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POSITIVE SOLUTIONS FOR NONLINEAR SECOND-ORDER m-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. By constructing a special cone and applying the fixed index theory in the cone, we prove the existence of positive solutions for a class of singular m-point boundary-value problems.

1. Introduction

This paper considers the existence of positive solutions for the second-order m-point boundary-value problem

$$(p(t)x'(t))' - q(t)x(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$ax(0) - bp(0)x'(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad cx(1) + dp(1)x'(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \tag{1.2}$$

where $a, c \in [0, +\infty)$, $b, d \in (0, +\infty)$ with $ac + ad + bc > 0$, $\xi_i \in (0, 1)$, $\alpha_i, \beta_i \in [0, +\infty)$ for $i \in \{1, 2, \dots, m-2\}$ are given constants, $p \in C^1([0, 1], (0, +\infty))$, $q \in C([0, 1], (0, +\infty))$ and $f \in C((0, 1) \times (0, +\infty), [0, +\infty))$, $f(t, x)$ is allowed to be singular at $t = 0$, $t = 1$ and $x = 0$.

If $p \equiv 1$, $q \equiv 0$, $\alpha_i, \beta_i = 0$, (for $i = 1, 2, \dots, m-2$), then (1.1)-(1.2) reduces to the two-point boundary-value problem

$$x''(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \tag{1.3}$$

$$ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0, \tag{1.4}$$

which has been intensively studied; see [5, 6].

In [7], by using the fixed index theory in a cone, positive solutions were obtained for differential systems

$$\begin{gathered} -x''(t) = f(t, y), \quad t \in (0, 1), \\ -y''(t) = g(t, x), \quad t \in (0, 1), \\ \alpha_1 x(0) - \beta_1 x'(0) = \gamma_1 x(1) + \delta_1 x'(1) = 0, \end{gathered}$$

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$$\alpha_2 y(0) - \beta_2 y'(0) = \gamma_2 y(1) + \delta_2 y'(1) = 0,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ and $\rho_i = \alpha_i\gamma_i + \alpha_i\gamma_i + \gamma_i\beta_i > 0$ $(i = 1, 2)$, $f(t, y)$ and $g(t, x)$ may be singular at $t = 0$, $t = 1$ and $x = 0$, $y = 0$, respectively.

In recent years, singular multi-point boundary-value problems have been extensively studied and many optimal results have been obtained, see [6, 11, 12, 13, 14] and references therein. In addition, many papers investigated the existence of solutions for the nonsingular multi-point boundary-value problems, for example, [1, 4, 5, 10].

Recently, Ma [8], Ma and Thompson [9] obtained excellent results about the existence of positive solutions for the more general m-point boundary-value problem (1.1)-(1.2), but in the above papers there are no studies for singularity of the nonlinearity $f(t, x)$ at the point $x = 0$. Recently, by using Nonlinear Alternative of Leray-Schauder with the properties of the associated vector field at the (u, u') plane, Galanis and Palamides [2] studied the problem

$$-[\phi_p(u')]' = q(t)f(t, u(t)), \quad 0 < t < 1$$

subject to

$$u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0,$$

or to

$$u(0) - \alpha u'(\eta) = 0, \quad u(1) + g(u'(1)) = 0,$$

where $f(t, u)$ is allowed to have singularity at $u = 0$, the obtained solutions remains away from the origin and avoid the singularity of the nonlinear term at $u = 0$.

Motivated by the above mentioned papers, we consider the existence of positive solutions for (1.1)-(1.2). Here we allow $f(t, x)$ to have a singularity at $t = 0, 1$, and at $x = 0$. As far as we know, there were only a few works when f has singularities at $t = 0, 1$ and $x = 0$. This paper attempts to fill part of this gap in the literature.

This work is organized as follows. In section 2, we present some lemmas that are used to prove our main results. Then in section 3, the existence of positive solution for (1.1)-(1.2) will be established by using the fixed point theory in the cone, which we state here for the convenience of the reader.

Lemma 1.1 ([3]). *Let P be a cone of the real Banach space E, Ω be a bounded open subset of E with $\theta \in \Omega$ and $T : \overline{\Omega} \cap P \to P$ is a completely continuous. Suppose that $Tu \neq \lambda u$, for all $u \in \partial\Omega \cap P$, $\lambda \geq 1$, then $i(T, \Omega \cap P, P) = 1$.*

Lemma 1.2 ([3]). *Let P be a cone of the real Banach space E, Ω be a bounded open subset of E with $\theta \in \Omega$ and $T : \overline{\Omega} \cap P \to P$ is a completely continuous. Suppose that*

(i) $\inf_{u \in P \cap \partial\Omega} \|Tu\| > 0$
(ii) *$Tu \neq \lambda u$, for all $u \in \partial\Omega \cap P$, $\lambda \in (0, 1]$,*

then $i(T, \Omega \cap P, P) = 0$.

