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HOMEOMORPHISMS AND FREDHOLM THEORY FOR PERTURBATIONS OF NONLINEAR FREDHOLM MAPS OF INDEX ZERO WITH APPLICATIONS

PETRONIJE S. MILOJEVIĆ

Abstract. We develop a nonlinear Fredholm alternative theory involving k-ball and k-set perturbations of general homeomorphisms and of homeomorphisms that are nonlinear Fredholm maps of index zero. Various generalized first Fredholm theorems and finite solvability of general (odd) Fredholm maps of index zero are also studied. We apply these results to the unique and finite solvability of potential and semilinear problems with strongly nonlinear boundary conditions and to quasilinear elliptic equations. The basic tools used are the Nussbaum degree and the degree theories for nonlinear C^1-Fredholm maps of index zero and their perturbations.

1. Introduction

Tromba [31] proved that a locally injective and proper Fredholm map T of index zero is a homeomorphism. The purpose of this paper is to give various extensions of this result to maps of the form $T + C$, where T is a homeomorphism and C is a k-set contractive map. We prove several Fredholm alternative results and various extensions of the first Fredholm theorem for this class of maps using either Nussbaum's degree or the degree theories for (non) compact perturbations of Fredholm maps as developed by Fitzpatrick, Pejsachowisz, Rabier, Salter [14, 23, 25] and Benevieri, Calamai, Furi [3, 4, 5]. Applications to potential equations with nonlinear boundary conditions and to Dirichlet problems for quasilinear elliptic equations are given.

Let us describe our main results in more detail. Throughout the paper, we assume that X and Y are infinite dimensional Banach spaces. In Section 2, we establish a number of nonlinear Fredholm alternatives involving k-ball and k-set perturbations $C : X \to Y$ of general homeomorphisms $T : X \to Y$ as well as of homeomorphisms that are nonlinear Fredholm maps of index zero. In particular, we obtain various homeomorphism results for $T + C$ assuming that it is locally injective, satisfies

Condition (+): $\{x_n\}$ is bounded whenever $\{(T + C)x_n\}$ converges,

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and $\alpha(C) < \beta(T)$, [15], using the set measure of noncompactness α,

$$\alpha(T) = \sup\{\alpha(T(A))/\alpha(A) : A \subset X \text{ bounded, } \alpha(A) > 0\},$$
$$\beta(T) = \inf\{\alpha(T(A))/\alpha(A) : A \subset X \text{ bounded, } \alpha(A) > 0\}.$$

$\alpha(T)$ and $\beta(T)$ are related to the properties of compactness and properness of the map T, respectively. In particular, these results show that such homeomorphisms are stable under k-set contractive perturbations. We also prove such results when $T + C$ is asymptotically close to a suitable map that is positively homogeneous outside some ball. In the last part of Section 2, we establish Fredholm alternatives for equations of the form $Tx + Cx + Dx = f$ with T either a homeomorphism or a Fredholm map of index zero, assuming that $\alpha(D) < \beta(T) - \alpha(C)$, $T + C$ is asymptotically close to a k-positive homogeneous map and D quasibounded. We show that these equations are either uniquely solvable or are finitely solvable for almost all right hand sides and that the cardinality of the solution set is constant on certain connected components in Y. In particular, we obtain such alternatives for $T + D$ when T is either a c-expansive homeomorphism or an expansive along rays local homeomorphism and D is quasibounded with $\alpha(D) < c$.

In Section 3, we study finite solvability of equations $Tx + Cx + Dx = f$ that are perturbations of odd Fredholm maps of index zero with $T + C$ odd and asymptotically close to a suitable k-positive homogeneous map. These results can be considered as generalized first Fredholm theorems. We complete this section by proving several Borsuk type results for (non) compact perturbations of odd Fredholm maps of index zero. All the results in this section are proved using the recent degree theories for nonlinear perturbations for Fredholm maps of index zero as defined by Fitzpatrick, Pejsachowicz-Rabier [14, 23], Benevieri-Furi [3, 4], Rabier-Salter [25] and Benevieri-Calamai-Furi [5].

In Section 4, we apply some of our results to the unique and finite solvability of potential and semilinear problems with (strongly) nonlinear boundary value conditions and the Dirichlet problems with strong nonlinearities. Problems of this kind arise in many applications like steady-state heat transfer, electromagnetic problems with variable electrical conductivity of the boundary, heat radiation and heat transfer (cf. [28] and the references therein). Except for [24], the earlier studies assume that the nonlinearities have at most a linear growth and were based on the boundary element method.

Finally, in Section 5, some of our results are applied to the finite solvability of quasilinear elliptic equations on a bounded domain. The Fredholm part is a C^1 map of type (S_+) that is asymptotically close to a k-homogeneous map and the perturbation is a k_1-set contraction.

2. Perturbations of homeomorphisms and nonlinear Fredholm alternatives

Let X, Y be infinite dimensional Banach spaces, U be an open subset of X and $T : U \to Y$ be as above. We recall the following properties (see [15]) of $\alpha(T)$ and $\beta(T)$ defined in the introduction. First, we note that $\alpha(T)$ is related to the property of compactness of the map T and the number $\beta(T)$ is related to the properness of T.

(1) $\alpha(\lambda T) = |\lambda|\alpha(T)$ and $\beta(\lambda T) = |\lambda|\beta(T)$ for each $\lambda \in \mathbb{R}$.
(2) $\alpha(T + C) \le \alpha(T) + \alpha(C)$.

