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INVERSE EIGENVALUE PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider the inverse nonlinear eigenvalue problem for the equation

$$-\Delta u + f(u) = \lambda u, \quad u > 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $f(u)$ is an unknown nonlinear term, $\Omega \subset \mathbb{R}^N$ is a bounded domain with an appropriate smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. Under basic conditions on f, for any given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\bar{\Omega})$ with $\|u_\alpha\|_2 = \alpha$. The curve $\lambda(\alpha)$ is called the L^2-bifurcation branch. Using a variational approach, we show that the nonlinear term $f(u)$ is determined uniquely by $\lambda(\alpha)$.

1. Introduction

We consider the nonlinear eigenvalue problem

$$-\Delta u + f(u) = \lambda u \quad \text{in } \Omega, \tag{1.1}$$
$$u > 0 \quad \text{in } \Omega, \tag{1.2}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with an appropriate smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter and $f(u)$ is *unknown* nonlinear term which is assumed to satisfy the following conditions:

- (A1) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$,
- (A2) $f(u)/u$ is strictly increasing for $u \geq 0$,
- (A3) $f(u)/u \to \infty$ as $u \to \infty$.
- (A4) There exists a constant $C > 0$ such that $F(u + v) \leq C(F(u) + F(v))$ for $u, v \geq 0$, where $F(u) := \int_0^u f(s)\,ds$.

Typical examples of functions satisfying (A1)–(A4) are $f(u) = u^p$ ($p > 1$, $u \geq 0$) and $f(u) = u^p \log(1 + u)$ ($p > 1$, $u \geq 0$). From [1, 2, 9, 10] we know that solutions to (1.1)–(1.3) have the following fundamental properties:

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(P1) If $f(u)$ satisfies (A1)–(A3), then for any given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\bar{\Omega})$ of (1.1)–(1.3) with $\|u_\alpha\|_2 = \alpha$, where $\|\cdot\|_2$ denotes usual L^2-norm. Furthermore, if we assume (A4), then this solution is obtained by a variational method.

(P2) $\lambda(\alpha)$ $(\alpha > 0)$ is an unbounded increasing curve of class C^1 and $\lambda(\alpha) \to \infty$ (resp. $\lambda(\alpha) \to \lambda_1$) as $\alpha \to \infty$ (resp. $\alpha \to 0$), where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω with Dirichlet zero boundary condition. $\lambda(\alpha)$ is called the L^2-bifurcation branch of positive solutions to (1.1)–(1.3).

The purpose of this paper is to show that unknown term $f(u)$ is *determined uniquely* from the L^2-bifurcation branch $\lambda = \lambda(\alpha)$. To explain our motivation and intention more precisely, we recall some famous problems in linear and nonlinear eigenvalue problems.

Linear eigenvalue problems have a long history and have been investigated intensively by many authors; among other, we have (i) Weyl formula (asymptotic distribution of eigenvalues [5]) and (ii) inverse eigenvalue problems (determination of the potential term by using the information of eigenvalues [8]).

As for the nonlinear problems, one of the most popular problem is the bifurcation problem; that is, to investigate the structure of the solution set of (1.1)–(1.3). Since (1.1)–(1.3) is regarded as the nonlinear *eigenvalue problems*, it is reasonable to treat it in L^2-framework like the linear eigenvalue problems.

Based on the properties (P1) and (P2), asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to \infty$ is one of the main interest of this field, and several asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to \infty$ have been obtained for the case, for instance, $f(u) = u^p$, $f(u) = u^p \log(1+u)$ $(p > 1, u \geq 0)$. We refer the reader to [11, 12] and the references therein. We also refer to [2, 3, 4, 6, 7] for the works from a viewpoint of bifurcation problems. The Weyl formula and bifurcation problems share certain similarities in that both deal with asymptotic properties of eigenvalues.

For the inverse problem, however, there are a few works in nonlinear problems. One of the setting of the inverse problem in nonlinear case is as follows. Assume that the unknown nonlinear term $f(u)$ satisfies (A1)–(A3). If we know the asymptotic expansion formula for $\lambda(\alpha)$ as $\alpha \to \infty$, which has the same leading and second terms as those for the case $f(u) = u^p$, then we conclude that $f(u) = u^p + g(u)$, where $g(u)$ is determined from the second term of $\lambda(\alpha)$ with suitable remainder term [13]. This sort of approach to the nonlinear inverse spectral problems seems to be new. However, as far as the author knows, the basic and important problem as we consider it here has not been treated yet.

To state our results, we introduce the following assumptions on the functions f_1 and f_2. Let $F_1(u) = \int_0^u f_1(s)\,ds$ and $F_2(u) = \int_0^u f_2(s)\,ds$.

(B1) $f_1(u) \leq f_2(u)$ for all $u \geq 0$.