2. Preliminaries

Let $E = C[0, 1]$ be a real Banach space, with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ for $x \in C[0, 1]$. Let $P = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. Clearly P is a cone in E.

The function x is said to be a positive solution of (1.1)-(1.2) if $x(t)$ is positive solution on $(0, 1)$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

The following lemmas play an important role when proving our main results.

Lemma 2.1 ([8, 9]). *Assume*

(H1) $p \in C^1([0,1],(0,+\infty))$, $q \in C([0,1],(0,+\infty))$.

Let ψ and ϕ be the solutions of the linear problems

$$(p(t)\psi'(t))'(t) - q(t)\psi(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$\psi(0) = b, \quad p(0)\psi'(0) = a, \tag{2.2}$$

and

$$(p(t)\phi'(t))'(t) - q(t)\phi(t) = 0, \quad t \in (0,1), \tag{2.3}$$

$$\phi(1) = d, \quad p(1)\phi'(1) = -c, \tag{2.4}$$

respectively. Then

(i) *ψ is strictly increasing on $[0,1]$, and $\psi(t) > 0$ on $[0,1]$;*

(ii) *ϕ is strictly decreasing on $[0,1]$, and $\phi(t) > 0$ on $[0,1]$.*

As in [9], set

$$\Delta = \det\begin{pmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i) \end{pmatrix}, \quad \rho = p(t)\det\begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}.$$

Then, by Liouville's formula, we have

$$\rho = p(0)\det\begin{pmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{pmatrix} = \text{constant}.$$

Define

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi(t)\psi(s), & 0 \le s \le t \le 1, \\ \phi(s)\psi(t), & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

It is easy to see that

$$0 \le G(t,s) \le G(s,s), \quad 0 \le s,\ t \le 1. \tag{2.6}$$

Remark 2.2. By (2.5) and Lemma 2.1, for any $t \in [0,1]$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{\phi(t)}{\phi(s)}, & 0 \le s \le t \le 1, \\ \frac{\psi(t)}{\psi(s)}, & 0 \le t \le s \le 1, \end{cases} \ge \begin{cases} \frac{d}{\phi(0)}, & 0 \le s \le t \le 1, \\ \frac{b}{\psi(1)}, & 0 \le t \le s \le 1. \end{cases}$$

Let $\gamma = \min\{\frac{d}{\phi(0)},\ \frac{b}{\psi(1)}\}$, then $G(t,s) \ge \gamma G(s,s)$, for $t,s \in [0,1]$.

Remark 2.3. Since $\gamma = \min\{\frac{d}{\phi(0)}, \frac{b}{\psi(1)}\}$, according to the monotonicity of $\psi(t)$, we have $\gamma \le \frac{b}{\psi(1)} = \frac{\psi(0)}{\psi(1)} \le \frac{\psi(t)}{\psi(1)}$, so $\psi(t) \ge \gamma\psi(1)$, for $t \in [0,1]$. Similarly, by the monotonicity of $\phi(t)$, we have $\gamma \le \frac{d}{\phi(0)} = \frac{\phi(1)}{\phi(0)} \le \frac{\phi(t)}{\phi(0)}$, so $\phi(t) \ge \gamma\phi(0)$, for $t \in [0,1]$.

Lemma 2.4 ([8, 9]). *Assume* (H1) *and that $\Delta \neq 0$. Then for any $y \in L[0,1]$, the problem*

$$(p(t)x'(t))'(t) - q(t)x(t) + y(t) = 0, \quad t \in (0,1), \tag{2.7}$$

$$ax(0) - bp(0)x'(0) = \sum_{i=1}^{m-2}\alpha_i x(\xi_i), \quad cx(1) + dp(1)x'(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i), \tag{2.8}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)y(s)ds + A(y)\psi(t) + B(y)\phi(t), \tag{2.9}$$

where

$$A(y)=\frac{1}{\Delta}\det\begin{pmatrix}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds & \rho-\sum_{i=1}^{m-2}\alpha_i\phi(\xi_i)\\ \sum_{i=1}^{m-2}\beta_i\int_0^1 G(\xi_i,s)y(s)ds & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i)\end{pmatrix} \tag{2.10}$$

$$B(y)=\frac{1}{\Delta}\det\begin{pmatrix}-\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds\\ \rho-\sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & \sum_{i=1}^{m-2}\beta_i\int_0^1 G(\xi_i,s)y(s)ds\end{pmatrix}. \tag{2.11}$$

