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EXISTENCE OF QUADRATIC-MEAN ALMOST PERIODIC SOLUTIONS TO SOME STOCHASTIC HYPERBOLIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we obtain the existence of quadratic-mean almost periodic solutions to some classes of partial hyperbolic stochastic differential equations. The main result of this paper generalizes in a natural fashion some recent results by authors. As an application, we consider the existence of quadratic-mean almost periodic solutions to the stochastic heat equation with divergence terms.

1. Introduction

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real Hilbert space which is separable and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t : t \in \mathbb{R}\}$, that is, a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$.

For the rest of this article, if $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ is a linear operator, we then define the operator $A : D(A) \subset L^2(\Omega, \mathbb{H}) \mapsto L^2(\Omega, \mathbb{H})$ as follows: $X \in D(A)$ and $AX = Y$ if and only if $X, Y \in L^2(\Omega, \mathbb{H})$ and $\mathcal{A}X(\omega) = Y(\omega)$ for all $\omega \in \Omega$.

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ be a sectorial linear operator. For $\alpha \in (0,1)$, let $\mathbb{H}_\alpha$ denote the intermediate Banach space between $D(\mathcal{A})$ and $\mathbb{H}$. Examples of those $\mathbb{H}_\alpha$ include, among others, the fractional spaces $D((-\mathcal{A})^\alpha)$, the real interpolation spaces $D_{\mathcal{A}}(\alpha, \infty)$ due to Lions and Peetre, and the Hölder spaces $D_{\mathcal{A}}(\alpha)$, which coincide with the continuous interpolation spaces that both Da Prato and Grisvard introduced in the literature.

In Bezandry and Diagana [2], the concept of quadratic-mean almost periodicity was introduced and studied. In particular, such a concept was, subsequently, utilized to study the existence and uniqueness of a quadratic-mean almost periodic solution to the class of stochastic differential equations

$$dX(t) = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset L^2(\Omega; \mathbb{H}) \mapsto L^2(\Omega; \mathbb{H})$ is a densely defined closed linear operator, and $F : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \mapsto L^2(\Omega; \mathbb{H})$, $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \mapsto L^2(\Omega; \mathcal{L}_2^0)$ are jointly continuous functions satisfying some additional conditions.

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Similarly, in [3], Bezandry and Diagana made extensive use of the same very concept of quadratic-mean almost periodicity to study the existence and uniqueness of a quadratic-mean almost periodic solution to the class of nonautonomous semilinear stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F(t, X(t))\,dt + G(t, X(t))\,dW(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operators satisfying the so-called Acquistapace and Terreni conditions [1], $F : \mathbb{R}\times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$, $G : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathcal{L}_2^0)$ are jointly continuous satisfying some additional conditions, and $W(t)$ is a Wiener process.

The present paper is definitely inspired by [2, 3, 6] and consists of studying the existence of quadratic-mean almost periodic solutions to the stochastic differential equation of the form

$$\begin{aligned} &d\Big(X(\omega, t) + f(t, \mathcal{B}X(\omega, t))\Big) \\ &= \big[\mathcal{A}X(\omega, t) + g(t, \mathcal{C}X(\omega, t))\big]dt + h(t, \mathcal{L}X(\omega, t))dW(\omega, t) \end{aligned} \tag{1.3}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$, where $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{L}$ are (possibly unbounded linear operators on $\mathbb{H}$) and $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_\beta (0 < \alpha < \frac{1}{2} < \beta < 1)$, $g : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, and $h : \mathbb{R} \times \mathbb{H} \to \mathcal{L}_2^0$ are jointly continuous functions.

To analyze (1.3), our strategy consists of studying the existence of quadratic-mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$d\Big(X(t) + F(t, BX(t))\Big) = \big[AX(t) + G(t, CX(t))\big]dt + H(t, LX(t))dW(t) \tag{1.4}$$

for all $t \in \mathbb{R}$, where $A : D(A) \subset L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(A) \cap i\mathbb{R} = \emptyset$, B, C, and L are (possibly unbounded linear operators on $L^2(\Omega, \mathbb{H})$) and $F : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H}_\beta)$ $(0 < \alpha < \frac{1}{2} < \beta < 1)$, $G : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$, and $H : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathcal{L}_2^0)$ are jointly continuous functions satisfying some additional assumptions.

It is worth mentioning that the main results of this paper generalize those obtained in Bezandry and Diagana [3].

The existence of almost periodic (respectively, periodic) solutions to autonomous stochastic differential equations has been studied by many authors, see, e.g., [1, 2, 9, 16] and the references therein. In particular, Da Prato and Tudor [5], have studied the existence of almost periodic solutions to (1.2) in the case when $A(t)$ is periodic. Though the existence and uniqueness of quadratic-mean almost periodic solutions to (1.4) in the case when A is sectorial is an important topic with some interesting applications, which is still an untreated question and constitutes the main motivation of the present paper. Among other things, we will make extensive use of the method of analytic semigroups associated with sectorial operators and the Banach's fixed-point principle to derive sufficient conditions for the existence and uniqueness of a quadratic-mean almost periodic solution to (1.4). To illustrate our abstract results, we study the existence of quadratic-mean almost periodic solutions to the stochastic heat equation with divergence coefficients.

2. Preliminaries

For details on this section, we refer the reader to [2, 4] and the references therein. Throughout the rest of this paper, we assume that $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ and $(\mathbb{H}, \|\cdot\|)$ are real separable Hilbert spaces, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. The space $L_2(\mathbb{K}, \mathbb{H})$ stands for the space of all Hilbert-Schmidt operators acting from $\mathbb{K}$ into $\mathbb{H}$, equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$.

