{s'(p+1)} \frac{D^{s(p+1)} \varphi^a_{\rho R}(\xi)}{(s(p+1))!} \\ &\lll \|u\| C_0\ a^{-1} \rho^{-1} (\rho R)^{-|\beta|} \Phi^a_{\rho,R}(t,x) \end{aligned}$$
Similarly, using the second estimate of Lemma 3.7, we prove the second part of Lemma 3.16. □

Lemma 3.17. *For all R and $\rho>0$, there exists positive constants $M_1(\rho,R)$, dependent on ρ and R, and $M_2(R)$, dependent only on R, such that*
$$\|\mathcal{B}\| \le \frac{1}{3} + a^{-1} M_1(\rho,R) + \rho^{-1} M_2(R) \quad in \quad E^{0,\omega}_{\rho,R,a}\,.$$

Proof. We choose $\eta>1$, then dependence on η will not be mentioned in all constants of the below estimations. We denote by $M_1(\rho,R)$ all positive constants dependent on ρ and R, and by $M_2(R)$ all positive constant dependent of R and independent on ρ. These constants are independent of the parameter a.

From Proposition 3.10, remark 3.3 and corollary 3.12, there exists positive constants $M_1(\rho,R)$ and $M_2(R)$ such that:
$$\text{if } j+|\beta|<m, \text{ then } \widetilde{b}_{j,\beta}(t,x) \lll M_1(\rho,R) \frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\,,$$

$$\text{if } j+|\beta|=m, \text{ then } \widetilde{b}_{j,\beta}(t,x) \lll M_2(R)\ (\rho R)^{|\beta|-1}\ \frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\,.$$

Substituting these results in Lemma 3.16, we obtain: if $j+|\beta|<m$, then

$$t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll M_1(\rho,R)\|u\|a^{-1}\frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\Phi^a_{\rho,R}(t,x)\,;$$

if $j+|\beta|=m$, then

$$t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll M_2(R)\|u\|\rho^{-1}\frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\Phi^a_{\rho,R}(t,x)$$

By (3.10), we get

$$\text{if } j+|\beta|<m, \text{ then} t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll M_1(\rho,R)\|u\|a^{-1}\Phi^a_{\rho,R}(t,x);$$
$$\text{if } j+|\beta|=m, \text{ then } t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll M_2(R)\|u\|\rho^{-1}\Phi^a_{\rho,R}(t,x)\,.$$

Which in turn gives

$$\sum_{j<m}\sum_{0<|\beta|<m-j} t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll \|u\|a^{-1}M_1(\rho,R)\Phi^a_{\rho,R}(t,x), \tag{3.19}$$
$$\sum_{j<m}\sum_{|\beta|=m-j} t\widetilde{b}_{j,\beta}(t,x)D_x^\beta\mathcal{H}_\nu^{m-j}u(t,x) \lll \|u\|\rho^{-1}M_2(R)\Phi^a_{\rho,R}(t,x)\,. \tag{3.20}$$

According to the estimations (3.15) and (3.6), the case $\beta=0$ gives

$$\sum_{j<m} b_j\mathcal{H}_\nu^{m-j}u(t,x) \lll \frac{\|u\|}{3}\Phi^a_{\rho,R}(t,x)\,. \tag{3.21}$$

Using the definition of the operator $\mathcal{B}$ and the estimations (3.19), (3.20) and (3.21), we deduce

$$\mathcal{B}u(t,x) \lll \|u\|\Big(\frac{1}{3}+a^{-1}M_1(\rho,R)+\rho^{-1}M_2(R)\Big)\Phi^a_{\rho,R}(t,x)$$

that means

$$\|\mathcal{B}\| \le \frac{1}{3}+a^{-1}M_1(\rho,R)+\rho^{-1}M_2(R) \quad \text{in } E^{0,\omega}_{\rho,R,a}\,.$$

□

Proof of Proposition 3.14. If we take, in the Lemma 3.17, parameters a and ρ large enough such that

$$a>3M_1(\rho,R), \quad \rho>3M_2(R),$$

we get $\|\mathcal{B}\|<1$, from which we deduce that operator $(I+\mathcal{B})$ is invertible in $E^{0,\omega}_{\rho,R,a}$ then for all $g\in E^{0,\omega}_{\rho,R,a}$, equation (3.8) admits a unique solution $u\in E^{0,\omega}_{\rho,R,a}$. □

3.2.3. *Final part in the proof of Theorem* 3.4.

I. Existence of solution in $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$. Let $f(t,x)=t^\nu g(t,x)$ with $g\in\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$. By property 3.11,$g\in E^{0,\omega}_{\rho,R,a}$ for all $R,\rho,a\in\mathbb{R}_+^*$. Applying Proposition 3.14, we can choose two increasing sequences $(\rho_n)_n$ and $(a_n)_n$ such that for all $n\in\mathbb{N}^*$ there exists a unique solution ψ_n of (3.8) in $E^{0,\omega}_{\rho_n,n,a_n}$. By property 3.9, we have

$$\psi_{n+1}\in E^{0,\omega}_{\rho_{n+1},(n+1),a_{n+1}} \subset E^{0,\omega}_{\rho_n,n,a_n}\,.$$

Since the uniqueness of solutions holds in $E^{0,\omega}_{\rho_n,n,a_n}$, we deduce

$$\psi_{n+1}=\psi_n \quad \text{on } \Delta_{\rho_n,n}$$

which allows us to define a solution u of (3.8) in $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$by

$$\psi=\psi_n,\quad \text{on}\quad \Delta_{\rho_n,n},\quad \forall n\in\mathbb{N}^*\,.$$