Lemma 2.5 ([8, 9]). *Assume* (H1) *and*

(H2) $\Delta<0,\ \rho-\sum_{i=1}^{m-2}\alpha_i\phi(\xi_i)>0,\ \rho-\sum_{i=1}^{m-2}\beta_i\psi(\xi_i)>0.$

Then for $y\in L[0,1]$ with $y\geq 0$, the unique solution x of (2.7)-(2.8) *satisfies $x(t)\geq 0$, for $t\in[0,1]$.*

Let $Q=\{x\in P: x(t)\geq\gamma\|x\|\}$. It is obvious that Q is a subcone of P. With Lemma 2.4, Problem (1.1)-(1.2) has a positive solution $x=x(t)$ if and only if $x\in Q\backslash\{\theta\}$ is a solution of the nonlinear integral equation

$$x(t)=\int_0^1 G(t,s)f(s,x(s))ds+A(f(s,x(s)))\psi(t)+B(f(s,x(s)))\phi(t), \tag{2.12}$$

where f satisfies the condition

(H3) $f\in C((0,1)\times(0,+\infty),[0,+\infty))$ and there exist $h\in C((0,1),[0,+\infty))$, $g\in C((0,+\infty),[0,+\infty))$ satisfying that for any $t\in(0,1)$, $u\in(0,+\infty)$ implies

$$f(t,u)\leq h(t)g(u),\quad t\in(0,1),\ u\in(0,+\infty),$$
$$0<\int_0^1 G(s,s)h(s)ds<+\infty.$$

Define an operator $T: Q\backslash\{\theta\}\to P$ by

$$(Tx)(t)=\int_0^1 G(t,s)f(s,x(s))ds+A(f(s,x(s)))\psi(t)+B(f(s,x(s)))\phi(t). \tag{2.13}$$

It is easy to prove that the existence of solutions to (1.1)-(1.2) is equivalent to the existence of solutions to (2.12). That is, the existence of a fixed point of operator T.

To overcome the singularity, we consider the following approximating equation of (2.13) with the boundary conditions (1.2).

$$(T_nx)(t)=\int_0^1 G(t,s)f_n(s,x(s))ds+A(f_n(s,x(s)))\psi(t)+B(f_n(s,x(s)))\phi(t), \tag{2.14}$$

where n is a positive integer and

$$f_n(t,x)=f(t,\max\{\frac{1}{n},x\}). \tag{2.15}$$

Remark 2.6. By (H3), there exists $\tau\in(0,\frac{1}{2})$ such that

$$0<\int_\tau^{1-\tau} G(s,s)h(s)ds<+\infty.$$

Lemma 2.7. *Assume* (H1)-(H3). *Then $T_n: P\to P$ is completely continuous for any fixed natural number n.*

Proof. First it is easy to see that T_n maps P into P. Then we prove that T_n maps bounded sets into bounded sets.

Suppose $D \subset P$ is an arbitrary bounded set. Then there exists a constant $M_1 > 0$ such that $\|x\| \le M_1$ for any $x \in D$. By (H1), for any $x \in D$ and $s \in [0,1]$, we have

$$
\begin{aligned}
|(T_nx)(t)| &= \int_0^1 G(t,s)f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(t) + B(f_n(s,x(s)))\phi(t)\\
&\le \int_0^1 G(s,s)h(s)g\big(\max\{\frac{1}{n},x(s)\}\big)ds\\
&\quad + A\Big(h(s)g\big(\max\{\frac{1}{n},x(s)\}\big)\Big)\psi(1)\\
&\quad + B\Big(h(s)g\big(\max\{\frac{1}{n},x(s)\}\big)\Big)\phi(0)\\
&\le M_2\Big[\int_0^1 G(s,s)h(s)ds + A\left(h(s)\right)\psi(1) + B\left(h(s)\right)\phi(0)\Big]\\
&\le M_2(1 + A\psi(1) + B\phi(0))\int_0^1 G(s,s)h(s)ds,
\end{aligned}
$$

where $M_2 = \sup_{x\in[\frac{1}{n},\frac{1}{n}+M_1]} g(x)$,

$$
A = \frac{1}{\Delta}\det\begin{pmatrix} \sum_{i=1}^{m-2}\alpha_i & \rho - \sum_{i=1}^{m-2}\alpha_i\phi(\xi_i)\\ \sum_{i=1}^{m-2}\beta_i & -\sum_{i=1}^{m-2}\beta_i\phi(\xi_i)\end{pmatrix}, \tag{2.16}
$$

$$
B = \frac{1}{\Delta}\det\begin{pmatrix} -\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i\\ \rho - \sum_{i=1}^{m-2}\beta_i\psi(\xi_i) & \sum_{i=1}^{m-2}\beta_i\end{pmatrix}. \tag{2.17}
$$

Therefore, $T_n(D)$ is uniformly bounded.