(B2) $F_1(u) \leq F_2(u)$ for all $u \geq 0$.

(B3) The connected components of the set $V := \{u \geq 0 : f_1(u) = f_2(u)\}$ are locally finite.

(B4) The connected components of the set $W := \{u \geq 0 : F_1(u) = F_2(u)\}$ are locally finite.

It is clear that (B1) implies (B2). However, there exist f_1 and f_2 which satisfy (B1) and do not satisfy (B3) (cf. Remark 1.3 below). Therefore, (B1) is listed for completeness. (B3) and (B4) imply that V and W consist of the intervals and the points $\{u_n\}_{n=1}^{\infty}$ whose accumulation point is only ∞. It is clear that (B3) and (B4) are satisfied if $f_1(u)$ and $f_2(u)$ are analytic in u.

Now we state our main results.

Theorem 1.1. *Assume that $f_1(u)$ and $f_2(u)$ satisfy* (A1)–(A4) *if $N \ge 2$, and* (A1)–(A3) *if $N = 1$. Let $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ be the L^2-bifurcation branches of* (1.1)–(1.3) *associated with the nonlinear term $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Assume that $\lambda_1(\alpha) = \lambda_2(\alpha)$ for any $\alpha > 0$. Then $f_1(u) \equiv f_2(u)$ provided one of the conditions* (B1)–(B4) *is satisfied.*

For the case $N = 1$, we obtain the desired conclusion under the following condition.

(A5) There exists a constant $0 < \delta \ll 1$ such that $f(u)u \ge (2+\delta)F(u)$ for $u \gg 1$.

Theorem 1.2. *Let $N = 1$. Assume that $f_1(u)$ and $f_2(u)$ satisfy* (A1)–(A3), (A5). *Let $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ be the L^2-bifurcation branches of* (1.1)–(1.3) *associated with the nonlinear term $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Assume that $\lambda_1(\alpha) = \lambda_2(\alpha)$ for any $\alpha > 0$. Furthermore, assume that there exists a constant $u_0 > 0$ such that $f_1(u) = f_2(u)$ for any $u \ge u_0$. Then $f_1(u) \equiv f_2(u)$.*

Remark 1.3. (1) Let $f_1(u) = u^p$ and $f_2(u) = u^p + u^q(1 + \sin(1/u))$ for $0 < u < \delta \ll 1$ ($f_2(0) := 0$) with $q > p + 1 > 3$. For $u > \delta$, it is possible for us to define f_1 and f_2 to satisfy (A1)–(A4) and $f_1(u) \le f_2(u)$, that is, f_1 and f_2 satisfy (A1)–(A4) and (B1) (cf. Appendix). Therefore, by Theorem 1.1, we conclude that $\lambda_1(\alpha) \not\equiv \lambda_2(\alpha)$. However, it is clear that (B3) is not satisfied. Indeed, $f_1(u_n) = f_2(u_n)$ for $u_n = ((3 + 4n)\pi/2)^{-1}$ ($n \in \mathbb{N}$), namely, $u_n \in V$ and $u_n \to 0$ as $n \to \infty$. So $u = 0$ is the accumulation point of $\{u_n\}_{n=1}^{\infty}$. Similarly, we also see that (B2) and (B4) are irrelevant each other.

(2) It should be mentioned that (B3) implies (B4). The proof will be given in Appendix. We also introduce an example of f_1 and f_2 which does not satisfy (B3) but satisfies (B4) in Appendix.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1. Theorem 1.2 will be proved in Section 3. Section 4 is an appendix.

2. Proof of Theorem 1.1

We prove Theorem 1.1 under the conditions (B2) and (B4). We first note that $f(u)$ which satisfies (A1)–(A3) is extended naturally to the odd C^1 function on $\mathbb{R}$ by $f(u) = -f(-u)$ for $u < 0$. Therefore, $F(u)$ is regarded as the even function on $\mathbb{R}$.

In the following subsections 2.1 and 2.2, we show that, for a given $\alpha > 0$, the unique solution pair $(\lambda_j(\alpha), u_{j,\alpha})$ of (1.1)–(1.3), which satisfies $\|u_{j,\alpha}\|_2 = \alpha$, is obtained by variational method.

