Electronic Journal of Differential Equations, Vol. 2009(2009), No. 104, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ERROR EVALUATION OF APPROXIMATE SOLUTIONS FOR SUM-DIFFERENCE EQUATIONS IN TWO VARIABLES

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ABSTRACT. This article presents estimates for the approximation of solutions of certain sum-difference equations in two independent variables with given initial conditions. A fundamental finite difference inequality with explicit estimate is used to establish our results.

1. INTRODUCTION

The method of approximation provides a very useful and important technique in the study of qualitative properties of solutions for mathematical models of various dynamic phenomena (see [2, 4, 5]). This paper focuses on the study of initial value problem (IVP, for short)

$$\Delta_2\Delta_1 u(m,n) = f(m,n,u(m,n),Gu(m,n)), \tag{1.1}$$

with

$$\begin{gathered} u(m,0)=\alpha(m), \quad u(0,n)=\beta(m), \quad n\in\mathbb{N}_0 \\ \alpha(0)=\beta(0), \end{gathered} \tag{1.2}$$

where

$$Gu(m,n) := \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} g(m,n,\sigma,\tau,u(\sigma,\tau)), \tag{1.3}$$

f, g are given functions and u is the unknown function to be found. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{N}_0 = \{0,1,2,\dots\}$ be the given subsets of $\mathbb{R}$. For the functions $z(m), w(m,n)$, $m,n\in\mathbb{N}_0$, we define the operators

$$\begin{gathered} \Delta z(m) = z(m+1)-z(m), \quad \Delta_1 w(m,n) = w(m+1,n)-w(m,n), \\ \Delta_2 w(m,n) = w(m,n+1)-w(m,n), \quad \Delta_2\Delta_1 w(m,n) = \Delta_2(\Delta_1 w(m,n)). \end{gathered}$$

We denote by $D(S_1,S_2)$ the class of discrete functions from the set S_1 to the set S_2 and use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. We assume that $f\in D(\mathbb{N}_0^2\times\mathbb{R}^2,\mathbb{R})$, $g\in D(\mathbb{N}_0^4\times\mathbb{R},\mathbb{R})$, $\alpha,\beta\in D(\mathbb{N}_0,\mathbb{R})$.

When dealing with the discrete event dynamical systems the basic questions to be answered are: (i) under what conditions the systems under considerations have

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2000 *Mathematics Subject Classification.* 26D10, 39A10, 65R20, 45G10.
Key words and phrases. Approximate solutions; sum-difference equation; finite difference inequality.

Submitted June 8, 2009. Published August 27, 2009.

solutions? (ii) how can we find the solutions or closely approximate them? (iii) what are their nature?. The study of such questions gives rise to new results and need a fresh outlook for handling such problems for (1.1)-(1.2). We note that the equation (1.1) under the conditions (1.2) admits a unique solution. In this paper we offer the conditions for the error evaluation of approximate solutions of equation (1.1) by establishing some new bounds and convergence properties on solutions of approximate problems. We also study the dependency of solutions of equations of the forms (1.1) on parameters. The main tool employed in the analysis is based on the application of a certain finite difference inequality with explicit estimate given in [8].

2. Main Results

The following is a variation of the finite difference inequality established in [8, Theorem 5.3.2] and [7, Theorem 4.3.2], and is crucial in the proof of our main results.

Lemma 2.1. *Let $u, a, p \in D(\mathbb{N}_0^2, \mathbb{R}_+)$, $q, \Delta_1 q, \Delta_2 q, \Delta_2\Delta_1 q \in D(\mathbb{N}_0^4, \mathbb{R}_+)$. If $a(m,n)$ is nondecreasing in each variable $m, n \in \mathbb{N}_0$, and*

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\Big[u(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} q(s,t,\sigma,\tau)u(\sigma,\tau)\Big], \tag{2.1}$$

for $m, n \in \mathbb{N}_0$, then

$$u(m,n) \le a(m,n)\Big[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\prod_{\xi=0}^{s-1}\Big[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + Tq(\xi,\eta)]\Big]\Big], \tag{2.2}$$

for $m, n \in \mathbb{N}_0$, where

$$\begin{aligned} Tq(m,n) := q(m+1,n+1,m,n) + \sum_{\sigma=0}^{m-1} \Delta_1 q(m,n+1,\sigma,n) \\ + \sum_{\tau=0}^{n-1} \Delta_2 q(m+1,n,m,\tau) + \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1} \Delta_2\Delta_1 q(m,n,\sigma,\tau). \end{aligned} \tag{2.3}$$

Let $u \in D(\mathbb{N}_0^2, \mathbb{R})$ and $\Delta_2\Delta_1 u(m,n) (m, n \in \mathbb{N}_0)$ exist and satisfy the inequality

$$|\Delta_2\Delta_1 u(m,n) - f(m,n,u(m,n),Gu(m,n))| \le \varepsilon,$$

for a given constant $\varepsilon \ge 0$, where it is assumed that (1.2) holds. Then we call $u(m,n)$ an ε-approximate solution of (1.1).