Now we show that $T_n(D)$ is equicontinuous on $[0,1]$. For any $\varepsilon > 0$, since $G(t,s),\psi(t)$ and $\phi(t)$ are uniformly continuous on $[0,1]\times[0,1]$ and $[0,1]$, respectively. There exists $\delta > 0$ such that for any $t_1,t_2 \in [0,1]$, $|t_1 - t_2| < \delta$ implies that

$$
\begin{gathered}
|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon\min_{0\le s\le 1} G(s,s)}{3M_2\int_0^1 G(s,s)h(s)ds},\\
|\psi(t_1) - \psi(t_2)| < \frac{\varepsilon}{3M_2A\int_0^1 G(s,s)h(s)ds},\\
|\phi(t_1) - \phi(t_2)| < \frac{\varepsilon}{3M_2B\int_0^1 G(s,s)h(s)ds}.
\end{gathered}
$$

Consequently, for any $x \in D$, $t_1,t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, we have

$$
\begin{aligned}
&|T_nx(t_1) - T_nx(t_2)|\\
&\le \int_0^1 |G(t_1,s) - G(t_2,s)|f_n(s,x(s))ds\\
&\quad + A(f_n(s,x(s)))|\psi(t_1) - \psi(t_2)| + B(f_n(s,x(s)))|\phi(t_1) - \phi(t_2)|\\
&\le \int_0^1 |G(t_1,s) - G(t_2,s)|h(s)g\big(\max\{\frac{1}{n},x(s)\}\big)ds\\
&\quad + A\Big(h(s)g\big(\max\{\frac{1}{n},x(s)\}\big)\Big)|\psi(t_1) - \psi(t_2)|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + B\Big(h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)\Big)|\phi(t_1) - \phi(t_2)| \\
&\le M_2 \int_0^1 |G(t_1, s) - G(t_2, s)|h(s)ds \\
&\quad + A(h(s)M_2)|\psi(t_1) - \psi(t_2)| + B(h(s)M_2)|\phi(t_1) - \phi(t_2)| \\
&\le M_2 \int_0^1 |G(t_1, s) - G(t_2, s)|h(s)ds \\
&\quad + M_2 A|\psi(t_1) - \psi(t_2)| \int_0^1 G(s, s)h(s)ds \\
&\quad + M_2 B|\phi(t_1) - \phi(t_2)| \int_0^1 G(s, s)h(s)ds \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus, $T_n(D)$ is equicontinuous on $[0, 1]$. According to Ascoli-Arzela Theorem, $T_n(D)$ is a relatively compact set.

In the end, we show T_n is continuous. Suppose $x_m, x \in D, x_m \to x$ $(m \to +\infty)$. Then there exists a constant $M_3 > 0$ such that $\|x\| \le M_3$, $\|x_m\| \le M_3$ $(m = 1, 2, \dots)$. Since $f_n(t, x)$ is uniformly continuous on $[0, 1] \times D$ for any fixed natural number n, hence,

$$
\lim_{m\to+\infty} f_n(t, x_m(t)) = f_n(t, x(t)), \quad \text{uniformly on } t \in [0, 1].
$$

According to the Lebesgue dominated convergence theorem,

$$
\lim_{m\to+\infty} \int_0^1 G(s, s)|f_n(s, x_m(s)) - f_n(s, x(s))|ds = 0.
$$

Thus for the above $\varepsilon > 0$, there exists a natural number M, such that $m > M$ implies that

$$
\int_0^1 G(s, s)|f_n(s, x_m(s)) - f_n(s, x(s))|ds < \frac{\varepsilon}{1 + A\psi(1) + B\phi(0)}. \tag{2.18}
$$

From (2.18), we obtain that for $m > M$,

$$
\begin{aligned}
&\|T_n u_m - T_n u\| \\
&= \max_{0\le t\le 1} \Big[\int_0^1 G(t, s)f_n(s, x_m(s))ds + A(f_n(s, x_m(s)))\psi(t) + B(f_n(s, x_m(s)))\phi(t) \\
&\quad - \int_0^1 G(t, s)f_n(s, x(s))ds - A(f_n(s, x(s)))\psi(t) - B(f_n(s, x(s)))\phi(t)\Big] \\
&\le \int_0^1 G(s, s)|f_n(s, x_m(s)) - f_n(s, x(s))|ds \\
&\quad + A\big(|f_n(s, x_m(s)) - f_n(s, x(s))|\big)\psi(1) \\
&\quad + B(|f_n(s, x_m(s)) - f_n(s, x(s))|)\phi(0) \\
&\le \big(1 + A\psi(1) + B\phi(0)\big) \int_0^1 G(s, s)|f_n(s, x_m(s)) - f_n(s, x(s))|ds < \varepsilon.
\end{aligned}
$$

Therefore, $T_n : P \to P$ is continuous. Thus $T_n : P \to P$ is a completely continuous operator. $\square$

Lemma 2.8. $T_n(Q) \subset Q$.