(3) $\beta(T)\beta(C) \leq \beta(ToC) \leq \alpha(T)\beta(C)$ (when defined)
(4) If $\beta(T) > 0$, then T is proper on bounded closed sets.
(5) $\beta(T) - \alpha(C) \leq \beta(T + C) \leq \beta(T) + \alpha(C)$.
(6) If T is a homeomorphism and $\beta(T) > 0$, then $\alpha(T^{-1})\beta(T) = 1$.

If $T : X \to Y$ is a homeomorphism, then (3) implies $1 = \beta(I) = \beta(T^{-1}oT) \leq \alpha(T^{-1})\beta(T)$. Hence, $\beta(T) > 0$.

If $L : X \to Y$ is a bounded linear operator, then $\beta(L) > 0$ if and only if $\operatorname{Im} L$ is closed and $\dim\ker L < \infty$ and $\alpha(L) \leq \|L\|$. Moreover, one can prove that L is Fredholm if and only if $\beta(L) > 0$ and $\beta(L^*) > 0$, where L^* is the adjoint of L.

Let $T : U \to Y$ be, as before, a map from an open subset U of a Banach space X into a Banach space Y, and let $p \in U$ be fixed. Let $B_r(p)$ be the open ball in X centered at p with radius r. Suppose that $B_r(p) \subset U$ and set

$$\alpha(T|_{B_r(p)}) = \sup\{\alpha T(A)/\alpha(A) : A \subset B_r(p) \text{ bounded}, \ \alpha(A) > 0\}.$$

This is non-decreasing as a function of r, and clearly $\alpha(T|_{B_r(p)}) \leq \alpha(T)$. Hence, the following definition makes sense:

$$\alpha_p(T) = \lim_{r\to 0} \alpha(T|_{B_r(p)}).$$

Similarly, we define $\beta_p(T)$. We have $\alpha_p(T) \leq \alpha(T)$ and $\beta_p(T) \geq \beta(T)$ for any p. If T is of class C^1, then $\alpha_p(T) = \alpha(T'(p))$ and $\beta_p(T) = \beta(T'(p))$ for any p [7]. Note that for a Fredholm map $T : X \to Y$, $\beta_p(T) > 0$ for all $p \in X$.

Recall that a map $T : X \to Y$ is a c-expansive map if $\|Tx - Ty\| \geq c\|x - y\|$ for all $x, y \in X$ and some $c > 0$.

Example 2.1. Let $T : X \to Y$ be continuous and c-expansive for some $c > 0$. Then $\alpha(T) \geq \beta(T) \geq c$. If T is also a homeomorphism, then $\alpha(T^{-1}) = 1/\beta(T) \leq 1/c$.

Example 2.2. Let $T : X \to Y$ be continuous and for each $p \in X$ there is an $r > 0$ such that $T : B(p, r) \to Y$ has the form $Tx = T(p) + L(x - p) + R(x)$, where $L : X \to Y$ is a continuous linear map such that $\|Lx\| \geq c_1\|x\|$ for some $c_1 > 0$ and all x, and R is a Lipschitz map with Lipchitz constant $c_2 < c_1$. Then $T : B \to T(B)$ is a homeomorphism. Moreover, for $c = c_1 - c_2$

$$\|Tx - Ty\| \geq c\|x - y\| \quad \text{for all } x, y \in B$$

and $\alpha(T|_B) \geq \beta(T|_B) \geq c$, $\beta_p(T) \geq c$, $\alpha(T^{-1}|_{T(B)}) \leq 1/c$ and $\alpha_{T(p)}(T^{-1}) = 1/\beta_p(T) \leq 1/c$.

Example 2.3. Let $T : X \to Y$ be a C^1 local homeomorphism. Then, for each $p \in X$, T has a representation as in Example 2.2 with a suitable $r > 0$ and therefore $\beta_p(T) \geq c$, $\alpha_{T(p)}(T^{-1}) = 1/\beta_p(T) \leq 1/c$ for some $c = c(p) > 0$.

For a continuous map $F : X \to Y$, let Σ be the set of all points $x \in X$ where F is not locally invertible and let $\operatorname{card} F^{-1}(\{f\})$ be the cardinal number of the set $F^{-1}(\{f\})$. We need the following result.

Theorem 2.4 (Ambrosetti)**.** *Let $F \in C(X, Y)$ be a proper map. Then the cardinal number* $\operatorname{card} F^{-1}(\{f\})$ *is constant, finite (it may be even 0) on each connected component of the set $Y \setminus F(\Sigma)$.*

In [20], using Browder's theorem [6], we have shown that if $T : X \to Y$ is closed on bounded closed subsets of X and is a local homeomorphism, then it is a homeomorphism if and only if it satisfies condition (+). Now, we shall look at perturbations of such maps.

Theorem 2.5 (Fredholm Alternative). *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$ be such that $\alpha(C) < \beta(T)$ (T be a c-expansive homeomorphism and C be k-ϕ-contraction with $k < c$, respectively). Then either*

(i) *$T + C$ is injective (locally injective, respectively), in which case it is an open map and $T + C$ is a homeomorphism if and only if either one of the following conditions holds*
   (a) *$T + C$ is closed (in particular, proper, or satisfies condition $(+)$),*
   (b) *$T{+}C$ is injective and $R(T + C)$ is closed, or*

(ii) *$T + C$ is not injective (not locally injective, respectively), in which case, assuming additionally that $T+tC$ satisfies condition $(+)$, the equation $Tx+Cx = f$ is solvable for each $f \in Y$ with $(T + C)^{-1}(f)$ compact and the cardinal number* $\operatorname{card}(T + C)^{-1}(f)$ *is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$.*