Our main result estimates the difference between the two approximate solutions of (1.1).

Theorem 2.2. *Suppose that f, g in (1.1) satisfy the conditions*

$$|f(m,n,u,v) - f(m,n,\bar{u},\bar{v})| \le p(m,n)[|u-\bar{u}| + |v-\bar{v}|], \tag{2.4}$$

$$|g(m,n,\sigma,\tau,u) - g(m,n,\sigma,\tau,\bar{u})| \le q(m,n,\sigma,\tau)|u-\bar{u}|, \tag{2.5}$$

where $p \in D(\mathbb{N}_0^2, \mathbb{R}_+)$, $q \in D(\mathbb{N}_0^4, \mathbb{R}_+)$ with $\Delta_1 q, \Delta_2 q, \Delta_2\Delta_1 q \in D(\mathbb{N}_0^4, \mathbb{R}_+)$. For $i = 1, 2$, let $u_i(m,n)$ $(m, n \in \mathbb{N}_0)$ be respectively ε_i-approximate solutions of (1.1) with

$$\begin{aligned} u_i(m,0) = \alpha_i(m), \quad u_i(0,n) = \beta_i(n), \quad n \in \mathbb{N}_0, \\ \alpha_i(0) = \beta_i(0), \end{aligned} \tag{2.6}$$

where α_i, β_i are in $D(\mathbb{N}_0, \mathbb{R})$ satisfy

$$|\alpha_1(m) - \alpha_2(m) + \beta_1(n) - \beta_2(n)| \leq \delta, \tag{2.7}$$

in which $\delta \geq 0$ is a constant. Then

$$\begin{aligned} &|u_1(m,n) - u_2(m,n)| \\ &\leq ((\varepsilon_1 + \varepsilon_2)mn + \delta)\Big[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t) \prod_{\xi=0}^{s-1}\Big[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + Tq(\xi,\eta)]\Big]\Big], \end{aligned} \tag{2.8}$$

for $m, n \in \mathbb{N}_0$, where $Tq(m,n)$ is given by (2.3).

Proof. Since $u_i(m,n)$ $(i = 1,2)$ for $m, n \in \mathbb{N}_0$ are respectively ε_i-approximate solutions of equation (1.1) with (2.6) we have

$$|\Delta_2\Delta_1 u_i(m,n) - f(m,n,u_i(m,n), Gu_i(m,n))| \leq \varepsilon_i. \tag{2.9}$$

Now keeping m fixed in (2.9), setting $n = t$ and taking sum on both sides over t from 0 to $n-1$, then keeping n fixed in the resulting inequality and setting $m = s$ and taking sum over s from 0 to $m-1$ and using (2.6), we observe that

$$\begin{aligned} \varepsilon_i mn &\geq \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |\Delta_2\Delta_1 u_i(s,t) - f(s,t,u_i(s,t), Gu_i(s,t))| \\ &\geq |\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \{\Delta_2\Delta_1 u_i(s,t) - f(s,t,u_i(s,t), Gu_i(s,t))\}| \\ &= \Big|\big\{u_i(m,n) - [\alpha_i(m) + \beta_i(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_i(s,t), Gu_i(s,t))\big\}\Big|. \end{aligned} \tag{2.10}$$