Proof. For any $x \in Q$, (H2) and (H3) imply $(T_n x)(t) \geq 0$. From (2.6), (2.14) and the monotonicity of $\psi(t)$ and $\phi(t)$, we have

$$(T_n x)(t) \leq \int_0^1 G(s,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(1) + B(f_n(s,x(s)))\phi(0),$$

which implies

$$\|T_n x\| \leq \int_0^1 G(s,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(1) + B(f_n(s,x(s)))\phi(0). \tag{2.19}$$

By Remarks 2.2 and 2.3, we have

$$\begin{aligned}(T_n x)(t) &= \int_0^1 G(t,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(t) + B(f_n(s,x(s)))\phi(t)\\ &\geq \gamma \int_0^1 G(s,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\gamma\psi(1) + B(f_n(s,x(s)))\gamma\phi(0)\\ &\geq \gamma \Big[\int_0^1 G(s,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(1) + B(f_n(s,x(s)))\phi(0)\Big].\end{aligned} \tag{2.20}$$

Then, (2.19) and (2.20) yield

$$(T_n x)(t) \geq \gamma \|T_n x\|.$$

Hence $T_n x \in Q$. □

3. Main results

In this section, we present our main results as follows.

Theorem 3.1. *Suppose that* (H1)–(H3) *hold and there exist numbers* $R > 0$ *and* $L > 0$ *such that*

$$\int_0^1 G(s,s)h(s)ds < \frac{R}{\widetilde{M}(1 + A\psi(1) + B\phi(0))}, \tag{3.1}$$

$$L\gamma^2 \int_\tau^{1-\tau} G(s,s)ds > 1, \quad \liminf_{x\to+\infty} \min_{\tau\leq t\leq 1-\tau} \frac{f(t,x)}{x} > L. \tag{3.2}$$

Then (1.1)-(1.2) *has at least one positive solution, where* $\widetilde{M} = \max_{u\in[\gamma R, 1+R]} g(u)$, γ *is defined in Remark 2.2 and* A, B *are defined by* (2.16) *and* (2.17)*, respectively.*

Proof. Firstly, we shall prove that when n is sufficiently large, we have

$$T_n x \neq \lambda x, \quad x \in \partial Q_R,\ \lambda \geq 1, \tag{3.3}$$

where $Q_R = \{x \in Q : \|x\| < R\}$ for $R > 0$. In fact, if there exists $x_0 \in \partial Q_R$, and $\lambda_0 \geq 1$ such that $\lambda_0 x_0 = T_n x_0$, then $x_0(t) \leq T_n x_0(t)$ for $t \in [0,1]$ and any n.

Choose a sufficiently large n satisfying $n > \frac{1}{\gamma R}$. Then we have

$$\begin{aligned}
x_0(t) &\leq (T_n x_0)(t) \\
&= \int_0^1 G(t,s) f_n(s, x_0(s)) ds + A(f_n(s, x_0(s)))\psi(t) + B(f_n(s, x_0(s)))\phi(t) \\
&\leq \int_0^1 G(s,s) f_n(s, x_0(s)) ds + A(f_n(s, x_0(s)))\psi(1) + B(f_n(s, x_0(s)))\phi(0) \\
&\leq (1 + A\psi(1) + B\phi(0)) \int_0^1 G(s,s) f_n(s, x_0(s)) ds \\
&\leq (1 + A\psi(1) + B\phi(0)) \int_0^1 G(s,s) h(s) g\big(\max\{\frac{1}{n}, x_0(s)\}\big) ds \\
&\leq (1 + A\psi(1) + B\phi(0)) \widetilde{M} \int_0^1 G(s,s) h(s) ds < R.
\end{aligned} \tag{3.4}$$

Therefore, by (3.4) we have $\|x_0\| < R$, which is a contradiction to $x_0 \in \partial Q_R$. So applying Lemma 1.1, $i(T_n, Q_R, Q) = 1$.