Proof. Since $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$, $T + C$ is proper on bounded closed subsets of X. Hence, if $T + C$ satisfies condition $(+)$, then it is a proper map and therefore closed. Assume that (i) holds. We shall show that $T + C$ is an open map. The equation $Tx + Cx = f$ is equivalent to $y + CT^{-1}y = f$, $y = Tx$. Then the map CT^{-1} is k/c-ϕ-contractive with $k/c < 1$ if T is expensive. If $\alpha(C) < \beta(T)$, then the map CT^{-1} is $\alpha(CT^{-1})$-contractive since

$$\alpha(CT^{-1}) \leq \alpha(C)\alpha(T^{-1}) = \alpha(C)/\beta(T) < 1.$$

Moreover, $I + CT^{-1}$ is (locally) injective since such is $T + C$. Hence, $I + CT^{-1}$ is an open map ([22], see also [11]). Thus, $T + C$ is an open map since $T + C = (I + CT^{-1})T$ and therefore it is a local homeomorphism. Hence, it is a homeomorphism by Browder's theorem [6] if (a) holds.

Let (i)(b) hold. Then $T + C$ is surjective since $(T + C)(X)$ is open and closed, and it is therefore a homeomorphism.

(ii) Suppose that $T + C$ is not injective (locally injective, respectively). We have seen above that CT^{-1} is k-contractive, $k < 1$, and $I + tCT^{-1}$ satisfies condition $(+)$ since C is bounded. Hence, using the homotopy $H(t, x) = x + tCT^{-1}x$ and the degree theory for condensing maps [22], we get that $I + CT^{-1}$ is surjective. Therefore, $T + C$ is surjective. Since $T + C$ is proper on X, the cardinal number $\operatorname{card}(T + C)^{-1}(f)$ is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$ by Theorem 2.4. □

Remark 2.6. Under the conditions of Theorem 2.5, we have that condition $(+)$ is equivalent to $R(T + C)(X)$ is closed when $T + C$ injective.

Remark 2.7. If X is an infinite dimensional Banach space and $T : X \to X$ is a homeomorphism, then T satisfies condition $(+)$ but it need not be coercive in the sense that $\|Tx\| \to \infty$ as $\|x\| \to \infty$. It was proved in [9, Corollary 8] that for every proper subset $U \subset X$, there is a homeomorphism $T : X \to X$ such that it maps $X \setminus U$ into U. Then, selecting U to be a ball in X, such a homeomorphism is not coercive.

Corollary 2.8. *Let $T : X \to Y$ be a homeomorphism and $C : X \to Y$ be continuous and uniformly bounded; i.e., $\|Cx\| \leq M$ for all x and some $M > 0$, and $\alpha(C) < \beta(T)$. Then either*

(i) *$T + C$ is injective, in which case $T + C$ is a homeomorphism, or*

(ii) *$T + C$ is not injective, in which case $T + C$ is surjective, $(T + C)^{-1}(f)$ is compact for each $f \in Y$, and the cardinal number* $\operatorname{card}(T + C)^{-1}(f)$ *is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$.*

Proof. We note first that the equation $Tx+Cx = f$ is equivalent to $y+CT^{-1}y = f$, with $y = Tx$. Using the uniform boundedness of C, it is easy to see that for each $f \in Y$, $\|H(t, y) = y + tCT^{-1}y - tf\| \to \infty$ as $\|y\| \to \infty$ uniformly in $t \in [0, 1]$. Hence, the Nussbaum degree $\deg(I - CT^{-1}, B(0, r), f) \neq 0$ for some $r > 0$ large. Thus, the equation $Tx+Cx = f$ is solvable for each $f \in Y$. Hence, (i) follows from Theorem 2.5(i)(b).

We have remarked before that $T+C$ is proper on bounded closed subsets. Next, we shall show that $T + C$ is a proper map. Let $K \subset Y$ be compact and $x_n \in (T+C)^{-1}(K)$. Then $y_n = Tx_n + Cx_n \in K$ and we may assume that $y_n \to y \in K$. Since C is uniformly bounded, $\{Tx_n\}$ is bounded. Set $z_n = Tx_n$ and note that $y_n = (I + CT^{-1})z_n \in K$. Hence, $z_n \in (I + CT^{-1})^{-1}(y_n) \subset (I + CT^{-1})^{-1}(K)$ and $\{z_n\}$ bounded. Since $I + CT^{-1}$ is proper on bounded closed subsets, we may assume that $z_n \to z \in B(0, r) \cap (I + CT^{-1})^{-1}(K)$ for some $r > 0$. Hence, $x_n = T^{-1}z_n \to T^{-1}z = x \in (T + C)^{-1}(K)$ since $T + C$ is proper on bounded closed subsets. Thus, $T + C$ is a proper map and (ii) follows by Theorem 2.4. □

When T is Fredholm of index zero, then the injectivity of $T + C$ can be replaced by the local injectivity.

Theorem 2.9 (Fredholm Alternative)**.** *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be such that $\alpha(C) < \beta(T)$. Then either*

(i) *$T + C$ is locally injective, in which case it is an open map and it is a homeomorphism if and only if one of the following conditions holds*
   (a) *$T + C$ is closed (in particular, proper, or satisfies condition $(+)$),*
   (b) *T+C is injective and $R(T + C)$ is closed, or*

(ii) *T+C is not locally injective, in which case, assuming additionally that T is locally injective and T+tC satisfies condition $(+)$, the equation $Tx+Cx = f$ is solvable for each $f \in Y$ with $(T + C)^{-1}(f)$ compact and the cardinal number* $\operatorname{card}(T+C)^{-1}(f)$ *is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$.*

Proof. As observed before, $T + C$ is proper on bounded closed subsets of X. If it satisfies condition $(+)$, then it is proper. Let (i) hold. Then $T + C$ is an open map by a theorem of Calamai [7]. Hence, $T + C$ is a local homeomorphism. If (a) holds, then $T + C$ is a homeomorphism if and only if it is a closed map by Browder's theorem [6]. If (b) holds, then T+C is surjective since $R(T+C)$ is open and closed. Hence, it is a homeomorphism.