2.1. Critical value of (1.1)–(1.3).

We first define the critical values $C_1(\alpha)$ and $C_2(\alpha)$ corresponding to f_1 and f_2, respectively. Let $H_0^1(\Omega)$ be the usual real Sobolev space. For $j = 1, 2$, let $X_j := \{u \in H_0^1(\Omega) : \int_\Omega F_j(u(x))\,dx < \infty\}$. We put $Y_j := H_0^1(\Omega) \bigcap X_j$. Then by (A4), Y_j is a subspace of $H_0^1(\Omega)$. For $j = 1, 2$ and $v \in Y_j$, let

$$\Phi_j(v) := \frac{1}{2}\|\nabla v\|_2^2 + \int_\Omega F_j(v(x))\,dx. \tag{2.1}$$

For $j = 1, 2$ and $\alpha > 0$, we put

$$C_j(\alpha) := \inf\{\Phi_j(v) : v \in M_{j,\alpha}\}, \tag{2.2}$$

where $M_{j,\alpha} := \{v \in Y_j : \|v\|_2 = \alpha\}$. Let $\phi \in M_{j,1}$. Since $\alpha\phi \in M_{j,\alpha}$ for $\alpha > 0$, by (2.1), as $\alpha \to 0$,

$$0 \le C_j(\alpha) \le \Phi_j(\alpha\phi) = \frac{1}{2}\alpha^2 \|\nabla\phi\|_2^2 + \int_\Omega F_j(\alpha\phi(x))\,dx \to 0.$$

Therefore, we put $C_j(0) = 0$.

2.2. **Existence of unique minimizer.** Secondly, we show the existence of unique minimizer for $C_j(\alpha)$ $(j = 1, 2)$. By choosing a minimizing sequence $\{v_{j,\alpha,k}\}_{k=1}^\infty \subset M_{j,\alpha}$, which is non-negative (since we can choose $|v_{j,\alpha,k}|$), and choosing a subsequence of $\{v_{j,\alpha,k}\}_{k=1}^\infty$ again if necessary, there exists $v_{j,\alpha,\infty} \in H_0^1(\Omega)$ such that as $k \to \infty$,

$$\Phi(v_{j,\alpha,k}) \to C_j(\alpha), \tag{2.3}$$

$$v_{j,\alpha,k} \to v_{j,\alpha,\infty} \quad \text{weakly in } H_0^1(\Omega), \tag{2.4}$$

$$v_{j,\alpha,k} \to v_{j,\alpha,\infty} \quad \text{in } L^2(\Omega), \tag{2.5}$$

$$v_{j,\alpha,k} \to v_{j,\alpha,\infty} \quad \text{a.e. in } \Omega. \tag{2.6}$$

Then by (2.1), (2.3), (2.6) and Fatou's Lemma, we obtain

$$\begin{aligned} \Phi_j(v_{j,\alpha,\infty}) &= \frac{1}{2}\|\nabla v_{j,\alpha,\infty}\|_2^2 + \int_\Omega F_j(v_{j,\alpha,\infty}(x))\,dx \\ &\le \liminf_{k\to\infty} \frac{1}{2}\|\nabla v_{j,\alpha,k}\|_2^2 + \liminf_{k\to\infty} \int_\Omega F_j(v_{j,\alpha,k}(x))\,dx \\ &\le \liminf_{k\to\infty} \Big(\frac{1}{2}\|\nabla v_{j,\alpha,k}\|_2^2 + \int_\Omega F_j(v_{j,\alpha,k}(x))\,dx\Big) \\ &= \lim_{k\to\infty} \Phi_j(v_{j,\alpha,k}) = C_j(\alpha). \end{aligned} \tag{2.7}$$

By the above inequality,

$$\int_\Omega F_j(v_{j,\alpha,\infty}(x))\,dx < \infty.$$

This implies that $v_{j,\alpha,\infty} \in Y_j$. By this, (2.2), (2.5) and (2.7), we obtain that $v_{j,\alpha,\infty} \in M_{j,\alpha}$ $(j = 1, 2)$ and $C_j(\alpha) = \Phi_j(v_{j,\alpha,\infty})$. Then by Lagrange multiplier theorem, we see that $v_{j,\alpha,\infty}$ is a non-negative weak solution (consequently, a positive classical solution by a standard regularity theorem and strong maximum principle) of (1.1)–(1.3) with $f = f_j$. Namely, there exists a Lagrange multiplier $\lambda_{j,\alpha}$ such that $v_{j,\alpha,\infty} \in M_{j,\alpha}$ satisfies (1.1)–(1.3) with $f = f_j$ and $\lambda = \lambda_{j,\alpha}$. Then by (P1), we see that $(\lambda_{j,\alpha}, v_{j,\alpha,\infty}) = (\lambda_j(\alpha), u_\alpha)$. This implies the uniqueness of the minimizer. We write $(\lambda_1(\alpha), u_{1,\alpha})$ and $(\lambda_2(\alpha), u_{2,\alpha})$ for the solutions pair associated with f_1 and f_2, respectively.

Remark. If $N = 1$, then let $Y_j = H_0^1(\Omega)$. Since $H_0^1(\Omega) \subset C(\bar{\Omega})$, we obtain the same conclusion as above without (A4).