From this inequality and using the elementary inequalities $|v - z| \leq |v| + |z|, |v| - |z| \leq |v - z|$, for $v, z \in \mathbb{R}$, we observe that

$$\begin{aligned} &(\varepsilon_1 + \varepsilon_2)mn \\ &\geq \Big|\{u_1(m,n) - [\alpha_1(m) + \beta_1(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_1(s,t), Gu_1(s,t))\}\Big| \\ &\quad + \Big|\{u_2(m,n) - [\alpha_2(m) + \beta_2(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_2(s,t), Gu_2(s,t))\}\Big| \\ &\geq \Big|\{u_1(m,n) - [\alpha_1(m) + \beta_1(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_1(s,t), Gu_1(s,t))\} \\ &\quad - \big\{u_2(m,n) - [\alpha_2(m) + \beta_2(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_2(s,t), Gu_2(s,t))\big\}\Big| \\ &\geq |u_1(m,n) - u_2(m,n)| - |\alpha_1(m) + \beta_1(n) - \{\alpha_2(m) + \beta_2(n)\}| \\ &\quad - \Big|\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_1(s,t), Gu_1(s,t)) - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t,u_2(s,t), Gu_2(s,t))\Big| \end{aligned} \tag{2.11}$$

Let $u(m,n) = |u_1(m,n) - u_2(m,n)|$ for $m,n \in \mathbb{N}_0$. From the above inequality and using the hypotheses, we observe that

$$\begin{aligned} u(m,n) &\le (\varepsilon_1+\varepsilon_2)mn + \delta \\ &\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,u_1(s,t),Gu_1(s,t)) - f(s,t,u_2(s,t),Gu_2(s,t))| \\ &\le (\varepsilon_1+\varepsilon_2)mn + \delta + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\Big[u(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} q(s,t,\sigma,\tau)u(\sigma,\tau)\Big]. \end{aligned} \tag{2.12}$$

Now an application of Lemma 2.1 yields (2.8).

Consider the initial-value problem (1.1)-(1.2) together with

$$\Delta_2\Delta_1 v(m,n) = \bar{f}(m,n,v(m,n),Gv(m,n)), \tag{2.13}$$

$$\begin{gathered} v(m,0) = \bar{\alpha}(m), \quad v(0,n) = \bar{\beta}(n), \quad n \in \mathbb{N}_0, \\ \bar{\alpha}(0) = \bar{\beta}(0), \end{gathered} \tag{2.14}$$

where G is given by (1.3) and $\bar{f} \in D(\mathbb{N}_0^2 \times \mathbb{R}^2, \mathbb{R})$, $\bar{\alpha}, \bar{\beta} \in D(\mathbb{N}_0, \mathbb{R})$. □

The following theorem concerns the closeness of solutions of (1.1)-(1.2) and of (2.13)-(2.14).

Theorem 2.3. *Suppose that f, g in* (1.1) *satisfy* (2.4), (2.5) *and there exist constants $\bar{\varepsilon} \ge 0$, $\bar{\delta} \ge 0$ such that*

$$|f(m,n,u,w) - \bar{f}(m,n,u,w)| \le \bar{\varepsilon}, \tag{2.15}$$

$$|\alpha(m) - \bar{\alpha}(m) + \beta(n) - \bar{\beta}(n)| \le \bar{\delta}, \tag{2.16}$$

where f, α, β and $\bar{f}, \bar{\alpha}, \bar{\beta}$ are as in (1.1)-(1.2) *and* (2.13)-(2.14). *Let $u(m,n)$ and $v(m,n)$ be respectively the solutions of* (1.1)-(1.2) *and of* (2.13)-(2.14) *for $m,n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &|u(m,n) - v(m,n)| \\ &\le (\bar{\varepsilon}mn + \bar{\delta})\Big[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t) \prod_{\xi=0}^{s-1}\Big[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + Tq(\xi,\eta)]\Big]\Big], \end{aligned} \tag{2.17}$$

for $m,n \in \mathbb{N}_0$.

Proof. Let $e(m,n) = |u(m,n) - v(m,n)|$ for $m,n \in \mathbb{N}_0$. Using the fact that $u(m,n), v(m,n)$ are the solutions of (1.1)-(1.2), and of (2.13)-(2.14), and the hypotheses, we observe that

$$\begin{aligned} e(m,n) &\le |\alpha(m) - \bar{\alpha}(m) + \beta(n) - \bar{\beta}(n)| \\ &\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,u(s,t),Gu(s,t)) - f(s,t,v(s,t),Gv(s,t))| \\ &\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,v(s,t),Gv(s,t)) - \bar{f}(s,t,v(s,t),Gv(s,t))| \\ &\le (\bar{\varepsilon}mn + \bar{\delta}) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\Big[e(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} q(s,t,\sigma,\tau)e(\sigma,\tau)\Big]. \end{aligned} \tag{2.18}$$

Now an application of Lemma 2.1 yields (2.17). □

Remark 2.4. The result given in Theorem 2.3 relates the solutions of (1.1)-(1.2) and of (2.13)-(2.14) in the sense that if f is close to $\bar{f}$, α is close to $\bar{\alpha}$, β is close to $\bar{\beta}$, then the solutions of (1.1)-(1.2) and of (2.13)-(2.14) are also close to each other.