Let (ii) hold. Then T is a homeomorphism by part (i)-a) (or by Tromba's theorem [31]) since it is proper on bounded closed subsets and satisfies condition $(+)$. Hence, the conclusions follow as in Theorem 2.5(ii). □

Condition $\alpha(C) < \beta(T)$ in (i) can be replaced by $\alpha_p(C) < \beta_p(T)$ for each $p \in X$ since T+C is also open in this case [7]. This condition always holds if T is a Fredholm map of index zero and C is compact. In view of this remark, we have the following extension of Tromba's homeomorphism result for proper locally injective Fredholm maps of index zero [31].

Corollary 2.10. *Let $T : X \to Y$ be a Fredholm map of index zero, $C : X \to Y$ be compact and $T+C$ be locally injective and closed. Then $T+C$ is a homeomorphism.*

Corollary 2.11. *Let $T : X \to Y$ be a locally injective closed Fredholm map of index zero, $C : X \to Y$ be continuous and uniformly bounded and $\alpha(C) < \beta(T)$. Then either*

- (i) *$T + C$ is locally injective, in which case $T + C$ is a homeomorphism, or*
- (ii) *$T+C$ is not locally injective, in which case $T+C$ is surjective, $(T+C)^{-1}(f)$ is compact for each $f \in Y$, and the cardinal number $\operatorname{card}(T + C)^{-1}(f)$ is positive, constant and finite on each connected component of the set $Y \setminus (T + C)(\Sigma)$.*

Proof. T is a homeomorphism by Corollary 2.10. We have shown in the proof of Corollary 2.8 that $T + C$ is proper and surjective. If $T + C$ is locally injective, then it is a homeomorphism by Theorem 2.9(i)(a). Part (ii) follows from Theorem 2.4. □

The following lemmas, needed later on, give a number of particular conditions on T and C that imply condition (+). Recall that a map C is quasibounded if, for some $k > 0$,

$$|C| = \limsup_{\|x\|\to\infty} \|Cx\|/\|x\|^k < \infty.$$

Lemma 2.12. *Suppose that $T, C : X \to Y$ and either one of the following conditions holds*

- (i) *$\|Cx\| \le a\|Tx\|+b$ for some constants $a \in [0, 1)$ and $b > 0$ and all $\|x\|$ large and $\|Tx\| \to \infty$ as $\|x\| \to \infty$.*
- (ii) *There exist constants $c, c_0, k > 0$ and $R > R_0$ such that*

$$\|Tx\| \ge c\|x\|^k - c_0 \quad \text{for all } \|x\| \ge R.$$

and C is quasibounded with the quasinorm $|C| < c$.

Then $T + tC$ satisfies condition (+) uniformly in $t \in [0, 1]$.

A map T is positive k-homogeneous outside some ball if $T(\lambda x) = \lambda^k T(x)$ for some $k \ge 1$, all $\|x\| \ge R$ and all $\lambda \ge 1$. We say that T is asymptotically close to a positive k-homogeneous map A if

$$|T - A| = \limsup_{\|x\|\to\infty} \|Tx - Ax\|/\|x\|^k < \infty.$$

We note that T is asymptotically close to a positively k-homogeneous map A if there is a functional $c : X \to [0, a]$ such that

$$\|T(tx)/t^k - Ax\| \le c(t)\|x\|^k,$$

In this case, $|T - A| \le a$.

Lemma 2.13 ([20])**.** *(a) Let $A : X \to Y$ be continuous, closed (in particular, proper) on bounded and closed subsets of X and for some $R_0 \ge 0$*

$$A(\lambda x) = \lambda^k (Ax) \tag{2.1}$$

for all $\|x\| \ge R_0$, $\lambda \ge 1$ and some $k \ge 1$. Suppose that either one of the following conditions holds

- (i) *There is a constant $M > 0$ such that if $Ax = 0$, then $\|x\| \le M$*
- (ii) *A is injective*

(iii) *A is locally injective and* (2.1) *holds for all* $\lambda > 0$.

Then there exist constants $c > 0$ *and* $R > R_0$ *such that*

$$\|Ax\| \geq c\|x\|^k \quad \text{for all } \|x\| \geq R \tag{2.2}$$

and, in addition, A^{-1} *is bounded when (ii) holds. Moreover, if A is positively k-homogeneous, then* $Ax = 0$ *has only the trivial solution if and only if* (2.2) *holds.*

(b) If $T : X \to Y$ *is asymptotically close to A with* $|T - A|$ *sufficiently small, then T also satisfies* (2.2) *with c replaced by* $c - |T - A|$.

Let us connect this with eigenvalue problems. Let T, C be asymptotically close to k-positive homogeneous maps T_0 and C_0, respectively. We say that μ is not an eigenvalue of T_0 relative to C_0 if $T_0x = \mu C_0x$ implies that $x = 0$. Then one can use Lemma 2.13 to show that $T - \mu C$ satisfies condition (2.2) provided that μ is not an eigenvalue of T_0 relative to C_0.