For a symmetric nonnegative operator $Q \in L_2(\mathbb{K}, \mathbb{H})$ with finite trace we assume that $\{W(t),\ t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{K}$. It is worth mentioning that the Wiener process W can obtained as follows: let $\{W_i(t),\ t \in \mathbb{R}\},\ i = 1, 2$, be independent $\mathbb{K}$-valued Q-Wiener processes, then

$$W(t) = \begin{cases} W_1(t) & \text{if } t \geq 0, \\ W_2(-t) & \text{if } t \leq 0, \end{cases}$$

is Q-Wiener process with the real number line as time parameter. We then let $\mathcal{F}_t = \sigma\{W(s), s \leq t\}$.

The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, will be denoted $L^2(\Omega, \mathbb{H})$. Of course, this is a Banach space when it is equipped with norm

$$\|X\|_{L^2(\Omega,\mathbb{H})} = \Big(\mathrm{E}\|X\|^2\Big)^{1/2},$$

where the expectation E is defined by

$$\mathrm{E}[g] = \int_\Omega g(\omega) d\mathbb{P}(\omega).$$

Let $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ and let $\mathcal{L}_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$ with respect to the norm

$$\|\Phi\|^2_{\mathcal{L}_2^0} = \|\Phi Q^{1/2}\|_2^2 = \mathrm{Trace}(\Phi Q \Phi^*)\,.$$

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. This setting requires the following preliminary definitions.

Definition 2.1. A stochastic process $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ is said to be continuous whenever

$$\lim_{t\to s} \mathrm{E}\|X(t) - X(s)\|^2 = 0.$$

Definition 2.2. A continuous stochastic process $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ is said to be quadratic mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathrm{E}\|X(t+\tau) - X(t)\|^2 < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ which are quadratic mean almost periodic is then denoted by $AP(\mathbb{R}; L^2(\Omega; \mathbb{B}))$.

The next lemma provides some properties of quadratic mean almost periodic processes.

Lemma 2.3. *If X belongs to $AP(\mathbb{R}; L^2(\Omega; \mathbb{B}))$, then*

(i) *the mapping $t \to \mathrm{E}\|X(t)\|^2$ is uniformly continuous;*
(ii) *there exists a constant $M > 0$ such that $\mathrm{E}\|X(t)\|^2 \leq M$, for all $t \in \mathbb{R}$.*

Let $\mathrm{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \mapsto L^2(\Omega;\mathbb{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathrm{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ is a Banach space when it is equipped with the norm:

$$\|X\|_\infty = \sup_{t\in\mathbb{R}} \Big(\mathrm{E}\|X(t)\|^2\Big)^{1/2}.$$

Lemma 2.4. *$AP(\mathbb{R}; L^2(\Omega;\mathbb{B})) \subset \mathrm{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ is a closed subspace.*

In view of the above, the space $AP(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ of quadratic mean almost periodic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1, \|\cdot\|_{\mathbb{B}_1})$ and $(\mathbb{B}_2, \|\cdot\|_{\mathbb{B}_2})$ be Banach spaces and let $L^2(\Omega;\mathbb{B}_1)$ and $L^2(\Omega;\mathbb{B}_2)$ be their corresponding L^2-spaces, respectively.

Definition 2.5. A function $F : \mathbb{R} \times L^2(\Omega;\mathbb{B}_1) \to L^2(\Omega;\mathbb{B}_2))$, $(t, Y) \mapsto F(t, Y)$, which is jointly continuous, is said to be quadratic mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$ where $\mathbb{K} \subset L^2(\Omega;\mathbb{B}_1)$ is a compact if for any $\varepsilon > 0$, there exists $l(\varepsilon, \mathbb{K}) > 0$ such that any interval of length $l(\varepsilon, \mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathrm{E}\|F(t+\tau, Y) - F(t, Y)\|^2_{\mathbb{B}_2} < \varepsilon$$

for each stochastic process $Y : \mathbb{R} \to \mathbb{K}$.

Theorem 2.6. *Let $F : \mathbb{R}\times L^2(\Omega;\mathbb{B}_1) \to L^2(\Omega;\mathbb{B}_2)$, $(t, Y) \mapsto F(t, Y)$ be a quadratic mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$, where $\mathbb{K} \subset L^2(\Omega;\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathrm{E}\|F(t, Y) - F(t, Z)\|^2_{\mathbb{B}_2} \le M\mathrm{E}\|Y - Z\|^2_{\mathbb{B}_1}$$

for all $Y, Z \in L^2(\Omega;\mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any quadratic mean almost periodic process $\Phi : \mathbb{R} \to L^2(\Omega;\mathbb{B}_1)$, the stochastic process $t \mapsto F(t, \Phi(t))$ is quadratic mean almost periodic.

3. Sectorial Operators on $\mathbb{H}$

In this section, we introduce some notations and collect some preliminary results from Diagana [7] that will be used later. If $\mathcal{A}$ is a linear operator on $\mathbb{H}$, then $\rho(\mathcal{A})$, $\sigma(\mathcal{A})$, $D(\mathcal{A})$, $\ker(\mathcal{A})$, $R(\mathcal{A})$ stand for the resolvent set, spectrum, domain, kernel, and range of $\mathcal{A}$. If $\mathbb{B}_1, \mathbb{B}_2$ are Banach spaces, then the notation $B(\mathbb{B}_1, \mathbb{B}_2)$ stands for the Banach space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. When $\mathbb{B}_1 = \mathbb{B}_2$, this is simply denoted $B(\mathbb{B}_1)$.

Definition 3.1. A linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\zeta \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that $S_{\theta,\zeta} \subset \rho(\mathcal{A})$,

$$S_{\theta,\zeta} := \{\lambda \in \mathbb{C} : \lambda \ne \zeta,]; |\arg(\lambda - \zeta)| < \theta\},$$

$$\|R(\lambda, \mathcal{A})\| \le \frac{M}{|\lambda - \zeta|}, \quad \lambda \in S_{\theta,\zeta}$$

where $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ for each $\lambda \in \rho(\mathcal{A})$.