Next, according to (3.2), there exists R_1 such that $x > R_1$ implies

$$f(t,x) > Lx, \quad t \in [\tau, 1-\tau]. \tag{3.5}$$

Choose $R' > \{R, \gamma^{-1} R_1\}$. When n being sufficiently large we can claim that

$$T_n x \neq \lambda x, \quad \forall x \in \partial Q_{R'},\ \lambda \in (0,1], \tag{3.6}$$

where $Q_R' = \{x \in Q : \|x\| < R'\}$. Suppose (3.6) is not true, then there exist $x_1 \in \partial Q_{R'}$ and $\lambda' \in (0,1]$ such that $\lambda' x_1 = T_n x_1$. Similarly, we choose sufficiently large n satisfying that $n > \frac{1}{\gamma R'}$. Therefore, by (3.5) we have

$$\begin{aligned}
x_1(t) &\geq (T_n x_1)(t) \\
&= \int_0^1 G(t,s) f_n(s, x_1(s)) ds + A(f_n(s, x_1(s)))\psi(t) + B(f_n(s, x_1(s)))\phi(t) \\
&\geq \int_0^1 G(t,s) f_n(s, x_1(s)) ds \\
&\geq \gamma \int_\tau^{1-\tau} G(s,s) f_n(s, x_1(s)) ds \\
&\geq L\gamma \int_\tau^{1-\tau} G(s,s) x_1(s) ds \\
&\geq LR'\gamma^2 \int_\tau^{1-\tau} G(s,s) ds.
\end{aligned}$$

This is a contradiction to $x_1 \in \partial Q_{R'}$. Consequently, (3.6) holds. Furthermore, for each $x \in \partial Q_R$,

$$\begin{aligned}
\|T_n x\| &\geq \int_0^1 G(t,s) f_n(s, x(s)) ds + A(f_n(s, x(s)))\psi(t) + B(f_n(s, x(s)))\phi(t) \\
&\geq \int_0^1 G(t,s) f_n(s, x(s)) ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \gamma \int_{\tau}^{1-\tau} G(s,s) f_n(s,x(s))ds \\
&\geq L\gamma \int_{\tau}^{1-\tau} G(s,s)x(s)ds \\
&\geq LR'\gamma^2 \int_{\tau}^{1-\tau} G(s,s)ds.
\end{aligned}
$$

So $\inf_{x\in\partial Q_{R'}} \|T_n x\| > 0$. Thus from Lemma 1.2, $i(T_n, Q_{R'}, Q) = 0$.

By the additivity of fixed point index, we know that

$$
i(T_n, Q_{R'} \setminus \overline{Q}_R, Q) = i(T_n, Q_{R'}, Q) - i(T_n, Q_R, Q) = -1.
$$

As a result, there exist $x_n \in Q_{R'} \setminus \overline{Q}_R$ satisfying $T_n x_n = x_n$ provided that n is sufficiently large.

Without loss of generality, suppose $T_n x_n = x_n$, $n \geq n_0$. Let $D = \{x_n\}_{n\geq n_0}$ be the sequence of solutions to (2.14). It is not difficult to prove that D is uniformly bounded. Next we show $\{x_n\}_{n\geq n_0}$ is equicontinuous on $[0,1]$. It is obvious that we only need to prove $\lim_{t\to 0+}(x_n(t) - x_n(0)) = 0$, $\lim_{t\to 1-}(x_n(t) - x_n(1)) = 0$ uniformly with respect to $n \geq n_0$ and D is equicontinuous on $[\sigma, 1-\sigma] \subset (0,1)$ for $\sigma \in (0, 1/2)$.

Now we prove that

$$
\lim_{t\to 0+} (x_n(t) - x_n(0)) = 0, \quad \text{uniformly with respect to } n \geq n_0. \tag{3.7}
$$