II. Uniqueness of this solution. Let ψ_1,ψ_2 be two solutions in $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$ of (3.8), then by property 3.11, for all $R,\rho,a>0$, the functions $\psi_1/\Delta_{\rho,R}$ and $\psi_2/\Delta_{\rho,R}$ are also solutions of (3.8) in $E^{0,\omega}_{\rho,R,a}$. By the uniqueness of the solution of the problem in this Banach space, we obtain

$$\forall|t|<\frac{R}{2^s},\quad \psi_1(t,\cdot)=\psi_2(t,\cdot)\quad \text{on } B^x_{\rho R/2q}\,.$$

Using analytic extension theorem we get

$$\forall R>0,\ \forall|t|<\frac{R}{2^s},\quad \psi_1(t,\cdot)=\psi_2(t,\cdot)\quad \text{on } \mathbb{C}^q$$

which implies the uniqueness of the solution of Problem (3.8) in the functional space $\mathcal{C}^{0,\omega}(\mathbb{R}\times\mathbb{C}^q)$.

To complete the proof of the Theorem 3.4, we note that the solution of (3.2) is the form $u(t,x)=t^\nu H^m_\nu\psi(t,x)$, where ψ is the unique solution of (3.8), then $u=O(t^\nu)$ in $\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$, because $H^m_\nu\psi\in\mathcal{C}^{0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$. □

Remark 3.18. Theorem 3.4 holds for any Fuchsian operator $\mathcal{P}$ with weight 0. Hence it also holds for operator $(D_tt)^h\mathcal{P}$ for all $h\in\mathbb{N}$, which allows us to deduce that: For all $f=O(t^\nu)$ in $\mathcal{C}^{0,\omega}_h(\mathbb{R}\times\mathbb{C}^q)$, there exists a unique solution $u=O(t^\nu)$ in $\mathcal{C}^{0,\omega}_{m+h}(\mathbb{R}\times\mathbb{C}^q)$ of (3.2) .

Proof of Theorem 3.1. Let $f\in\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$ and $\nu_0>0$ satisfies (3.6). For $\nu\ge\nu_0$, the Mac-Laurin expansion of f of order ν gives

$$f(t,x)=\sum_{l=0}^{\nu-1}D^l_tf(0,x)\frac{t^l}{l!}+t^\nu f_\nu(t,x)\,.$$

By simple calculations we prove that $f_\nu\in\mathcal{C}^{0,\omega}_\nu(\mathbb{R}\times\mathbb{C}^q)$. We look for solution of (3.2) in the form

$$U^{(\nu)}(t,x)=\sum_{l=0}^{\nu-1}D^l_tU^{(\nu)}(0,x)\frac{t^l}{l!}+t^\nu u_\nu(t,x)\,.$$

By the decomposition (3.5), problem (3.2)) is equivalent to system:

$$\begin{gathered}0\le l\le\nu-1,\\ \mathcal{C}(l)\frac{D^l_tU^{(\nu)}(0,x)}{l!}=\frac{D^l_tf(0,x)}{l!}+\sum_{j=0}^{m-1}\frac{1}{(l-j-1)!}\big[D^{l-1}_tB_{m-j}U^{(\nu)}(t,x)\big]\big|_{t=0},\\ \mathcal{P}\big[t^\nu u_\nu(t,x)\big]=t^\nu f_\nu(t,x)\,.\end{gathered}\tag{3.22}$$

(a) Since $\mathcal{C}(l)\neq0$, for all $l\in\mathbb{N}$, then the functions $D^l_tU^{(\nu)}(0,x)$ $(l=0,\dots,\nu-1)$ are determined uniquely.

(b) We have $f_\nu\in\mathcal{C}^{0,\omega}_\nu(\mathbb{R}\times\mathbb{C}^q)$. By remark 3.18 there exists a unique solution $u_\nu\in\mathcal{C}^{0,\omega}_{m+\nu}(\mathbb{R}\times\mathbb{C}^q)$ of (3.22), then $t^\nu u_\nu\in\mathcal{C}^{\nu,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$.

From (a) and (b), we deduce there exists a unique solution $U^{(\nu)}\in\mathcal{C}^{\nu,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$ of (3.2). Since for any $\nu_1>\nu_0$, we have $\mathcal{C}^{\nu_1,\omega}_m(\mathbb{R}\times\mathbb{C}^q)\subset\mathcal{C}^{\nu_0,\omega}_m(\mathbb{R}\times\mathbb{C}^q)$, by the existence and uniqueness of the solution of (3.2) in $\mathcal{C}_m^{\nu_1,\omega}(\mathbb{R}\times\mathbb{C}^q)$ and $\mathcal{C}_m^{\nu_0,\omega}(\mathbb{R}\times\mathbb{C}^q)$ we deduce that

$$\forall\,\nu_1>\nu_0,\quad U^{(\nu_0)}\in\mathcal{C}_m^{\nu_1,\omega}(\mathbb{R}\times\mathbb{C}^q);$$

hence $U^{(\nu_0)}\in\mathcal{C}^{\infty,\,\omega}(\mathbb{R}\times\mathbb{C}^q)$.

For uniqueness, we remark that $\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)\subset\mathcal{C}^{\nu_0,\omega}(\mathbb{R}\times\mathbb{C}^q)$ where the uniqueness of solutions holds, then if U is solution of (3.2)) in $\mathcal{C}^{\infty,\omega}(\mathbb{R}\times\mathbb{C}^q)$, we deduce that $U=U^{(\nu_0)}$, which completes the proof of Theorem 3.1. □

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