Lemma 2.14. *(i) Let* $A : X = X^{**} \to X^*$ *be k-positive homogeneous such that* $(Ax, x) \geq m\|x\|^k$ *for all* $x \in X$ *and some* $m > 0$, $k \geq 2$, $G : X \to X^*$ *be k-positive homogeneous,* $g(x) = (Gx, x)$ *be weakly continuous and* $f(x) = (Ax + Gx, x)$ *be weakly lower semicontinuous and positive definite, i.e.,* $f(x) > 0$ *for* $x \neq 0$. *Then*

$$(Ax + Gx, x) \geq c\|x\|^k \quad \text{for all } x \in X \text{ and some } c > 0.$$

(ii) If $T, C : X \to X^*$ *are asymptotically close to A and G, respectively with* $|T - A|$ *and* $|C - G|$ *sufficiently small, then there is a* $c_1 > 0$ *and an* $R > 0$ *such that*

$$\|Tx + Cx\| \geq c_1\|x\|^{k-1} \quad \text{for all } \|x\| \geq R. \tag{2.3}$$

Proof. Let $c = \inf_{\|x\|=1} f(x)$. If $c = 0$, then there is a sequence $\{x_n\}$ such that $f(x_n) \to c = 0$. We may assume that $x_n \rightharpoonup x_0$. Since f is weakly lower semicontinuous, we get

$$\lim_{n\to\infty} f(x_n) \geq f(x_0).$$

Hence, $f(x_0) \leq 0$ and therefore $x_0 = 0$ by the positive definiteness of f. On the other hand,

$$\lim_{n\to\infty} f(x_n) \geq m \lim_{n\to\infty} \|x\|^k - \lim_{n\to\infty} |(Gx_n, x_n)| \geq m > 0$$

since $g(x)$ is weakly continuous and $\|x_n\| = 1$. This is a contradiction, and therefore $c > 0$. By the k-positive homogeneity of f we get that $f(x/\|x\|) \geq c$ for each $x \neq 0$. Thus, $f(x) \geq c\|x\|^k$.

(ii) Let $\epsilon > 0$ be such that $2\epsilon + |T - A| + |C - G| < c$ with $c > 0$ in part (i). Then there is an $R > 0$ such that $\|Tx - Ax\| \leq (\epsilon + |T - A|)\|x\|^{k-1}$ and $\|Cx - Gx\| \leq (\epsilon + |C - G|)\|x\|^{k-1}$ for all $\|x\| \geq R$. This implies (2.3) with $c_1 = c - 2\epsilon - |T - A| - |C - G|$. $\square$

For $G : X \to X^*$, define $g(x) = (Gx, x)$. If G is weakly continuous, g need not be weakly lower semicontinuous. It is easy to show that

(1) g is weakly lower semicontinuous if G is weakly continuous and monotone, i.e., $(Gx - Gy, x - y) \geq 0$, or G is completely continuous, i.e., $Gx_n \to Gx$ if $x_n \rightharpoonup x$.

(2) If $A : X \to X^*$ is such that $h(x) = (Ax, x)$ is weakly lower semicontinuous and G is completely continuous, then $f(x) = (Ax + Gx, x)$ is weakly lower semicontinuous.

Lemma 2.15 ([20]). *Let* $T : X \to Y$ *be a homeomorphism and* $C : X \to Y$.

(i) *If $I+CT^{-1}: Y \to Y$ is proper on bounded closed subsets of Y and if either T or C is bounded, then $T+C$ is proper on bounded closed subsets of X.*
(ii) *If T^{-1} is bounded and $I+CT^{-1}: Y \to Y$ satisfies condition (+), then $T+C$ satisfies condition (+). Conversely, if either T or C is bounded, and $T+tC$, $t \in [0,1]$, satisfies condition (+), or C has a linear growth and T^{-1} is quasibounded with a sufficiently small quasinorm, then $I+tCT^{-1}: Y \to Y$ satisfies condition (+), $t \in [0,1]$.*

Lemma 2.16. *Let $T: X \to Y$ be a C^1 local homeomorphism and $C: X \to Y$ be c-expansive. Then $T+C$ is locally expansive and therefore it is locally injective on X.*

Proof. Let $p \in X$ be fixed. By Example 2.3, there is an $r = r(p) > 0$ such that $T: B(p,r) \to Y$ is $c(p)$-expansive for some $c(p) \in (0,c)$. Hence, $T+C$ is $c-c(p)$-expansive on B and is therefore locally injective on X. □

Next, we shall prove some nonlinear extensions of the Fredholm alternative to set contractive like perturbations of homeomorphisms as well as of Fredholm maps of index zero that are asymptotically close to positive k-homogeneous maps.

Theorem 2.17 (Fredholm alternative)**.** *Let $T: X \to Y$ be a homeomorphism and $C, D: X \to Y$ be continuous maps such that $\alpha(D) < \beta(T) - \alpha(C)$ with $|D|$ sufficiently small (T be a c-expansive homeomorphism and C be a k-ϕ-contraction with $k < c$, respectively), where*

$$|D| = \limsup_{\|x\| \to \infty} \|Dx\|/\|x\|^k < \infty.$$

Assume that $T+C$ is injective (locally injective, respectively) and either $\|Tx + Cx\| \geq c\|x\|^k - c_0$ for all $\|x\| \geq R$ for some R, c and c_0, or $T+C$ is asymptotically close to a continuous, closed (proper, in particular) on bounded and closed subsets of X positive k-homogeneous map A outside some ball in X; i.e., there are $k \geq 1$ and $R_0 > 0$ such that $A(\lambda x) = \lambda^k Ax$ for all $\|x\| \geq R_0$, all $\lambda \geq 1$ with $A^{-1}(0)$ bounded and $|T+C-A|$ sufficiently small. Then either
(i) *$T+C+D$ is injective, in which case $T+C+D$ is a homeomorphism, or*
(ii) *$T+C+D$ is not injective, in which case the solution set $(T+C+D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $\operatorname{card}(T+C+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T+C+D)(\Sigma)$.*

Proof. Since T is a homeomorphism, it is proper and $\beta(T) > 0$. Moreover, $T+C$ satisfies condition (+) by Lemma 2.13(i)(a). Since $\alpha(C) < \beta(T)$, $T+C$ is proper on bounded closed subsets of X and is therefore proper by condition (+). It follows that $T+C$ is a homeomorphism by Theorem 2.5(i)(a). Next, we claim that $H_t = T+C+tD$ satisfies condition (+).