Now we consider (1.1)-(1.2) and sequence of initial-value problems

$$\Delta_2\Delta_1 w(m,n) = f_k(m,n,w(m,n),Gw(m,n)), \tag{2.19}$$

$$\begin{gathered} w(m,0)=\alpha_k(m), \quad w(0,n)=\beta_k(n), \quad n\in\mathbb{N}_0, \\ \alpha_k(0)=\beta_k(0), \end{gathered} \tag{2.20}$$

for $m,n\in\mathbb{N}_0$, $(k=1,2,\dots)$ where G is given by (1.3) and $f_k\in D(\mathbb{N}_0^2\times\mathbb{R}^2,\mathbb{R})$, $\alpha_k,\beta_k\in D(\mathbb{N}_0,\mathbb{R})$.

As an immediate consequence of Theorem 2.3, we have the following corollary.

Corollary 2.5. *Suppose that f,g in* (1.1) *satisfy* (2.4), (2.5) *and*

$$|f(m,n,u,v)-f_k(m,n,u,v)|\le\varepsilon_k, \tag{2.21}$$

$$|\alpha(m)-\alpha_k(m)+\beta(n)-\beta_k(n)|\le\delta_k, \tag{2.22}$$

with $\varepsilon_k\to 0$ and $\delta_k\to 0$ as $k\to\infty$, where f,α,β and f_k,α_k,β_k are as in (1.1)-(1.2) *and in* (2.19)-(2.20)*. If $w_k(m,n)$ $(k=1,2,\dots)$ and $u(m,n)$ are respectively the solutions of* (2.19)-(2.20) *and of* (1.1)-(1.2) *for $m,n\in\mathbb{N}_0$, then $w_k(m,n)\to u(m,n)$ as $k\to\infty$.*

Proof. For $k=1,2,\dots$, the conditions of Theorem 2.3 hold. An application of Theorem 2.3 yields

$$\begin{aligned} &|w_k(m,n)-u(m,n)| \\ &\le(\varepsilon_k mn+\delta_k)\Big[1+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}p(s,t)\prod_{\xi=0}^{s-1}\Big[1+\sum_{\eta=0}^{t-1}[p(\xi,\eta)+Tq(\xi,\eta)]\Big]\Big], \end{aligned} \tag{2.23}$$

for $m,n\in\mathbb{N}_0$ and $k=1,2,\dots$. The required result follows from (2.23). □

Remark 2.6. We note that the result obtained in Corollary 2.5 provides sufficient conditions that ensures, solutions of (2.19)-(2.20) will converge to the solutions to (1.1)-(1.2).

3. Dependency on parameters

In this section, we present results on the dependency of solutions of equation (1.1) and its variants on given initial conditions and pure parameters.

The following theorem shows the dependency of solutions of (1.1) on given initial conditions.

Theorem 3.1. *Suppose that f,g in* (1.1) *satisfy* (2.4), (2.5)*. Let $u(m,n)$ and $z(m,n)$ be respectively the solutions of* (1.1) *with the initial conditions* (1.2) *and*

$$\begin{gathered} z(m,0)=\alpha_0(m), \quad z(0,n)=\beta_0(n), \quad n\in\mathbb{N}_0 \\ \alpha_0(0)=\beta_0(0), \end{gathered} \tag{3.1}$$

where $\alpha_0,\beta_0\in D(\mathbb{N}_0,\mathbb{R})$ and

$$|\alpha(m)-\alpha_0(m)+\beta(n)-\beta_0(n)|\le k, \tag{3.2}$$

in which $k \geq 0$ is a constant. Then

$$|u(m,n) - z(m,n)| \leq k\Big[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\prod_{\xi=0}^{s-1}\Big[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + Tq(\xi,\eta)]\Big]\Big], \tag{3.3}$$

for $m, n \in \mathbb{N}_0$.