2.3. **The relationship between $C_j(\alpha)$ and $\lambda_j(\alpha)$.** In section 2.2, we see that $C_j(\alpha) = \Phi(u_{j,\alpha})$. By this, (1.1), (2.1) and integration by parts, we have [11]

$$\begin{aligned} \frac{dC_j(\alpha)}{d\alpha} &= \int_\Omega \nabla u_{j,\alpha} \frac{d}{d\alpha} \nabla u_{j,\alpha} + \int_\Omega f(u_{j,\alpha}) \frac{d}{d\alpha} u_{j,\alpha}\, dx \\ &= \int_\Omega \{-\Delta u_{j,\alpha} + f(u_{j,\alpha})\} \frac{d}{d\alpha} u_{j,\alpha}\, dx \\ &= \int_\Omega \lambda_j(\alpha) u_{j,\alpha} \frac{d}{d\alpha} u_{j,\alpha}\, dx \\ &= \frac{1}{2}\lambda_j(\alpha) \frac{d}{d\alpha} \int_\Omega u_{j,\alpha}^2\, dx = \lambda_j(\alpha)\alpha. \end{aligned} \tag{2.8}$$

Lemma 2.1. *$C_1(\alpha) = C_2(\alpha)$ for $\alpha \ge 0$.*

Proof. Since $C_1(0) = C_2(0) = 0$, by (2.8),

$$\begin{aligned} C_1(\alpha) &= \int_0^\alpha \frac{d}{ds} C_1(s)\, ds = \int_0^\alpha \lambda_1(s) s\, ds \\ &= \int_0^\alpha \lambda_2(s) s\, ds \\ &= \int_0^\alpha \frac{d}{ds} C_2(s)\, ds = C_2(\alpha). \end{aligned} \tag{2.9}$$

Thus, the proof is complete. □

Lemma 2.2. *Assume* (B2). *Then $f_1(u) \equiv f_2(u)$ for $u \ge 0$.*

Proof. If $F_1(u) = F_2(u)$ for all $u \ge 0$, then it is clear that $f_1(u) \equiv f_2(u)$ for $u \ge 0$. Assume that there exists $0 < u_0 < \infty$ such that $F_1(u_0) < F_2(u_0)$. Then there exists $\alpha > 0$ such that $\|u_{2,\alpha}\|_\infty = u_0$. Then by (B2),

$$\begin{aligned} C_1(\alpha) &\le \Phi_1(u_{2,\alpha}) \\ &= \frac{1}{2}\|\nabla u_{2,\alpha}\|_2^2 + \int_\Omega F_1(u_{2,\alpha}(x))\, dx \\ &< \frac{1}{2}\|\nabla u_{2,\alpha}\|_2^2 + \int_\Omega F_2(u_{2,\alpha}(x))\, dx = C_2(\alpha). \end{aligned} \tag{2.10}$$

This contradicts Lemma 2.1. Therefore, we obtain that $F_1(u) = F_2(u)$ for all $u \ge 0$, which implies our conclusion. Thus the proof is complete. □

Lemma 2.3. *Assume* (B4). *Then $f_1(u) \equiv f_2(u)$ for $u \ge 0$.*

Proof. By using the fact that $0 \in W$, we show that $W = [0, \infty)$, namely, $F_1(u) \equiv F_2(u)$ for $u \ge 0$, which implies our conclusion. There are two cases to consider.

(a) Assume that $u = 0$ is contained in the interval $[0, \epsilon] \subset W$ for some constant $0 < \epsilon \ll 1$. This implies that $F_1(u) = F_2(u)$ for $0 \le u \le \epsilon$. We consider the connected component K of W such that $[0, \epsilon] \subset K$. We show that $K = [0, \infty)$. If there exists $u_1 > 0$ such that $K = [0, u_1]$, then without loss of generality, there exists a constant $0 < \epsilon_1 \ll 1$ such that $F_1(u) < F_2(u)$ for $u_1 < u < u_1 + \epsilon_1$ by (B4).

We choose $\alpha > 0$ satisfying $\|u_{2,\alpha}\|_\infty = u_1 + \epsilon_1$. Then

$$\begin{aligned} C_1(\alpha) \le \Phi_1(u_{2,\alpha}) &= \frac{1}{2}\|\nabla u_{2,\alpha}\|_2^2 + \int_\Omega F_1(u_{2,\alpha}(x))\,dx \\ &< \frac{1}{2}\|\nabla u_{2,\alpha}\|_2^2 + \int_\Omega F_2(u_{2,\alpha}(x))\,dx = C_2(\alpha). \end{aligned} \tag{2.11}$$

This contradicts Lemma 2.1. Therefore, we see that $u_1 = \infty$ and $K = [0, \infty)$. This implies that $F_1(u) = F_2(u)$ for all $u \ge 0$ and obtain our conclusion. This proves case (a).

(b) Assume that $u = 0$ is an isolated point in W. Then by (B4), without loss of generality, there exists a constant $0 < \epsilon \ll 1$ such that $F_1(u) < F_2(u)$ for $0 < u < \epsilon$. Then by the same argument as that in (a) just above, we can derive a contradiction. Therefore, the case (b) does not occur.