Let $y_n = (T+C+t_nD)x_n \to y$ as $n \to \infty$ with $t_n \in [0,1]$, and suppose that $\|x_n\| \to \infty$. Then, by Lemma 2.13(b) with $c_1 = c - |T+C|$,

$$c_1\|x_n\|^k - c_0 \leq \|(T+C)x_n\| \leq \|y_n\| + (|D| + \epsilon)\|x_n\|^k$$

for all n large and any $\epsilon > 0$ fixed. Dividing by $\|x_n\|^k$ and letting $n \to \infty$, we get that $c \leq |D|$. This contradicts our assumption that $|D|$ is sufficiently small and therefore condition (+) holds for H_t. Let (i) hold. Since $\alpha(D) < \beta(T) - \alpha(C) \leq$

$\beta(T+C)$ and $H_1 = T+C+D$ satisfies condition (+), $T+C+D$ is a homeomorphism by Theorem 2.5(i)(a).

Next, let (ii) hold. Since $\alpha(D) < \beta(T+C)$, $T+C$ is a homeomorphism and H_t satisfies condition (+), the equation $Tx + Cx + Dx = f$ is solvable for each f by Theorem 2.5(ii). Moreover, $T + C + D$ is proper on closed bounded sets since $\beta(T + C + D) \geq \beta(T) - \alpha(C + D) \geq \beta(T) - \alpha(C) - \alpha(D) > 0$. Hence, the map $T + C + D$ is proper on X by condition (+), and the other conclusions follow from Theorem 2.4. $\square$

If $C = 0$ in Theorem 2.17, then the injectivity of $T + D$ can be weaken to local injectivity when T is c-expansive.

Corollary 2.18. *Let $T : X \to Y$ be a c-expansive homeomorphism and $D : X \to Y$ be a continuous map such that $\alpha(D) < c$ and*

$$|D| = \limsup_{\|x\|\to\infty} \|Dx\|/\|x\| < c.$$

Then either

(i) *$T + D$ is locally injective, in which case $T + D$ is a homeomorphism, or*

(ii) *$T+D$ is not locally injective, in which case the solution set $(T+D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $\mathrm{card}(T+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + D)(\Sigma)$.*

Proof. Let (i) hold. Then T is a homeomorphism, $\beta(T) \geq c$ and $\alpha(D) < \beta(T)$. Since $T + D$ satisfies condition (+) by Lemma 2.12(ii), $T + D$ is a homeomorphism by Theorem 2.5(i)(a). Part (ii) follows from Theorem 2.17(ii) with $C = 0$. $\square$

Next, we shall look at various conditions on a c-expansive map that make it a homeomorphism. They came about when some authors tried to give a positive answer to the Nirenberg problem on surjectivity of a c-expansive map with $T(0) = 0$ and mapping a neighborhood of zero onto a neighborhood of zero.

Corollary 2.19. *Let $T : X \to Y$ be a c-expansive map and $D : X \to Y$ be a continuous maps such that $\alpha(D) < c$ and*

$$|D| = \limsup_{\|x\|\to\infty} \|Dx\|/\|x\| < c.$$

Suppose that either one of the following conditions holds

(a) *Y is reflexive, T is Fréchet differentiable and*

$$\limsup_{x\to x_0} \|T'(x) - T'(x_0)\| < c \quad \textit{for each } x_0 \in X$$

(b) *$T : X \to X$ is Fréchet differentiable and such that the logarithmic norm $\mu(T'(x))$ of $T'(x)$ is strictly negative for all $x \in X$, where*

$$\mu(T'(x)) = \lim_{t\to 0^+} (\|I + tT'(x)\| - 1)/t$$

(c) *$X = Y = H$ is a Hilbert space, T is Fréchet differentiable and such that either*

$$\inf_{\|h\|=1} Re(T'(x)h, h) > 0 \quad \textit{for all } x \in H,$$

or

$$\sup_{\|h\|=1} Re(T'(x)h, h) < 0 \quad \textit{for all } x \in H,$$

(d) *X is reflexive and $T : X \to X^*$ is a C^1 potential map.*

Then either

(i) *$T + D$ is locally injective, in which case $T + D$ is a homeomorphism, or*
(ii) *$T+D$ is not locally injective, in which case the solution set $(T+D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $\mathrm{card}(T+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + D)(\Sigma)$.*

Proof. T is a homeomorphism by Chang-Shujie [8] in parts (a), (d), and by Hernandez-Nashed [16] in parts (b), (c). Since $\|Tx\| \geq c\|x\| - \|T(0)\|$, Corollary 2.18 applies. □

Recall that a map $T : X \to Y$ is expansive along rays if for each $y \in Y$, there is a $c(y) > 0$ such that $\|Tx - Ty\| \geq c(y)\|x - y\|$ for all $x, y \in T^{-1}([0, y])$, where $[0, y] = \{ty : 0 \leq t \leq 1\}$.