Remark 3.2. If the operator $\mathcal{A}$ is sectorial, then it generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathbb{H})$ and such that there exist constants M_0, $M_1 > 0$ such that

$$\|T(t)\| \leq M_0 e^{\zeta t}, \quad t > 0 \tag{3.1}$$

$$\|t(\mathcal{A} - \zeta I)T(t)\| \leq M_1 e^{\zeta t}, \quad t > 0 \tag{3.2}$$

Definition 3.3. A semigroup $(T(t))_{t\geq 0}$ is hyperbolic; that is, there exist a projection P and constants M, $\delta > 0$ such that $T(t)$ commutes with P, $\ker(P)$ is invariant with respect $T(t)$, $T(t) : R(S) \to R(S)$ is invertible, and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\|, \quad t > 0, \tag{3.3}$$

$$\|T(t)Sx\| \leq Me^{\delta t}\|x\|, \quad t \leq 0, \tag{3.4}$$

where $S := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t\geq 0}$ associated with the linear operator $\mathcal{A}$ is hyperbolic if and only if $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Definition 3.4. Let $\alpha \in (0,1)$. A Banach space $(\mathbb{H}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(\mathcal{A})$ and $\mathbb{H}$, or a space of class $\mathcal{J}_\alpha$, if $D(\mathcal{A}) \subset \mathbb{H}_\alpha \subset \mathbb{H}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq c\|x\|^{1-\alpha}\|x\|^\alpha_{[D(\mathcal{A})]}, \quad x \in D(\mathcal{A}), \tag{3.5}$$

where $\|\cdot\|_{[D(\mathcal{A})]}$ is the graph norm of $\mathcal{A}$. Here, $\|u\|_{[D(\mathcal{A})]} = \|u\| + \|\mathcal{A}u\|$, for each $u \in D(\mathcal{A})$.

Concrete examples of $\mathbb{H}_\alpha$ include $D((-\mathcal{A})^\alpha)$ for $\alpha \in (0,1)$, the domains of the fractional powers of $\mathcal{A}$, the real interpolation spaces $D_{\mathcal{A}}(\alpha,\infty)$, $\alpha \in (0,1)$, defined as the space of all $x \in \mathbb{H}$ such that

$$[x]_\alpha = \sup_{0\leq t\leq 1} \|t^{1-\alpha}(\mathcal{A} - \zeta I)e^{-\zeta t}T(t)x\| < \infty,$$

with the norm

$$\|x\|_\alpha = \|x\| + [x]_\alpha,$$

and the abstract Holder spaces $D_{\mathcal{A}}(\alpha) := \overline{D(\mathcal{A})}^{\|\cdot\|_\alpha}$.

Lemma 3.5 ([6, 7])**.** *For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, there exist constants $C(\alpha) > 0$, $\delta > 0$, $M(\alpha) > 0$, and $\gamma > 0$ such that*

$$\|T(t)Sx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \quad \text{for } t \leq 0, \tag{3.6}$$

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \quad \text{for } t > 0. \tag{3.7}$$

The next Lemma is crucial for the rest of the paper. A version of it in a general Banach space can be found in Diagana [6, 7].

Lemma 3.6 ([6, 7])**.** *Let $0 < \alpha < \beta < 1$. For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, there exist constants $c > 0$, $\delta > 0$, and $\gamma > 0$ such that*

$$\|\mathcal{A}T(t)Qx\|_\alpha \leq n(\alpha,\beta)e^{\delta t}\|x\| \leq n'(\alpha,\beta)e^{\delta t}\|x\|_\beta, \quad \text{for } t \leq 0 \tag{3.8}$$

$$\|\mathcal{A}T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \leq M'(\alpha)t^{-\alpha}e^{-\gamma t}\|x\|_\beta, \quad \text{for } t > 0. \tag{3.9}$$

Also, for $\Xi \in \mathcal{L}_2^0$,

$$\|\mathcal{A}T(t)Q\Xi\|_{\mathcal{L}_2^0} \leq n_1(\alpha,\beta)e^{\delta t}\|\Xi\|_{\mathcal{L}_2^0}, \quad \text{for } t \leq 0 \tag{3.10}$$

$$\|\mathcal{A}T(t)P\Xi\|_{\mathcal{L}_2^0} \leq M_1(\alpha)t^{-\alpha}e^{-\gamma t}\|\Xi\|_{\mathcal{L}_2^0}, \quad \text{for } t > 0. \tag{3.11}$$

4. Existence of Quadratic-Mean Almost Periodic Solutions

This section is devoted to the existence and uniqueness of a quadratic-mean almost periodic solution to the stochastic hyperbolic differential equation (1.4)

Definition 4.1. Let $\alpha \in (0,1)$. A continuous random function, $X : \mathbb{R} \to L^2(\Omega; \mathbb{H}_\alpha)$ is said to be a bounded solution of (1.4) provided that the function $s \to AT(t-s)PF(s, BX(s))$ is integrable on $(-\infty, t)$, $s \to AT(t-s)QF(s, BX(s))$ is integrable on (t, ∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned}
X(t) = -F(t, BX(t)) &- \int_{-\infty}^{t} AT(t-s)PF(s, BX(s))\, ds \\
&+ \int_{t}^{\infty} AT(t-s)SF(s, BX(s))\, ds \\
&+ \int_{-\infty}^{t} T(t-s)PG(s, CX(s))\, ds - \int_{t}^{\infty} T(t-s)SG(s, CX(s))\, ds \\
&+ \int_{-\infty}^{t} T(t-s)PH(s, LX(s))\, dW(s) - \int_{t}^{\infty} T(t-s)SH(s, LX(s))\, dW(s)
\end{aligned}$$

for each $t \in \mathbb{R}$.