Combining (a) and (b), we obtain our conclusion. Thus the proof is complete. □

3. Proof of Theorem 1.2

In this section, let $\Omega = I = (0,1)$, $\lambda_\alpha = \lambda_1(\alpha) = \lambda_2(\alpha)$ and $M_\alpha = M_{1,\alpha} = M_{2,\alpha}$ for simplicity. Further, C denotes various positive constants independent of $\alpha \gg 1$.

Lemma 3.1. *Assume that there exists $\alpha > 0$ such that*

$$\int_I F_1(u_{1,\alpha}(x))\ dx \ge \int_I F_2(u_{1,\alpha}(x))\ dx. \tag{3.1}$$

Then $f_1(u) = f_2(u)$ for $0 \le u \le \|u_{1,\alpha}\|_\infty$.

Proof. Since $u_{1,\alpha} \in M_\alpha$, by (2.2), Lemma 2.1 and (3.1),

$$\begin{aligned} C_1(\alpha) &= \frac{1}{2}\|u'_{1,\alpha}\|_2^2 + \int_I F_1(u_{1,\alpha}(x))\,dx \\ &\ge \frac{1}{2}\|u'_{1,\alpha}\|_2^2 + \int_I F_2(u_{1,\alpha}(x))\,dx \ge C_2(\alpha). \end{aligned} \tag{3.2}$$

This along with Lemma 2.1 implies that $C_2(\alpha) = \Phi_2(u_{1,\alpha})$. Then by section 2.2, we obtain $u_{1,\alpha} \equiv u_{2,\alpha}$. By this and (1.1), $f_1(u_{1,\alpha}(x)) = f_2(u_{1,\alpha}(x))$ for $x \in I$. Thus the proof is complete. □

corollary 3.2. *Assume that there exists $\{\alpha_k\}_{k=1}^\infty$ such that $\alpha_k \to \infty$ as $k \to \infty$, and satisfies*

$$\int_I F_1(u_{1,\alpha_k}(x))\,dx \ge \int_I F_2(u_{1,\alpha_k}(x))\,dx \tag{3.3}$$

or

$$\int_I F_2(u_{2,\alpha_k}(x))\,dx \ge \int_I F_1(u_{2,\alpha_k}(x))\,dx. \tag{3.4}$$

Then $f_1(u) \equiv f_2(u)$.

Finally, we consider the case where neither (3.3) nor (3.4) holds.

Lemma 3.3. *Assume that there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$,*

$$\int_I F_1(u_{1,\alpha}(x))\,dx < \int_I F_2(u_{1,\alpha}(x))\,dx, \tag{3.5}$$

$$\int_I F_2(u_{2,\alpha}(x))\,dx < \int_I F_1(u_{2,\alpha}(x))\,dx. \tag{3.6}$$

Then $f_1(u) \equiv f_2(u)$.

Proof. For $u > u_0$,

$$\int_{u_0}^{u} f_1(s)\,ds = F_1(u) - \int_0^{u_0} f_1(s)\,ds, \tag{3.7}$$

$$\int_{u_0}^{u} f_2(s)\,ds = F_2(u) - \int_0^{u_0} f_2(s)\,ds. \tag{3.8}$$

Since $f_1(u) = f_2(u)$ for $u \ge u_0$, by (3.7) and (3.8), for $u \ge u_0$,

$$F_1(u) = F_2(u) + C_0, \tag{3.9}$$

where $C_0 := \int_0^{u_0}(f_1(s) - f_2(s))ds$. We consider the cases where $C_0 = 0$, $C_0 > 0$ and $C_0 < 0$. To this end, we need some useful tools. It is well known [2] that for $j = 1, 2$,

$$u_{j,\alpha}(x) = u_{j,\alpha}(1-x), \quad x \in \bar{I}, \tag{3.10}$$

$$u'_{j,\alpha}(x) > 0, \quad x \in (0, 1/2), \tag{3.11}$$

$$\|u_{j,\alpha}\|_\infty = u_{j,\alpha}(1/2), \tag{3.12}$$

$$\frac{f_j(\|u_{j,\alpha}\|_\infty)}{\|u_{j,\alpha}\|_\infty} \le \lambda_\alpha \le \frac{f_j(\|u_{j,\alpha}\|_\infty)}{\|u_{j,\alpha}\|_\infty} + \pi^2. \tag{3.13}$$

Multiply (1.1) by $u'_{1,\alpha}(x)$. Then for $x \in \bar{I}$, we obtain

$$[u''_{1,\alpha}(x) + \lambda_\alpha u_{1,\alpha}(x) - f_1(u_{1,\alpha}(x))]u'_{1,\alpha}(x) = 0.$$