Theorem 2.20. *An expansive along rays local homeomorphism $T : X \to Y$ is a homeomorphism. A locally expansive local homeomorphism is a homeomorphism if $m = \inf_x c(x) > 0$. In general, a locally expansive local homeomorphism need not be a homeomorphism.*

Proof. Note first that if T is a c-expansive local homeomorphism, then it is an open map. Since $R(T)$ is also closed, T is surjective. Since it is injective, it is a homeomorphism. Next, assume that T is just expansive along rays. We may assume that $T(0) = 0$. Since T is a local homeomorphism, there is a ball B about 0 in Y and a continuous local inverse $g : B \to X$ of T with $g(0) = 0$. Next, we shall continue the local inverse g along each ray from 0 as far out as possible. To that end, let D be the set of all points $y \in Y$ such that there is a continuous inverse g of T defined on the ray $[0, y] = \{ty : 0 \leq t \leq 1\}$ and $g(0) = 0$. It is known that D is an open subset of Y, the value of $g(y)$ depends uniquely on T and y, the map T^{-1} defined by $T^{-1}(y) = g(y)$ is an inverse of T on D and T^{-1} is continuous (see John [18] for details). Next, it has been shown in Hernandez-Nashed [16] that $D = Y$. Hence, T is a homeomorphism. If T is a locally expensive local homeomorphism with $m > 0$, then T is a homeomorphism by John's theorem [18] since the scalar derivative of T is $D_x^- T = \liminf_{y\to x} \|Tx - Ty\|/\|x - y\| = c(x)$ and $m = \inf_x D_x^- T > 0$.

A C^1 local homeomorphism is locally expansive by Example 2.3, but it need not be a homeomorphism. □

Corollary 2.21. *Let $T : X \to Y$ be an expansive along rays local homeomorphism and $D : X \to Y$ be a continuous maps such that $\alpha(D) < \beta(T)$ and $\|Dx\| \leq M$ for all $x \in X$ and some $M > 0$. Then either*

(i) *$T + D$ is injective, in which case $T + D$ is a homeomorphism, or*
(ii) *$T + D$ is not injective, in which case the solution set $(T + D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number $\mathrm{card}(T + D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + D)(\Sigma)$.*

The proof of the above corollary follows from Theorem 2.20 and Corollary 2.8. Next, we shall give another extension of the Fredholm Alternative to perturbations of nonlinear Fredholm maps of index zero.

Theorem 2.22 (Fredhom Alternative). *Let $T : X \to Y$ be a Fredholm map of index zero and $C, D : X \to Y$ be continuous maps such that $\alpha(D) < \beta(T) - \alpha(C)$ with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\|\to\infty} \|Dx\|/\|x\|^k < \infty.$$

Assume that either $\|Tx + Cx\| \geq c\|x\|^k - c_0$ for all $\|x\| \geq R$ for some R, c and c_0, or $T + C$ is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X, i.e., there are $k \geq 1$ and $R_0 > 0$ such that $A(\lambda x) = \lambda^k Ax$ for all $\|x\| \geq R_0$, all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded with $|T + C - A|$ sufficiently small. Then either

- (i) *$T+C+D$ is locally injective, in which case $T+C+D$ is a homeomorphism, or*
- (ii) *$T + C + D$ is not locally injective, in which case, assuming additionally that $T + C$ is locally injective, the solution set $(T + C + D)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number* $\operatorname{card}(T + C+D)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the set $Y \setminus (T + C + D)(\Sigma)$.*

Proof. Let (i) hold. Since $\alpha(D) < \beta(T)-\alpha(C) \leq \beta(T+C)$, we get $\beta(T+C+D) > 0$ and therefore $T + C + D$ is proper on bounded closed subsets of X. We have that $T + C + D$ satisfies condition (+) as in Theorem 2.17. Hence, $T + C + D$ is proper on X and so it is closed on X. Since $\alpha(C + D) < \beta(T)$ and T is Fredholm of index zero, $T + C + D$ is an open map by Calamai's theorem [7]. Hence, $T + C + D$ is surjective since the range $(T+C+D)(X)$ is both open and closed. Since $T+C+D$ is a local homeomorphism, it is a homeomorphism by the Banach-Mazur theorem.

Let (ii) hold. Since $\alpha(C) < \beta(T)$ and $T + C$ is locally injective and satisfies condition (+) by Lemma 2.13(i)(a), $T+C$ is a homeomorphism by Theorem 2.9(i). Since $\alpha(D) < \beta(T) - \alpha(C) \leq \beta(T + C)$, the conclusions follow as in Theorem 2.17(ii). □

Corollary 2.23. *Let $T, C : X \to Y$ and $T + C$ be Fredholm maps of index zero such that $\alpha(C) < \beta(T)$ and $|C|$ sufficiently small, where*

$$|C| = \limsup_{\|x\|\to\infty} \|Cx\|/\|x\|^k < \infty.$$

Assume that either $\|Tx\| \geq c\|x\|^k - c_0$ for all $\|x\| \geq R$ for some R, c and c_0, or T is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X; i.e., there are $k \geq 1$ and $R_0 > 0$ such that

$$A(\lambda x) = \lambda^k Ax$$

for all $\|x\| \geq R_0$, all $\lambda \geq 1$ and $(A)^{-1}(0)$ bounded with $|T + C - A|$ sufficiently small. Then either

- (i) *T+C is locally injective, in which case T+C is a homeomorphism, or*
- (ii) *T+C is not locally injective, in which case, assuming additionally that T is locally injective, the solution set $(T + C)^{-1}(\{f\})$ is nonempty and compact for each $f \in Y$ and the cardinal number* $\operatorname{card}(T + C)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the open and dense set*

of regular values $R_{T+C} = Y \setminus (T+C)(S)$ of Y, where S is the set of singular points of $T + C$.