According to (2.12),

$$
\begin{aligned}
&\big|x_n(t) - x_n(0)\big| \\
&= \Big| \int_0^1 G(t,s) f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(t) + B(f_n(s,x(s)))\phi(t) \\
&\quad - \int_0^1 G(0,s) f_n(s,x(s))ds - A(f_n(s,x(s))\psi(0) - B(f_n(s,x(s))\phi(0)\Big| \\
&= \frac{1}{\rho}\phi(t) \int_0^t \psi(s) f_n(s,x(s))ds + \frac{1}{\rho}(\psi(t) - \psi(0)) \int_t^1 \phi(s) f_n(s,x(s))ds \qquad (3.8)\\
&\quad - \frac{1}{\rho}\phi(0) \int_0^t \psi(s) f_n(s,x(s))ds + A(f_n(s,x(s)))(\psi(t) - \psi(0)) \\
&\quad + B(f_n(s,x(s)))(\phi(t) - \phi(0)) \\
&\leq \frac{1}{\rho}\phi(t) \int_0^t \psi(s)h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)ds \\
&\quad + \frac{1}{\rho}(\psi(t) - \psi(0)) \int_t^1 \phi(s)h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)ds \\
&\quad - \frac{1}{\rho}\phi(0) \int_0^t \psi(s)h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)ds \\
&\quad + A\Big(h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)\Big)(\psi(t) - \psi(0)) \\
&+ B\Big(h(s)g\big(\max\{\frac{1}{n}, x(s)\}\big)\Big)(\phi(0) - \phi(t)) \\
&\leq \frac{1}{\rho}M_4\phi(t) \int_0^t \psi(s)h(s)ds + \frac{1}{\rho}M_4(\psi(t) - \psi(0)) \int_t^1 \phi(s)h(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\rho}\phi(0)M_4\int_0^t\psi(s)h(s)ds+AM_4(\psi(t)-\psi(0))\int_0^1G(s,s)h(s)ds\\
&+BM_4(\phi(0)-\phi(t))\int_0^1G(s,s)h(s)ds.
\end{aligned}
$$

Since $\{x_n(t)\}_{n\geq n_0}$ is uniformly bounded, it follows that $\{g(x_n)\}_{n\geq n_0}$ is bounded. Therefore, there exists a constant M_4 such that $\|g(x_n)\|<M_4$ for $n\geq n_0$. This together with (H3) and (3.8) show that we need to prove only that

$$\lim_{t\to 0+}\frac{1}{\rho}\phi(t)\int_0^t\psi(s)h(s)ds=0, \tag{3.9}$$

$$\lim_{t\to 0+}\frac{1}{\rho}\phi(0)\int_0^t\psi(s)h(s)ds=0, \tag{3.10}$$

$$\lim_{t\to 0+}\frac{1}{\rho}(\psi(t)-\psi(0))\int_t^1\phi(s)h(s)ds=0, \tag{3.11}$$

$$\lim_{t\to 0+}(\psi(t)-\psi(0))=0,\quad \lim_{t\to 0+}(\phi(0)-\phi(t))=0. \tag{3.12}$$

Since $\psi(t)$ and $\phi(t)$ are continuous on $[0,1]$, (3.12) holds. For all $\varepsilon>0$, by the absolutely continuity of integral function and (H3), there exists $\delta_1\in(0,\frac{1}{2})$ such that $t_1,t_2\in[0,1],|t_1-t_2|<\delta_1$ implies

$$\Big|\int_{t_1}^{t_2}G(s,s)h(s)ds\Big|<\varepsilon. \tag{3.13}$$

Therefore, from (2.5) and (3.13), we have

$$
\begin{gathered}
\frac{1}{\rho}\phi(t)\int_0^t\psi(s)h(s)ds\leq\int_0^tG(s,s)h(s)ds<\varepsilon,\quad t\in(0,\delta_1],\\
\frac{1}{\rho}\phi(0)\int_0^t\psi(s)h(s)ds\leq\frac{\phi(0)}{\phi(\delta_1)}\int_0^tG(s,s)h(s)ds<\varepsilon,\quad t\in(0,\delta_1];
\end{gathered}
$$

i.e., (3.9) and (3.10) hold.

$$
\begin{aligned}
&\frac{1}{\rho}(\psi(t)-\psi(0))\int_t^1\phi(s)h(s)ds\\
&\leq\frac{1}{\rho}\psi(t)\int_t^{\delta_1}\phi(s)h(s)ds+\frac{1}{\rho}(\psi(t)-\psi(0))\int_{\delta_1}^1\phi(s)h(s)ds\\
&\leq\int_t^{\delta_1}G(s,s)h(s)ds+\frac{\psi(t)-\psi(0)}{\psi(\delta_1)}\int_{\delta_1}^1G(s,s)h(s)ds\leq 2\varepsilon.
\end{aligned}
$$

