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MULTIPLICITY OF SOLUTIONS FOR SOME DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We show the existence of infinitely many solutions for a symmetric quasilinear problem whose principal part is degenerate.

1. Introduction and statement of main result

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$. We are interested in the solvability of the quasilinear elliptic problem

$$\begin{gathered} -\operatorname{div}(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = g(x,u) \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$j(x,s,\xi) = \frac{1}{2}\, a(x,s)|\xi|^2\,.$$

A feature of quasilinear problems like (1.1) is that, for a general $u \in H_0^1(\Omega)$, the term $j_s(x,u,\nabla u)$ belongs to $L^1(\Omega)$ under reasonable assumptions, but not to $H^{-1}(\Omega)$. As a consequence, the functional

$$f(u) = \int_\Omega j(x,u,\nabla u) - \int_\Omega G(x,u)\,, \quad G(x,s) = \int_0^s g(x,t)\,dt\,,$$

whose Euler-Lagrange equation is represented by (1.1), is continuous on $H_0^1(\Omega)$, but not locally Lipschitz, in particular not of class C^1.

In spite of this fact, existence and multiplicity results have been already obtained for this class of problems, also by means of variational methods, starting from the case in which a is bounded and bounded away from zero (see [1, 4]). Actually, in [1] the nonsmoothness of the functional is overcome by a suitable approximation procedure, while in [4] a direct approach, based on a critical point theory for continuous functionals developed in [5, 6, 7, 8], is used. However, it seems to be hard, in the approach of [1], to get multiplicity results when, e.g., f is even.

Both approaches have been extended to the case in which a is still bounded away from zero, but possibly unbounded (unbounded case), in [2, 9], respectively. Again, multiplicity results when f is even are proved only in the second paper.

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Finally, in [3] the case in which a is bounded, but not bounded away from zero (degenerate case) is addressed in the line of [1, 2]. Also in this case, no multiplicity result is proved when f is even.

The main purpose of this paper is to prove that, in the model case mentioned in [3] and with a symmetry assumption, problem (1.1) possesses infinitely many solutions. We also show that it is not necessary to restart all the machinery of the previous cases to get the result. By a change of variable, the degenerate case can be reduced to that of [4]. Apart from the model case, it would be interesting to check if, up to a change of variable, the degenerate case can be reduced to a suitable form of the unbounded case and viceversa.

To state our result, consider the model case

$$a(x,s) = \frac{1}{(b(x)+s^2)^{\alpha}}\,,$$

where b is a measurable function satisfying $0 < \beta_1 \le b(x) \le \beta_2$ a.e. in Ω and $\alpha \in [0, \frac{n}{2n-2})$.

Let also $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following assumptions:

- there exists $b, d > 0$ and $2 < p < 2^*(1-\alpha)$ such that
$$|g(x,s)| \le b|s|^{p-1} + d \tag{1.2}$$
for almost every $x \in \Omega$ and every $s \in \mathbb{R}$
- for almost every $x \in \Omega$ and every $s \in \mathbb{R}$,
$$g(x,-s) = -g(x,s). \tag{1.3}$$

We set $G(x,s) = \int_0^s g(x,t)\,dt$ and we assume that there exist $\nu > 2, R > 0$ such that

$$0 < \nu G(x,s) \le s g(x,s). \tag{1.4}$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \ge R$.

It easily follows that the function a satisfies the following conditions:

- there exist $c_1, c_2 > 0$ such that
$$\frac{c_1}{(1+|s|)^{2\alpha}} \le a(x,s) \le c_2 \tag{1.5}$$
for almost every $x \in \Omega$, for every $s \in \mathbb{R}$,
- for almost every x in Ω the function $a(x,.)$ is differentiable on $\mathbb{R}$ and there exist $c_3 > 0$ such that, for almost every $x \in \Omega$, its derivative
$$a_s(x,s) \equiv \frac{\partial a}{\partial s}(x,s) = \frac{-2\alpha\, s\, a(x,s)}{b(x)+s^2}$$
satisfies
$$|a_s(x,s)| \le c_3 \quad \forall s \in \mathbb{R} \tag{1.6}$$
- for almost every $x \in \Omega$ and all $s \in \mathbb{R}$
$$a(x,s) = a(x,-s). \tag{1.7}$$

Definition 1.1. We say that u is a weak solution of (1.1) if $u \in H_0^1(\Omega)$ and

$$\int_\Omega j_\xi(x,u,\nabla u)\nabla v + j_s(x,u,\nabla u)v = \int_\Omega g(x,u)\,v$$

for every $v \in C_0^\infty(\Omega)$.

We are now able to state our main result.

Theorem 1.2. *Assume that conditions* (1.2), (1.3) *and* (1.4) *hold. Then there exists a sequence* $\{u_h\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ *of weak solutions of (1.1) such that*

$$\int_\Omega j(x, u_h, \nabla u_h) - \int_\Omega G(x, u_h)$$

approaches $+\infty$ *as* $h \to +\infty$.

2. Proof of the main result

Let $\varphi \in C^2(\mathbb{R})$ be defined as

$$\varphi(s) = s\,(1+s^2)^{\frac{\alpha}{2(1-\alpha)}}.$$

Remark 2.1. We observe that φ is odd and that there exists $\gamma > 0$ such that

$$\varphi'(s) \geq \gamma(1 + |\varphi(s)|)^\alpha. \tag{2.1}$$

Moreover we have

$$\lim_{s\to\pm\infty} \frac{s\,\varphi'(s)}{\varphi(s)} = \lim_{s\to\pm\infty} \Big(1 + s\frac{\varphi''(s)}{\varphi'(s)}\Big) = \frac{1}{1-\alpha}. \tag{2.2}$$

Let us consider the change of variable $u = \varphi(v)$. We can define on $H_0^1(\Omega)$ the functional

$$\tilde{f}(v) = \frac{1}{2}\int_\Omega A(x, v)|\nabla v|^2 - \int_\Omega \widetilde{G}(x, v),$$

where

$$A(x, s) = a(x, \varphi(s)) \cdot (\varphi'(s))^2, \quad \widetilde{G}(x, s) = G(x, \varphi(s)) = \int_0^s \tilde{g}(x, t)dt,$$

with $\tilde{g}(x, s) = g(x, \varphi(s)) \cdot \varphi'(s)$.