In the rest of this article, we denote by Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 the nonlinear integral operators defined by

$$\begin{aligned}
(\Gamma_1 X)(t) &:= \int_{-\infty}^{t} AT(t-s)PF(s, BX(s))\, ds, \\
(\Gamma_2 X)(t) &:= \int_{t}^{\infty} AT(t-s)SF(s, BX(s))\, ds, \\
(\Gamma_3 X)(t) &:= \int_{-\infty}^{t} T(t-s)PG(s, CX(s))\, ds \\
(\Gamma_4 X)(t) &:= \int_{t}^{\infty} T(t-s)SG(s, CX(s))\, ds, \\
(\Gamma_5 X)(t) &:= \int_{-\infty}^{t} T(t-s)PH(s, LX(s))\, dW(s), \\
(\Gamma_6 X)(t) &:= \int_{t}^{\infty} T(t-s)SH(s, LX(s))\, dW(s).
\end{aligned}$$

To discuss the existence of quadratic-mean almost periodic solution to (1.4) we need to set some assumptions on A, B, C, L, F, G, and H. First of all, note that for $0 < \alpha < \beta < 1$, then

$$L^2(\Omega, \mathbb{H}_\beta) \hookrightarrow L^2(\Omega, \mathbb{H}_\alpha) \hookrightarrow L^2(\Omega; \mathbb{H})$$

are continuously embedded and hence there exist constants $k_1 > 0$, $k(\alpha) > 0$ such that

$$\mathrm{E}\|X\|^2 \leq k_1 \mathrm{E}\|X\|_\alpha^2 \quad \text{for each } X \in L^2(\Omega, \mathbb{H}_\alpha),$$
$$\mathrm{E}\|X\|_\alpha^2 \leq k(\alpha)\mathrm{E}\|X\|_\beta^2 \quad \text{for each } X \in L^2(\Omega, \mathbb{H}_\beta).$$

(H1) The operator $\mathcal{A}$ is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\geq 0}$.

(H2) Let $\alpha \in (0, \frac{1}{2})$. Then $\mathbb{H}_\alpha = D((-\mathcal{A})^\alpha)$, or $\mathbb{H}_\alpha = D_{\mathcal{A}}(\alpha, p), 1 \leq p \leq \infty$, or $\mathbb{H}_\alpha = D_{\mathcal{A}}(\alpha)$, or $\mathbb{H}_\alpha = [\mathbb{H}, D(\mathcal{A})]_\alpha$. We also assume that $B, C, L : L^2(\Omega, \mathbb{H}_\alpha) \to L^2(\Omega; \mathbb{H})$ are bounded linear operators and set

$$\varpi := \max\Big(\|B\|_{B(L^2(\Omega,\mathbb{H}_\alpha),L^2(\Omega;\mathbb{H}))}, \|C\|_{B(L^2(\Omega,\mathbb{H}_\alpha),L^2(\Omega;\mathbb{H}))}, \|L\|_{B(L^2(\Omega,\mathbb{H}_\alpha),L^2(\Omega;\mathbb{H}))}\Big).$$

(H3) Let $\alpha \in (0, \frac{1}{2})$ and $\alpha < \beta < 1$. Let $F : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega, \mathbb{H}_\beta)$, $G : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathbb{H})$ and $H : \mathbb{R} \times L^2(\Omega; \mathbb{H}) \to L^2(\Omega; \mathcal{L}_2^0)$ are quadratic-mean almost periodic. Moreover, the functions F, G, and H are uniformly Lipschitz with respect to the second argument in the following sense: there exist positive constants K_F, K_G, and K_H such that

$$\mathrm{E}\|F(t,\psi_1) - F(t,\psi_2)\|_\beta^2 \leq K_F \mathrm{E}\|\psi_1 - \psi_2\|^2,$$
$$\mathrm{E}\|G(t,\psi_1) - G(t,\psi_2)\|^2 \leq K_G \mathrm{E}\|\psi_1 - \psi_2\|^2,$$
$$\mathrm{E}\|H(t,\psi_1) - H(t,\psi_2)\|_{\mathcal{L}_2^0}^2 \leq K_H \mathrm{E}\|\psi_1 - \psi_2\|^2,$$

for all stochastic processes $\psi_1, \psi_2 \in L^2(\Omega; \mathbb{H})$ and $t \in \mathbb{R}$.

Theorem 4.2. *Under assumptions* (H1)–(H3)*, the evolution equation* (1.4) *has a unique quadratic-mean almost periodic mild solution whenever* $\Theta < 1$*, where*

$$\Theta := \varpi\Big[k'(\alpha)K_F'\Big\{1 + c\Big(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}} + \frac{1}{\delta}\Big)\Big\} + k_1' \cdot K_G'\Big(M'(\alpha)\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}} + \frac{C'(\alpha)}{\delta}\Big) + c\sqrt{\operatorname{Tr} Q} \cdot K_H' \cdot k_1' \cdot \Big\{\frac{K'(\alpha,\beta)}{\sqrt{\delta}} + 2K'(\alpha,\gamma,\delta,\Gamma)\Big\}\Big].$$

To prove this Theorem 4.2, we will need the following lemmas, which will be proven under our initial assumptions.

Lemma 4.3. *Under assumptions* (H1)–(H3)*, the integral operators* Γ_1 *and* Γ_2 *defined above map* $AP(\mathbb{R}; L^2(\Omega, \mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the quadratic-mean almost periodicity of $\Gamma_2 X$ is similar to that of $\Gamma_1 X$ and hence will be omitted. Let $X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Since $B \in B(L^2(\Omega; \mathbb{H}_\alpha), L^2(\Omega; \mathbb{H}))$ it follows that the function $t \to BX(t)$ belongs to $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}))$. Using Theorem 2.6 it follows that $\Psi(\cdot) = F(\cdot, BX(\cdot))$ is in $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$ whenever $X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. We can now show that $\Gamma_1 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Indeed, since $X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $t \in [\xi, \xi + l(\varepsilon)]$ with the property:

$$\mathrm{E}\|\Psi X(t+\tau) - \Psi X(t)\|_\beta^2 < \nu^2\varepsilon \quad \text{for each } t \in \mathbb{R},$$

where $\nu = \frac{\gamma^{1-\alpha}}{M'(\alpha)\Gamma(1-\alpha)}$ with $\Gamma(\cdot)$ being the classical gamma function.