Proof. Using the facts that $u(m,n)$ and $z(m,n)$ are respectively the solutions of (1.1)-(1.2) and of (1.1)-(3.1) and the hypotheses, we have

$$\begin{aligned}
&|u(m,n) - z(m,n)| \\
&\leq |\alpha(m) - \alpha_0(m) + \beta(n) - \beta_0(n)| \\
&\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,u(s,t),Gu(s,t)) - f(s,t,z(s,t),Gz(s,t))| \\
&\leq k + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\Big[|u(s,t) - z(s,t)| + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} q(s,t,\sigma,\tau)|u(\sigma,\tau) - z(\sigma,\tau)|\Big].
\end{aligned} \tag{3.4}$$

Now a suitable application of Lemma 2.1 yields (3.3), which shows the dependency of solutions of (1.1) on the given initial values. $\square$

We now consider the sum-difference equations

$$\Delta_2\Delta_1 u(m,n) = f(m,n,u(m,n),Gu(m,n),\mu), \tag{3.5}$$

$$\Delta_2\Delta_1 u(m,n) = f(m,n,u(m,n),Gu(m,n),\mu_0), \tag{3.6}$$

with the initial conditions (1.2), where G is given by (1.3), $f \in D(\mathbb{N}_0^2 \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ and μ, μ_0 are parameters.

The next theorem shows the dependency of solutions of (3.5)-(1.2) and of (3.6)-(1.2) on the parameters μ, μ_0.

Theorem 3.2. *Suppose that g, f in (3.5), (3.6) satisfy respectively (2.5) and*

$$|f(m,n,u,v,\mu) - f(m,n,\bar{u},\bar{v},\mu)| \leq p(m,n)[|u - \bar{u}| + |v - \bar{v}|], \tag{3.7}$$

$$|f(m,n,u,v,\mu) - f(m,n,u,v,\mu_0)| \leq r(m,n)|\mu - \mu_0|, \tag{3.8}$$

where $p, r \in D(\mathbb{N}_0^2, \mathbb{R}_+)$. Let $u_1(m,n)$ and $u_2(m,n)$ be the solutions of (3.5)-(1.2) and of (3.6)-(1.2) respectively. Then

$$\begin{aligned}
&|u_1(m,n) - u_2(m,n)| \\
&\leq \bar{a}(m,n)\Big[1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\prod_{\xi=0}^{s-1}\Big[1 + \sum_{\eta=0}^{t-1}[p(\xi,\eta) + Tq(\xi,\eta)]\Big]\Big],
\end{aligned} \tag{3.9}$$

for $m, n \in \mathbb{N}_0$, where

$$\bar{a}(m,n) = |\mu - \mu_0|\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} r(s,t), \tag{3.10}$$

for $m, n \in \mathbb{N}_0$.

Proof. Let $h(m,n) = |u_1(m,n) - u_2(m,n)|$ for $m,n \in \mathbb{N}_0$. Using the facts that $u_1(m,n)$ and $u_2(m,n)$ are respectively the solutions of (3.5)-(1.2) and of (3.6)-(1.2) and the hypotheses, we observe that

$$\begin{aligned} h(m,n) &\le \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,u_1(s,t),Gu_1(s,t),\mu) - f(s,t,u_2(s,t),Gu_2(s,t),\mu)| \\ &\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |f(s,t,u_2(s,t),Gu_2(s,t),\mu) - f(s,t,u_2(s,t),Gu_2(s,t),\mu_0)| \\ &\le \bar{a}(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\Big[h(s,t) + \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} q(s,t,\sigma,\tau)h(\sigma,\tau)\Big]. \end{aligned} \tag{3.11}$$

Now an application of Lemma 2.1 yields (3.9), which shows the dependency of solutions of (3.5)-(1.2) and of (3.6)-(1.2) on the parameters μ, μ_0. □

Remark 3.3. We note that the results given in this paper can be extended very easily to study the sum-difference equation

$$\begin{aligned} &\Delta_2\Delta_1 u(m,n) + \Delta_2(b(m,n)u(m,n)) \\ &= f(m,n,u(m,n),Gu(m,n),Hu(m,n)), \end{aligned} \tag{3.12}$$

with the given initial conditions in (1.2), where G is given by (1.3) and H is given by

$$Hu(m,n) := \sum_{\sigma=0}^{\infty}\sum_{\tau=0}^{\infty} h(m,n,\sigma,\tau,u(\sigma,\tau)),$$

under some suitable conditions on b, f, g, h involved in (3.12) by making use of the finite difference inequality given in [8, Theorem 5.2.3].

For further results on the qualitative properties of solutions of various finite difference equations, see [1, 6, 7, 8, 9].

Acknowledgements. The author is grateful to the anonymous referee and Professor Julio G. Dix whose comments and suggestions helped to improve this article.

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