This implies that for $x \in \bar{I}$,

$$\begin{aligned}
&\frac{1}{2}u'_{1,\alpha}(x)^2 + \frac{1}{2}\lambda_\alpha u_{1,\alpha}(x)^2 - F_1(u_{1,\alpha}(x)) \equiv \text{constant} \\
&= \frac{1}{2}\lambda_\alpha \|u_{1,\alpha}\|_\infty^2 - F_1(\|u_{1,\alpha}\|_\infty) \quad (\text{put } x = 1/2).
\end{aligned}$$

By this and (3.11), for $0 \le x \le 1/2$,

$$u'_{1,\alpha}(x) = \sqrt{\lambda_\alpha(\|u_{1,\alpha}\|_2^2 - u_{1,\alpha}(x)^2) - 2(F_1(\|u_{1,\alpha}\|_\infty) - F(u_{1,\alpha}(x)))}. \tag{3.14}$$

Case 1. Assume that $C_0 = 0$ in (3.9). By (3.5), (3.9), (3.10), (3.12) and (3.14), for $\alpha \gg 1$,

$$\begin{aligned}
0 &< \int_I (F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))\,dx \\
&= 2\int_0^{1/2} (F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))\,dx \\
&= 2\int_0^{1/2} \frac{(F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))u'_{1,\alpha}(x)}{\sqrt{\lambda_\alpha(\|u_{1,\alpha}\|_\infty^2 - u_{1,\alpha}(x)^2) - 2(F_1(\|u_{1,\alpha}\|_\infty) - F(u_{1,\alpha}(x)))}}\,dx \\
&= 2\int_0^{\|u_{1,\alpha}\|_\infty} \frac{F_2(\theta) - F_1(\theta)}{\sqrt{\lambda_\alpha(\|u_{1,\alpha}\|_\infty^2 - \theta^2) - 2(F_1(\|u_{1,\alpha}\|_\infty) - F_1(\theta))}}\,d\theta \\
&= \frac{2}{\sqrt{\lambda_\alpha}\|u_{1,\alpha}\|_\infty}\int_0^{u_0} \frac{F_2(\theta) - F_1(\theta)}{\sqrt{L_\alpha(\theta)}}\,d\theta,
\end{aligned} \tag{3.15}$$

where

$$L_\alpha(\theta) := 1 - \frac{\theta^2}{\|u_{1,\alpha}\|_\infty^2} - \frac{2}{\lambda_\alpha\|u_{1,\alpha}\|_\infty^2}(F_1(\|u_{1,\alpha}\|_\infty) - F_1(\theta)).$$

By this, (A5) and (3.13), there exists a constant $0 < \epsilon \ll 1$ such that for $\alpha \gg 1$ and $0 \le \theta \le u_0$,

$$\begin{aligned} 1 &\ge L_\alpha(\theta) \\ &\ge 1 - \frac{\theta^2}{\|u_{1,\alpha}\|_\infty^2} - \frac{2f_1(\|u_{1,\alpha}\|_\infty)}{(2+\delta)\lambda_\alpha\|u_{1,\alpha}\|_\infty} \\ &\ge \Big(1 - \frac{2}{2+\delta}\Big)(1-\epsilon) := 1 - \epsilon_0. \end{aligned} \tag{3.16}$$

By this and (3.15), for $\alpha \gg 1$,

$$\int_0^{u_0} (F_2(\theta) - F_1(\theta))\, d\theta > 0. \tag{3.17}$$

On the other hand, by (3.6) and the same arguments as just above, we obtain

$$\int_0^{u_0} (F_2(\theta) - F_1(\theta))\, d\theta < 0. \tag{3.18}$$

This contradicts (3.17). Therefore, we obtain that either (3.3) or (3.4) must hold. This implies that $f_1(u) \equiv f_2(u)$ by Corollary 3.2.

Case 2. Assume that $C_0 > 0$. Then by (3.5), (3.9) and (3.15),

$$\begin{aligned} 0 &< \int_I (F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))\, dx \\ &= 2\int_0^{1/2} (F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))\, dx \\ &= 2\int_0^{1/2} \frac{(F_2(u_{1,\alpha}(x)) - F_1(u_{1,\alpha}(x)))u_{1,\alpha}'(x)\, dx}{\sqrt{\lambda_1(\alpha)(\|u_{1,\alpha}\|_\infty^2 - u_{1,\alpha}(x)^2) - 2(F_1(\|u_{1,\alpha}\|_\infty) - F(u_{1,\alpha}(x)))}} \\ &= 2\int_0^{\|u_{1,\alpha}\|_\infty} \frac{F_2(\theta) - F_1(\theta)}{\sqrt{\lambda_\alpha(\|u_{1,\alpha}\|_\infty^2 - \theta^2) - 2(F_1(\|u_{1,\alpha}\|_\infty) - F_1(\theta))}}\, d\theta \\ &= \frac{2}{\sqrt{\lambda_\alpha}\|u_{1,\alpha}\|_\infty}\Big(\int_{u_0}^{\|u_{1,\alpha}\|_\infty} \frac{-C_0}{\sqrt{L_\alpha(\theta)}}\, d\theta + \int_0^{u_0} \frac{F_2(\theta) - F_1(\theta)}{\sqrt{L_\alpha(\theta)}}\, d\theta\Big). \end{aligned} \tag{3.19}$$