Now, the estimate in (3.9) yields

$$\begin{aligned}
&\mathrm{E}\|\Gamma_1 X(t+\tau)-\Gamma_1 X(t)\|_\alpha^2\\
&\le \mathrm{E}\Big(\int_0^\infty \|AT(s)P[\Psi(t-s+\tau)-\Psi(t-s)]\|_\alpha ds\Big)^2\\
&\le M'(\alpha)^2\Big(\int_0^\infty s^{-\alpha}e^{-\gamma s}ds\Big)\Big(\int_0^\infty s^{-\alpha}e^{-\gamma s}\mathrm{E}\|\Psi(t-s+\tau)-\Psi(t-s)\|_\beta^2 ds\Big)\\
&\le \Big(\frac{M'(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2\sup_{t\in\mathbb{R}}\mathrm{E}\|\Psi(t+\tau)-\Psi(t)\|_\beta^2 ds<\varepsilon
\end{aligned}$$

for each $t\in\mathbb{R}$, and hence $\Gamma_1 X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. □

Lemma 4.4. *Under assumptions* (H1)–(H3)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the quadratic-mean almost periodicity of $\Gamma_4 X$ is similar to that of $\Gamma_3 X$ and hence will be omitted. Note, however, that for $\Gamma_4 X$, we make use of (3.6) rather than (3.7).

Let $X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$. Since $C\in B(L^2(\Omega;\mathbb{H}_\alpha),L^2(\Omega;\mathbb{H}))$, it follows that $CX\in AP(\mathbb{R},L^2(\Omega;\mathbb{H})))$. Setting $\Phi(t)=G(t,CX(t))$ and using Theorem 2.6 it follows that $\Phi\in AP(\mathbb{R};L^2(\Omega,\mathbb{H})))$. We can now show that $\Gamma_3 X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$. Indeed, since $\Phi\in AP(\mathbb{R};L^2(\Omega,\mathbb{H})))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathrm{E}\|\Phi(t+\tau)-\Phi(t)\|^2<\mu^2\cdot\varepsilon\quad\text{for each } t\in\mathbb{R},$$

where $\mu=\frac{\gamma^{1-\alpha}}{M(\alpha)\Gamma(1-\alpha)}$. Now using the expression

$$(\Gamma_3X)(t+\tau)-(\Gamma_3X)(t)=\int_0^\infty T(s)P[\Phi(t-s+\tau)-\Phi(t-s)]\,ds$$

and (3.7) it easily follows that

$$\mathrm{E}\|(\Gamma_3X)(t+\tau)-(\Gamma_3X)(t)\|_\alpha^2<\varepsilon\quad\text{for each } t\in\mathbb{R},$$

and hence, $\Gamma_3 X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. □

Lemma 4.5. *Under assumptions* (H1)–(H3)*, the integral operators* Γ_5 *and* Γ_6 *map* $AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. Let $X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. Since $L\in B(L^2(\Omega;\mathbb{H}_\alpha),L^2(\Omega;\mathbb{H}))$, it follows that $LX\in AP(\mathbb{R},L^2(\Omega;\mathbb{H})))$. Setting $\Lambda(t)=H(t,LX(t))$ and using Theorem 2.6 it follows that $\Lambda\in AP(\mathbb{R};L^2(\Omega;\mathcal{L}_2^0))$. We claim that $\Gamma_5X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. Indeed, since $\Lambda\in AP(\mathbb{R};L^2(\Omega;\mathcal{L}_2^0))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathrm{E}\|\Lambda(t+\tau)-\Lambda(t)\|_{\mathcal{L}_2^0}^2<\zeta\cdot\varepsilon\quad\text{for each } t\in\mathbb{R},\tag{4.1}$$

where

$$\zeta=\frac{1}{2c^2\operatorname{Tr}Q\cdot K(\alpha,\gamma,\delta,\Gamma)}.$$

Now using the expression

$$(\Gamma_5X)(t+\tau)-(\Gamma_5X)(t)=\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s),$$

Equation (3.5), the arithmetic-geometric inequality, and Ito isometry we have

$$
\begin{aligned}
&\mathrm{E}\|(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\|_\alpha^2\\
&=\Big\|\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)\Big\|_\alpha^2\\
&\leq c^2\mathrm{E}\Big\{(1-\alpha)\Big\|\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)\Big\|\\
&\quad+\alpha\Big\|\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)\Big\|_{[D(A)]}\Big\}^2\\
&\leq c^2\mathrm{E}\Big\{\Big\|\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)\Big\|\\
&\quad+\Big\|A\int_0^\infty T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)\Big\|\Big\}^2\\
&\leq 2c^2\operatorname{Tr}Q\Big\{\int_0^\infty \mathrm{E}\|T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\|_{\mathcal{L}_2^0}^2ds\\
&\quad+\int_0^\infty \mathrm{E}\|AT(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\|_{\mathcal{L}_2^0}^2ds\Big\}.
\end{aligned}
$$

Now

$$
\mathrm{E}\|T(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\|_{\mathcal{L}_2^0}^2\leq M^2e^{-2\delta s}\mathrm{E}\|\Lambda(t-s+\tau)-\Lambda(t-s)\|_{\mathcal{L}_2^0}^2,
$$

and

$$
\begin{aligned}
&\mathrm{E}\|AT(s)P[\Lambda(t-s+\tau)-\Lambda(t-s)]\|_{\mathcal{L}_2^0}^2\\
&\leq M_1^2(\alpha)s^{-2\alpha}e^{-2\gamma s}\mathrm{E}\|\Lambda(t-s+\tau)-\Lambda(t-s)\|_{\mathcal{L}_2^0}^2.
\end{aligned}
$$