Now let us consider the integrand $\tilde{j} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{j}(x, s, \xi) = \frac{1}{2} A(x, s)|\xi|^2.$$

Remark 2.2. It is readily seen that (2.1) and the left inequality of (1.5) imply that for almost every $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, there holds

$$\tilde{j}(x, s, \xi) \geq \alpha_0 |\xi|^2,$$

where $\alpha_0 = \frac{c_1\,\gamma^2}{2}$.

Remark 2.3. By Remark 2.1 there exists $\Lambda > 0$ such that, for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, we have

$$A(x, s) \leq \Lambda, \tag{2.3}$$
$$|A_s(x, s)| \leq \Lambda. \tag{2.4}$$

Proposition 2.4. *Assume condition* (1.4). *Then*

$$\frac{\nu}{1-\alpha}\,\widetilde{G}(x, s) \leq s\tilde{g}(x, s) \tag{2.5}$$

for every $s \in \mathbb{R}$ *with* $|s| \geq R$.

Proof. Condition (1.4) implies

$$\nu\widetilde{G}(x,s)\,\varphi'(s) \le \varphi(s)\,\widetilde{G}_s(x,s)$$

hence

$$\nu\frac{s\varphi'(s)}{\varphi(s)}\widetilde{G}(x,s) \le s\widetilde{G}_s(x,s)$$

and taking into account Remark 2.1, we get the thesis. ☐

Proposition 2.5. *There exists $\mu < \frac{\nu}{1-\alpha} - 2$ such that, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^n$, for every $s \in \mathbb{R}$ with $|s| \ge R$, we have*

$$0 \le s\tilde{j}_s(x,s,\xi) \le \mu\,\tilde{j}(x,s,\xi). \tag{2.6}$$

Proof. Indeed

$$\begin{aligned}
\tilde{j}_s(x,s,\xi) &= \frac{1}{2}A_s(x,s)|\xi|^2 \\
&= \frac{1}{2}[a_s(x,\varphi(s))\cdot(\varphi'(s))^3 + 2\varphi'(s)\cdot\varphi''(s)\cdot a(x,\varphi(s))]|\xi|^2 \\
&= a(x,\varphi(s))\cdot\varphi'(s)\Big[\frac{-\alpha\,\varphi(s)\,(\varphi'(s))^2}{b(x)+(\varphi(s))^2} + \varphi''(s)\Big]|\xi|^2.
\end{aligned} \tag{2.7}$$

Let $s > 0$. Then recalling that $a(x,\varphi(s))$ and $\varphi'(s)$ are positive functions, it suffices to prove that the square bracket is non negative. Note that the expression is equal to

$$\frac{\alpha s(1+s^2)^{\frac{\alpha}{2(1-\alpha)}}}{(1-\alpha)(1+s^2)^2}\Big[\frac{-(s^2+1-\alpha)(1+s^2)^{\frac{\alpha}{(1-\alpha)}}}{b(x)+(1+s^2)^{\frac{\alpha}{(1-\alpha)}}\,s^2} + \frac{(s^2+1-\alpha)\,b(x)}{(1-\alpha)(b(x)+(1+s^2)^{\frac{\alpha}{1-\alpha}}\,s^2)} + 2\Big].$$

Observing that the second term in square bracket is positive and the sum of the first and third is equal to

$$\frac{(1+s^2)^{\frac{\alpha}{1-\alpha}}(s^2-(1-\alpha))+2\,b(x)}{b(x)+(1+s^2)^{\frac{\alpha}{1-\alpha}}\,s^2},$$

the assertion follows if we assume $R = \sqrt{1-\alpha}$. On the other hand if $s < 0$, taking into account that $\varphi''(s)$ is an odd function, we deduce that the square bracket in (2.7) is negative. Now we prove the right inequality. Since $\varphi'(s) \ge 0$ and $a_s(x,\varphi(s)) \le 0$, we have

$$s\,\tilde{j}_s(x,s,\xi) \le 2\,\tilde{j}(x,s,\xi)\Big(\frac{\varphi''(s)\,s}{\varphi'(s)}\Big)$$

and by Remark 2.1 it follows the assertion with R large enough and $\mu = \frac{2\alpha}{1-\alpha}$. ☐

We now are able to prove the main result of the paper.

Proof of Theorem 1.2. By Remark 2.2 and 2.3 and Proposition 2.4 and 2.5 we are able to apply Theorem 2.6 in [4]. So obtaining a sequence of weak solutions $\{v_h\} \subset H_0^1(\Omega)$ of the problem

$$\int_\Omega A(x,v)\nabla v\nabla w + \frac{1}{2}\int_\Omega A_s(x,v)|\nabla u|^2\,w = \int_\Omega \tilde{g}(x,v)w$$

for every $w \in C_0^\infty(\Omega)$ with $f(v_h) \to \infty$. By Theorem 7.1 in [9] these solutions belong to $L^\infty(\Omega)$. If we set $u_h = \varphi(v_h)$, it is clear that $u_h \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and an easy calculation shows that each u_h is a weak solution of (1.1) with

$$\int_\Omega j(x, u_h, \nabla u_h) - \int_\Omega G(x, u_h) \to +\infty$$

as $h \to +\infty$. $\square$

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