By this, we obtain

$$0 < A + B := \int_{u_0}^{\|u_{1,\alpha}\|_\infty} \frac{-C_0}{\sqrt{L_\alpha(\theta)}}\, d\theta + \int_0^{u_0} \frac{F_2(\theta) - F_1(\theta)}{\sqrt{L_\alpha(\theta)}}\, d\theta. \tag{3.20}$$

It is clear by (3.16) that $|B| \le C_2$ for $\alpha \gg 1$. Let C_3 be a constant sufficiently large. If $\alpha \gg 1$, then (3.16) is also valid for $0 \le \theta \le u_0 + C_3$, we obtain

$$A < \int_{u_0}^{u_0+C_3} \frac{-C_0}{\sqrt{L_\alpha(\theta)}}\, d\theta < -C_0C_3 < -C_2. \tag{3.21}$$

This contradicts (3.20). Thus we obtain the same conclusion as that in Case 1. The case $C_0 < 0$ can be treated by using (3.6) instead of (3.5), and the same argument as that in Case 2. Thus the proof is complete. $\square$

4. Appendix

(B3) implies (B4). Let $u_0 \in W$, that is, $F_1(u_0) = F_2(u_0)$. Without loss of generality, we may assume that $u_0 > 0$, since the case $u_0 = 0$ can be treated similarly to the case $u_0 > 0$.

Case 1. Assume that $f_1(u_0) < f_2(u_0)$. Then there exists $\delta > 0$ such that $f_1(u) < f_2(u)$ for $u_0 - \delta \le u \le u_0 + \delta$. This implies that $F_1(u) < F_2(u)$ for $u_0 < u \le u_0 + \delta$ and $F_1(u) > F_2(u)$ for $u_0 - \delta \le u < u_0$. Therefore, u_0 is not an accumulation point of W.

Case 2. Assume that $f_1(u_0) > f_2(u_0)$. Then by the same argument as that in Case 1, we conclude that $F_1(u) > F_2(u)$ for $u_0 < u \le u_0 + \delta$ and $F_1(u) < F_2(u)$ for $u_0 - \delta \le u < u_0$. Therefore, u_0 is not an accumulation point of W.

Case 3. Assume that $f_1(u_0) = f_2(u_0)$. If $f_1(u) < f_2(u)$ (resp. $f_1(u) > f_2(u)$) for $u_0 < u \le u_0 + \delta$, then as in the Case1 and Case 2 above, we obtain that $F_1(u) < F_2(u)$ (resp. $F_1(u) > F_2(u)$) for $u_0 < u \le u_0 + \delta$.

If $f_1(u) > f_2(u)$ (resp. $f_1(u) < f_2(u)$) for $u_0 - \delta \le u < u_0$, then as just above, we obtain that $F_1(u) < F_2(u)$ (resp. $F_1(u) > F_2(u)$) for $u_0 - \delta \le u < u_0$. Therefore, u_0 is not an accumulation point of W.

Assume that there exists a constant $\epsilon > 0$ such that $f_1(u) = f_2(u)$ for $u_0 \le u \le u_0 + \epsilon$. Then we obtain that $F_1(u) \equiv F_2(u)$ for $u_0 \le u \le u_0 + \epsilon$. Now, there are three cases to consider: there exists a constant $0 < \delta \ll 1$ such that for $u \in [u_0 - \delta, u_0)$, (i) $f_1(u) < f_2(u)$, (ii) $f_1(u) > f_2(u)$, (iii) $f_1(u) = f_2(u)$. These three cases can be treated by the same arguments as those in Cases 1 and 2 above, and we obtain that u_0 is either included in the interval in W or not an accumulation point of W. Thus the proof is complete. ☐

Example of f_1 and f_2 which satisfies (B4) but does not satisfy (B1) and (B3). For $n \in \mathbb{N}$ sufficiently large, let $\delta_n := 2/(\pi(1+2n))$. We put

$$f_1(u) = u^p, \quad u \ge 0, \tag{4.1}$$

$$f_2(u) = \begin{cases} u^p + u^q\big(\frac{1}{2} + \sin\frac{1}{u}\big), & 0 < u < \delta_n, \\ u^p + \frac{3}{2}u^q, & u \ge \delta_n, \end{cases} \tag{4.2}$$

where $q > p + 1 > 3$. We define $f_2(0) = 0$. Then f_2 is C^1 and for $0 \le u < \delta_n$,

$$\big(\frac{f_2(u)}{u}\big)' = u^{p-2}\Big(p - 1 + (q-1)u^{q-p}(\sin\frac{1}{u} + \frac{1}{2}) - u^{q-p-1}\cos\frac{1}{u}\Big) > 0.$$

This implies that $f_2(u)$ satisfies (A1)–(A3) for $0 \le u < \delta_n$, and consequently, for $u \ge 0$. Clearly, the pair $f_1(u)$ and $f_2(u)$ does not satisfy (B1) near $u = 0$ and (B3) at $u = 0$.