Hence,

$$
\mathrm{E}\|(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\|_\alpha^2\leq 2c^2\operatorname{Tr}Q\cdot K(\alpha,\gamma,\delta,\Gamma)\sup_{t\in\mathbb{R}}\mathrm{E}\|\Lambda(t+\tau)-\Lambda(t)\|_{\mathcal{L}_2^0}^2.
$$

where

$$
K(\alpha,\gamma,\delta,\Gamma)=\frac{M^2}{2\delta}+\frac{M_1^2(\alpha)\Gamma(1-2\alpha)}{\gamma^{1-2\alpha}},
$$

and it follows from (4.1) that $\Gamma_5 X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$.

As for $\Gamma_6 X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$, since $\Lambda\in AP(\mathbb{R};L^2(\Omega;\mathcal{L}_0^2))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$
\mathrm{E}\|\Lambda(t+\tau)-\Lambda(t)\|_{\mathcal{L}_0^2}^2<\kappa\cdot\varepsilon\quad\text{for each } t\in\mathbb{R}, \tag{4.2}
$$

where $\kappa=\frac{\delta}{c^2\cdot\operatorname{Tr}Q\cdot K(\alpha,\beta)}$. Now using the expression

$$
(\Gamma_6 X)(t+\tau)-(\Gamma_6 X)(t)=\int_{-\infty}^0 T(s)S[\Lambda(t-s+\tau)-\Lambda(t-s)]\,dW(s)
$$

Equation (3.5), the arithmetic-geometric inequality, and Ito isometry we have

$$
\begin{aligned}
&\mathrm{E}\|(\Gamma_6 X)(t+\tau) - (\Gamma_6 X)(t)\|_\alpha^2 \\
&= \Big\| \int_{-\infty}^0 T(s)S[\Lambda(t-s+\tau) - \Lambda(t-s)]\, dW(s)\Big\|_\alpha^2 \\
&\le 2c^2 \operatorname{Tr} Q \Big\{ \int_{-\infty}^0 \mathrm{E}\|T(s)S[\Lambda(t-s+\tau) - \Lambda(t-s)]\|_{\mathcal{L}_2^0}^2 ds \\
&\quad + \int_{-\infty}^0 \mathrm{E}\|AT(s)S[\Lambda(t-s+\tau) - \Lambda(t-s)]\|_{\mathcal{L}_2^0}^2 ds \Big\}
\end{aligned}
$$

However,

$$
\mathrm{E}\|T(s)S[\Lambda(t-s+\tau) - \Lambda(t-s)]\|_{\mathcal{L}_2^0}^2 \le M^2 e^{2\delta s} \mathrm{E}\|\Lambda(t-s+\tau) - \Lambda(t-s)\|_{\mathcal{L}_2^0}^2,
$$

$$
\mathrm{E}\|AT(s)S[\Lambda(t-s+\tau) - \Lambda(t-s)]\|_{\mathcal{L}_2^0}^2 \le n_1^2(\alpha,\beta) e^{2\delta s} \mathrm{E}\|\Lambda(t-s+\tau) - \Lambda(t-s)\|_{\mathcal{L}_2^0}^2
$$

Thus,

$$
\mathrm{E}\|(\Gamma_6 X)(t+\tau) - (\Gamma_6 X)(t)\|_\alpha^2 \le c^2 \cdot \operatorname{Tr} Q \cdot \frac{K(\alpha,\beta)}{\delta} \sup_{t\in\mathbb{R}} \mathrm{E}\|\Lambda(t+\tau) - \Lambda(t)\|_{\mathcal{L}_2^0}^2 ds,
$$

where $K(\alpha,\beta)) = M^2 + n_1^2(\alpha,\beta)$ is a constant depending on α and β and it follows from (4.2) that $\Gamma_6 X \in AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$. □

We are ready for the proof of Theorem 4.2.

Proof. Consider the nonlinear operator $\mathbb{M}$ on the space $AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$ equipped with the α-sup norm $\|X\|_{\infty,\alpha} = \sup_{t\in\mathbb{R}} (\mathrm{E}\|X(t)\|_\alpha^2)^{1/2}$ and defined by

$$
\begin{aligned}
\mathbb{M}X(t) = &-F(t, BX(t)) - \int_{-\infty}^t AT(t-s)PF(s, BX(s))\, ds \\
&+ \int_t^\infty AT(t-s)SF(s, BX(s))\, ds \\
&+ \int_{-\infty}^t T(t-s)PG(s, CX(s))\, ds - \int_t^\infty T(t-s)SG(s, CX(s))\, ds \\
&+ \int_{-\infty}^t T(t-s)PH(s, LX(s))\, dW(s) - \int_t^\infty T(t-s)SH(s, LX(s))\, dW(s)
\end{aligned}
$$

for each $t \in \mathbb{R}$.

As we have previously seen, for every $X \in AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$, $f(\cdot, BX(\cdot)) \in AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\beta)) \subset AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$. In view of Lemmas 4.3, 4.4, and 4.5, it follows that $\mathbb{M}$ maps $AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed point.