That is, (3.11) holds. By (3.9)-(3.12), (3.7) holds. Since

$$
\begin{aligned}
&|x_n(t)-x_n(1)|\\
&=\Big|\int_0^1G(t,s)f_n(s,x(s))ds+A(f_n(s,x(s)))\psi(t)+B(f_n(s,x(s)))\phi(t)\\
&\quad-\int_0^1G(1,s)f_n(s,x(s))ds-A(f_n(s,x(s))\psi(1)-B(f_n(s,x(s))\phi(1)\Big|\\
&\leq\frac{1}{\rho}(\phi(t)-\phi(1))\int_0^t\psi(s)f_n(s,x(s))ds+\frac{1}{\rho}\psi(t)\int_t^1\phi(s)f_n(s,x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\rho}\phi(1)\int_t^1 \psi(s)f_n(s,x(s))ds + A(f_n(s,x(s)))(\psi(t)-\psi(1)) \\
&+ B(f_n(s,x(s)))(\phi(t)-\phi(1)) \\
&\le \frac{1}{\rho}M_4(\phi(t)-\phi(1))\int_0^t \psi(s)h(s)ds + \frac{1}{\rho}M_4\psi(t)\int_t^1 \phi(s)h(s)ds \\
&+ \frac{1}{\rho}\phi(1)M_4\int_0^t \psi(s)h(s)ds + AM_4(\psi(1)-\psi(t))\int_0^1 G(s,s)h(s)ds \\
&+ BM_4(\phi(t)-\phi(1))\int_0^1 G(s,s)h(s)ds,
\end{aligned}
$$

similar to the above, we can easily prove that

$$
\lim_{t\to 1-}(x_n(t)-x_n(1)) = 0, \quad \text{uniformly with respect to } n \ge n_0. \tag{3.14}
$$

Next we prove that D is equicontinuous on $[\sigma, 1-\sigma]$ for any $\sigma \in (0,1/2)$. In fact, for $n \ge n_0$, $t_1, t_2 \in [\sigma, 1-\sigma]$ with $t_2 > t_1$, we have

$$
\begin{aligned}
&|x_n(t_2)-x_n(t_1)| \\
&= \Big| \int_0^1 G(t_2,s)f_n(s,x(s))ds + A(f_n(s,x(s)))\psi(t_2) + B(f_n(s,x(s)))\phi(t_2) \\
&\quad - \int_0^1 G(t_1,s)f_n(s,x(s))ds - A(f_n(s,x(s)))\psi(t_1) - B(f_n(s,x(s)))\phi(t_1)\Big| \\
&\le \frac{1}{\rho}(\phi(t_2)-\phi(t_1))\int_0^{t_1} \psi(s)f_n(s,x(s))ds + \frac{1}{\rho}\phi(t_2)\int_{t_1}^{t_2} \psi(s)f_n(s,x(s))ds \\
&\quad + \frac{1}{\rho}(\psi(t_2)-\psi(t_1))\int_{t_2}^1 \phi(s)f_n(s,x(s))ds - \frac{1}{\rho}\psi(t_1)\int_{t_1}^{t_2} \phi(s)f_n(s,x(s))ds \\
&\quad + A(f_n(s,x(s)))(\psi(t_2)-\psi(t_1)) + B(f_n(s,x(s)))(\phi(t_2)-\phi(t_1)).
\end{aligned} \tag{3.15}
$$

By (2.5) and the monotonicity of $\psi(t)$ and $\phi(t)$, we have

$$
\frac{1}{\rho}\int_0^{t_1} \psi(s)h(s)ds \le \frac{1}{d}\int_0^1 G(s,s)h(s)ds, \tag{3.16}
$$

$$
\frac{1}{\rho}\int_{t_2}^1 \phi(s)h(s)ds \le \frac{1}{b}\int_0^1 G(s,s)h(s)ds, \tag{3.17}
$$

$$
\frac{1}{\rho}\phi(t_2)\int_{t_1}^{t_2} \psi(s)h(s)ds \le \int_{t_1}^{t_2} G(s,s)h(s)ds, \tag{3.18}
$$

$$
\frac{1}{\rho}\psi(t_1)\int_{t_1}^{t_2} \phi(s)h(s)ds \le \int_{t_1}^{t_2} G(s,s)h(s)ds. \tag{3.19}
$$

By (3.15)-(3.19), we have

$$
\begin{aligned}
&|x_n(t_2)-x_n(t_1)| \\
&\le M_4(\frac{1}{b}+A)(\psi(t_2)-\psi(t_1))\int_0^1 G(s,s)h(s)ds \\
&\quad + M_4(\frac{1}{d}+B)(\phi(t_1)-\phi(t_2))\int_0^1 G(s,s)h(s)ds + 2M_4\int_{t_1}^{t_2} G(s,s)h(s)ds.
\end{aligned}
$$

By the above inequality, (H3), (3.9)-(3.12), and continuity of $\psi(t)$, $\phi(t)$, D is equicontinuous on $[\sigma, 1-\sigma]$.

From the above proof, we can know D is equicontinuous on $[0,1]$. It follows from Ascoli-Arzela's theorem that the sequence $\{x_n\}_{n\geq n_0}$ has a subsequence which uniformly converges on $[0,1]$. Without loss of generality, we assume that $\{x_n\}$ itself uniformly converges to x on $[0,1]$. According to the Lebesgue's dominated theorem, we know that x is the positive solution of (1.1)-(1.2). $\square$

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