We show that $f_2(u)$ satisfies (A4). Indeed, we have

$$\begin{aligned} F_2(u) &= \frac{1}{p+1}u^{p+1} + \int_0^u x^q\big(\frac{1}{2} + \sin\frac{1}{x}\big)\,dx \\ &= \frac{1}{p+1}u^{p+1} + \frac{1}{2(q+1)}u^{q+1} + \frac{1}{q+1}u^{q+1}\sin\frac{1}{u} + \frac{1}{q+1}\int_0^u x^{q-1}\cos\frac{1}{x}\,dx. \end{aligned} \tag{4.3}$$

Note that $(u+v)^p \le 2^p(u^p+v^p)$ for $u, v \ge 0$. Then by (4.3), for $0 < u+v < \delta_n$,

$$
\begin{aligned}
&F_2(u+v)\\
&\le C_1(u^{p+1}+v^{p+1}) + C_2(u^{q+1}+v^{q+1}) + C_3(u^q+v^q)\\
&\le C_4(u^{p+1}+v^{p+1})\\
&\le C_5\Big(\frac{1}{p+1}u^{p+1} - \frac{1}{2(q+1)}u^{q+1} - \frac{1}{q(q+1)}u^q\Big)\\
&\quad + C_5\Big(\frac{1}{p+1}v^{p+1} - \frac{1}{2(q+1)}v^{q+1} - \frac{1}{q(q+1)}v^q\Big)\\
&\le C_6\Big(\frac{1}{p+1}u^{p+1} + \frac{1}{2(q+1)}u^{q+1} + \frac{1}{q+1}u^{q+1}\sin\frac{1}{u} + \frac{1}{q+1}\int_0^u x^{q-1}\cos\frac{1}{x}\,dx\Big)\\
&\quad + C_6\Big(\frac{1}{p+1}v^{p+1} + \frac{1}{2(q+1)}v^{q+1} + \frac{1}{q+1}v^{q+1}\sin\frac{1}{v} + \frac{1}{q+1}\int_0^v x^{q-1}\cos\frac{1}{x}\,dx\Big)\\
&\le C_6(F_2(u)+F_2(v)).
\end{aligned} \tag{4.4}
$$

By (4.1) and (4.2), we easily see that (4.4) holds for $u+v \ge u_1 \gg 1$ if we choose $C_6 > 0$ and u_1 sufficiently large. Moreover, it is clear that (4.4) holds for $\delta_n \le u+v \le u_1$ if we choose $C_6 > 0$ suitably. Consequently, we obtain that $f_2(u)$ satisfies (A4). Note that $F_1(u) = u^{p+1}/(p+1)$. We show that $F_1(u) < F_2(u)$ for $u > 0$. Namely, (B2) (and consequently, (B4)) holds. To this end, we show that for $0 < u < \delta_n$,

$$
\int_0^u x^q\Big(\frac{1}{2} + \sin\frac{1}{x}\Big)\,dx \ge Cu^{q+1}. \tag{4.5}
$$

By the above inequality,

$$
\int_0^u x^q\Big(\frac{1}{2} + \sin\frac{1}{x}\Big)\,dx = \frac{1}{2(q+1)}u^{q+1} + \int_0^u x^q \sin\frac{1}{x}\,dx := G + H. \tag{4.6}
$$

By putting $\theta = 1/x$, we have

$$
H = \int_{1/u}^{\infty} \theta^{-(q+2)}\sin\theta\,d\theta = u^{q+2}\cos(1/u) - (q+2)\int_{1/u}^{\infty}\theta^{-(q+3)}\cos\theta\,d\theta. \tag{4.7}
$$

Then clearly,

$$
\Big|\int_{1/u}^{\infty}\theta^{-(q+3)}\cos\theta\,d\theta\Big| \le Cu^{q+2}. \tag{4.8}
$$

By this and (4.7), for $0 < u < \delta_n$, we have $|H| \le Cu^{q+2}$. This along with (4.6) implies (4.5). By (4.3) and (4.5), we obtain that $F_1(u) < F_2(u)$ for $0 < u < \delta_n$. Since $f_1(u) < f_2(u)$ for $u \ge \delta_n$, by the result above, we find that $F_1(u) < F_2(u)$ for $u > 0$. Thus the proof is complete.

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