Let $X, Y \in AP(\mathbb{R}; L^2(\Omega;\mathbb{H}_\alpha))$. By (H1), (H2), and (H3), we obtain

$$
\begin{aligned}
\mathrm{E}\|F(t, BX(t)) - F(t, BY(t))\|_\alpha^2 &\le k(\alpha) K_F \mathrm{E}\|BX(t) - BY(t)\|^2 \\
&\le k(\alpha) \cdot K_F \varpi^2 \|X - Y\|_{\infty,\alpha}^2,
\end{aligned}
$$

which implies

$$
\|F(\cdot, BX(\cdot)) - F(\cdot, BY(\cdot))\|_{\infty,\alpha} \le k'(\alpha) \cdot K_F' \varpi \|X - Y\|_{\infty,\alpha}.
$$

Now for Γ_1 and Γ_2, we have the following evaluations

$$\begin{aligned}
&\mathrm{E}\|(\Gamma_1 X)(t)-(\Gamma_1 Y)(t)\|_\alpha^2\\
&\le \mathrm{E}\Big(\int_{-\infty}^t \|AT(t-s)P[F(s,BX(s))-F(s,BY(s))]\|_\alpha ds\Big)^2\\
&\le c^2\Big(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}ds\Big)\\
&\quad\times\Big(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\mathrm{E}\|[F(s,BX(s))-F(s,BY(s))]\|_\alpha^2 ds\Big)\\
&\le c^2 k(\alpha)K_F\varpi^2\|X-Y\|_{\infty,\alpha}^2\Big(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}ds\Big)^2\\
&= c^2 k(\alpha)K_F\Big(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2\varpi^2\|X-Y\|_{\infty,\alpha}^2,
\end{aligned}$$

which implies

$$\|\Gamma_1 X-\Gamma_1 Y\|_{\infty,\alpha}\le c\cdot k'(\alpha)\cdot K_F'\frac{\Gamma(1-\alpha))}{\gamma^{1-\alpha}}\varpi\|X-Y\|_{\infty,\alpha}.$$

Similarly,

$$\begin{aligned}
&\mathrm{E}\|(\Gamma_2 X)(t)-(\Gamma_2 Y)(t)\|_\alpha^2\\
&\le \mathrm{E}\Big(\int_t^\infty \|AT(t-s)S[F(s,BX(s))-F(s,BY(s))]\|_\alpha ds\Big)^2\\
&\le \frac{c^2 k(\alpha)K_F}{\delta^2}\varpi^2\|X-Y\|_{\infty,\alpha}^2,
\end{aligned}$$

which implies

$$\|\Gamma_2 X-\Gamma_2 Y\|_{\infty,\alpha}\le \frac{c\cdot k'(\alpha)\cdot K_F'}{\delta}\varpi\|X-Y\|_{\infty,\alpha}.$$

As to Γ_3 and Γ_4, we have the following evaluations

$$\begin{aligned}
&\mathrm{E}\|(\Gamma_3 X)(t)-(\Gamma_3 Y)(t)\|_\alpha^2\\
&\le \mathrm{E}\Big(\int_{-\infty}^t \|T(t-s)P[G(s,CX(s))-G(s,CY(s))]\|_\alpha ds\Big)^2\\
&\le k_1\cdot M^2(\alpha)\Big(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}ds\Big)\\
&\quad\times\Big(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\mathrm{E}\|G(s,CX(s))-G(s,CY(s))\|_\alpha^2 ds\Big)\\
&\le k_1\cdot K_G\cdot M^2(\alpha)\Big(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\Big)^2\varpi^2\|X-Y\|_{\infty,\alpha}^2,
\end{aligned}$$

which implies

$$\|\Gamma_3 X-\Gamma_3 Y\|_{\infty,\alpha}\le k_1'\cdot K_G'\cdot M'(\alpha)\frac{\Gamma(1-\alpha))}{\gamma^{1-\alpha}}\varpi\|X-Y\|_{\infty,\alpha}.$$

Similarly,

$$
\begin{aligned}
&\mathrm{E}\|(\Gamma_4 X)(t)-(\Gamma_4 Y)(t)\|_\alpha^2 \\
&\leq \mathrm{E}\Big(\int_t^\infty \|T(t-s)S[G(s,CX(s))-G(s,CY(s))]\|_\alpha ds\Big)^2 \\
&\leq \frac{k_1 K_G C(\alpha)}{\delta^2}\varpi^2\|X-Y\|_{\infty,\alpha}^2,
\end{aligned}
$$

which implies

$$
\|\Gamma_4 X-\Gamma_4 Y\|_{\infty,\alpha} \leq \frac{k_1'\cdot K_G'\cdot C'(\alpha)}{\delta}\varpi\|X-Y\|_{\infty,\alpha}.
$$

Finally for Γ_5 and Γ_6, we have the following evaluations

$$
\begin{aligned}
&\mathrm{E}\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\|_\alpha^2 \\
&\leq 2c^2\,\mathrm{Tr}\,Q\{\int_0^\infty \mathrm{E}\|T(s)P[H(t,LX(t))-H(t,LY(t))\|_{\mathcal{L}_2^0}^2 ds \\
&\leq 2c^2\cdot \mathrm{Tr}\,Q\cdot k_1\cdot K(\alpha,\gamma,\delta,\Gamma)\cdot K_H\cdot \varpi^2\|X-Y\|_{\infty,\alpha}^2,
\end{aligned}
$$

which implies

$$
\|\Gamma X_5-\Gamma_5 Y\|_{\infty,\alpha} \leq 2c\cdot\sqrt{\mathrm{Tr}\,Q}\cdot k_1'\cdot K'(\alpha,\gamma,\delta,\Gamma)\cdot K_H'\cdot\varpi\|X-Y\|_{\infty,\alpha}.
$$

Similarly,

$$
\mathrm{E}\|(\Gamma_6 X)(t)-(\Gamma_6 Y)(t)\|_\alpha^2 \leq c^2\cdot\mathrm{Tr}\,Q\cdot k_1\cdot K_H\cdot\frac{K(\alpha,\beta)}{\delta}\varpi^2\|X-Y\|_{\infty,\alpha}^2,
$$

which implies

$$
\|\Gamma X_6-\Gamma_6 Y\|_{\infty,\alpha} \leq c\cdot\sqrt{\mathrm{Tr}\,Q}\cdot k_1'\cdot K_H'\cdot\frac{K'(\alpha,\beta)}{\sqrt{\delta}}\cdot\varpi\|X-Y\|_{\infty,\alpha}.
$$

Consequently,

$$
\|\mathbb{M}X-\mathbb{M}Y\|_{\infty,\alpha} \leq \Theta\cdot\|X-Y\|_{\infty,\alpha}.
$$

Clearly, if $\Theta < 1$, then (1.4) has a unique fixed-point by Banach fixed point theorem, which is obviously the only quadratic-mean almost periodic solution to it. □

5. Example

Let $\Gamma \subset \mathbb{R}^N$ ($N \geq 1$) be a open bounded subset with C^2 boundary $\partial\Gamma$. To illustrate our abstract results, we study the existence of quadratic mean almost periodic solutions to the stochastic heat equation in divergence given by

$$
\begin{gathered}
\partial\Big[\Phi+F(t,\widehat{\mathrm{div}}\Phi)\Big] = \Big[\Delta\Phi+G(t,\widehat{\mathrm{div}}\Phi)\Big]\partial_t + H(t,\Phi)\partial W(t), \quad \text{in } \Gamma \\
\Phi = 0, \quad \text{on } \partial\Gamma
\end{gathered} \tag{5.1}
$$

where the unknown Φ is a function of $\omega\in\Omega$, $t\in\mathbb{R}$, and $x\in\Gamma$, the symbols $\widehat{\mathrm{d}iv}$ and Δ stand respectively for the first and second-order differential operators defined by

$$
\widehat{\mathrm{d}iv} := \sum_{j=1}^N \frac{\partial}{\partial x_j}, \quad \Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2},
$$

and the coefficients $F, G : \mathbb{R}\times L^2(\Omega,\mathbb{H}_0^\alpha(\Gamma)\cap\mathbb{H}^{2\alpha}(\Gamma)) \mapsto L^2(\Omega,L^2(\Gamma))$ and $H : \mathbb{R}\times L^2(\Omega,\mathbb{H}_0^\alpha(\Gamma)\cap\mathbb{H}^{2\alpha}(\Gamma)) \to L^2(\Omega,\mathcal{L}_2^0)$ are quadratic-mean almost periodic.

Define the linear operator appearing in (5.1) as follows:

$$AX = \Delta X \quad \text{for all } u \in D(A) = L^2(\Omega; \mathbb{H}_0^1(\Gamma) \cap \mathbb{H}^2(\Gamma)).$$

Using the fact that the operator $\mathcal{A}$, defined in $L^2(\Gamma)$ by

$$\mathcal{A}u = \Delta u \quad \text{for all } u \in D(\mathcal{A}) = \mathbb{H}_0^1(\Gamma) \cap \mathbb{H}^2(\Gamma),$$

is sectorial and whose corresponding analytic semigroup is hyperbolic, one easily sees that the operator A defined above is sectorial and hence is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$. Moreover, the semigroup $(T(t))_{t \geq 0}$ is hyperbolic as

$$\sigma(A) \cap i\mathbb{R} = \emptyset.$$

For each $\mu \in (0,1)$, we take $\mathbb{H}_\mu = D((-\Delta)^\mu) = L^2(\Omega, \mathbb{H}_0^\mu(\Gamma) \cap \mathbb{H}^{2\mu}(\Gamma))$ equipped with its μ-norm $\|\cdot\|_\mu$. Moreover, since $\alpha \in (0, \frac{1}{2})$, we suppose that $\frac{1}{2} < \beta < 1$. Letting $L = I$, and $BX = CX = \operatorname{div} X$ for all $X \in L^2(\Omega, \mathbb{H}_\alpha) = L^2(\Omega, D((-\Delta)^\alpha)) = L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma))$, one easily see that both B and C are bounded from $L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma))$ in $L^2(\Omega, L^2(\Gamma))$ with $\varpi = 1$.

We require the following assumption:

(H4) Let $\frac{1}{2} < \beta < 1$, and

$$F : \mathbb{R} \times L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma)) \mapsto L^2(\Omega, \mathbb{H}_0^\beta(\Gamma) \cap \mathbb{H}^{2\beta}(\Gamma))$$

be quadratic-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma))$, $G : \mathbb{R} \times L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma)) \mapsto L^2(\Omega, L^2(\Gamma))$ be quadratic-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in L^2(\Omega, \mathbb{H}_0^\alpha(\Gamma) \cap \mathbb{H}^{2\alpha}(\Gamma))$. Moreover, the functions F, G are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K' > 0$ such that

$$\begin{gathered} \mathrm{E}\|F(t, \Phi_1) - F(t, \Phi_2)\|_\beta \leq K' \mathrm{E}\|\Phi_1 - \Phi_2\|_{L^2(\Gamma)}, \\ \mathrm{E}\|G(t, \Phi_1) - G(t, \Phi_2)\|_{L^2(\Gamma)} \leq K' \mathrm{E}\|\Phi_1 - \Phi_2\|_{L^2(\Gamma)}, \\ \mathrm{E}\|H(t, \psi_1) - H(t, \psi_2)\|^2_{\mathcal{L}_2^0} \leq K' \mathrm{E}\|\psi_1 - \psi_2\|^2_{L^2(\Gamma)} \end{gathered}$$

for all $\Phi_1, \Phi_2, \psi_1, \psi_2 \in L^2(\Omega; L^2(\Gamma))$ and $t \in \mathbb{R}$.

As a final result, we have the following theorem.

Theorem 5.1. *Under the above assumptions including* (H4)*, the N-dimensional stochastic heat equation* (5.1) *has a unique quadratic-mean almost periodic solution $\Phi \in L^2(\Omega, \mathbb{H}_0^1(\Gamma) \cap \mathbb{H}^2(\Gamma))$ whenever K' is small enough.*

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