Proof. Part (i) and the surjectivity of $T + C$ follow from Theorem 2.17 (with C replaced by D). As before, we have that $T + C$ is proper on bounded and closed set and satisfies condition (+). Hence, it is proper and the other conclusions follow from the general theorem on nonlinear Fredholm maps of index zero (see [33]). $\square$

3. Finite solvability of equations with perturbations of odd Fredholm maps of index zero

In this section, we shall study perturbations of Fredholm maps of index zero assuming that the maps are odd. We shall first look at compact perturbations and use the Fitzpatrick-Pejsachowisz-Rabier-Salter degree.

Theorem 3.1 (Generalized First Fredholm Theorem). *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on bounded and closed subsets of X and $C, D : X \to Y$ be compact maps with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\|\to\infty} \|Dx\|/\|x\|^k < \infty.$$

Assume that $T+C$ is odd, asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X, i.e., there exists $R_0 > 0$ such that

$$A(\lambda x) = \lambda^k Ax$$

for all $||x|| \ge R_0$, for all $\lambda \ge 1$ and some $k \ge 1$, and $\|x\| \le M < \infty$ if $Ax = 0$ and $|T + C - A|$ sufficiently small. Then the equation $Tx + Cx + Dx = f$ is solvable for each $f \in Y$ with $(T + C + D)^{-1}(\{f\})$ compact and the cardinal number $\operatorname{card}(T+C+D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + C + D)(\Sigma)$.

Proof. Step 1. Let p be a base point of T. By Lemma 2.13, there is an $R > 0$ such that condition (2.2) holds for $T + C$. Define the homotopy $H(t, x) = Tx + Cx + tDx - tf$ for $t \in [0, 1]$. Since $T + C$ satisfies (2.2), it is easy to show that $H(t, x)$ satisfies condition (+) and therefore $H(t, x) \neq 0$ for $(t, x) \in [0, 1] \times \partial B(0, R_1)$ for some $R_1 \ge R$.

Next, we note that $H(t, x)$ is a compact perturbation tD of the odd map $T + C$ with T Fredholm of index zero. Then, by the homotopy theorem [25, Corollary 7.2] and the Borsuk theorem for such maps of Rabier-Salter [25], we get that the Fitzpatrick-Pejsachowisz-Rabier-Salter degree

$$\begin{aligned} \deg_{T,p}(H_1, B(R_1, 0), 0) &= \deg_{T,p}(T + C + D - f, B(R_1, 0), 0) \\ &= \nu \deg_{T,p}(T + C, B(R_1, 0), 0) \neq 0 \end{aligned}$$

where ν is 1 or -1.

Step 2. T has no base point. Then pick a point $q \in X$ and let $A : X \to Y$ be a continuous linear map with finite dimensional range such that $T'(q) + A$ is invertible. Then $T + A$ is Fredholm of index zero, proper on $\bar{B}(0, r)$ and q is a base point of $T + A$. Moreover, $T + C + D = (T + A) + (C - A) + D$. We have reduced the problem to the case when there is a base point for $T + A$ and the maps $T + A$,

$C - A$ and D satisfy the same conditions of the theorem as the maps T, C and D. Then, as in Step 1,

$$\begin{aligned}\deg_{T+A,q}(H_1, B(R_1,0),0) &= \deg_{T+A,q}((T+A)+(C-A)+D-f, B(R_1,0),0)\\ &= \nu \deg_{T+A,q}(T+C, B(R_1,0),0) \neq 0\end{aligned}$$

where ν is 1 or -1.

Hence, by the existence theorem of this degree, we have that $Tx + Cx + Dx = f$ is solvable in either case. Next, since $T+C+D$ is continuous and satisfies condition (+), it is proper since it is proper on bounded closed sets as a compact perturbation of such a map. Hence, the second part of the theorem follows from Theorem 2.4. □

Remark 3.2. Earlier generalizations of the first Fredholm theorem to condensing vector fields, maps of type (S_+), monotone like maps and (pseudo) A-proper maps assumed the homogeneity of T with $Tx = 0$ only if $x = 0$ (see [17, 19, 21] and the references therein).

Next, we provide some generalizations of the Borsuk-Ulam principle for odd compact perturbations of the identity. The first result generalizes Theorem 3.1 when $D = 0$.

Theorem 3.3. *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on closed bounded subsets of X and $C : X \to Y$ be compact such that $T + C$ is odd outside some ball $B(0,R)$. Suppose that $T + C$ satisfies condition (+). Then $Tx + Cx = f$ is solvable, $(T + C)^{-1}(f)$ is compact for each $f \in Y$ and the cardinal number* $\operatorname{card}(T+C)^{-1}(f)$ *is positive and constant on each connected component of $Y \setminus (T + C)(\Sigma)$.*

Proof. Condition (+) implies that for each $f \in Y$ there is an $r = r_f > R$ and $\gamma > 0$ such that

$$\|Tx + Cx - tf\| \geq \gamma \quad \text{for all } t \in [0,1], \|x\| = r.$$

The homotopy $H(t,x) = Tx + Cx - tf$ is admissible for the Rabier-Salter degree and $H(t,x) \neq 0$ on $[0,1] \times \partial B(0,r)$. Hence, by the homotopy [25, Corollary 7.2], if $p \in X$ is a base point of T, then

$$\deg_{T,p}(T + C - f, B(0,r),0) = \nu \deg_{T,p}(T + C, B(0,r),0) \neq 0$$

since ν is plus or minus one and the second degree is odd by the generalized Borsuk theorem in [25]. If T has no base point, then proceed as in Step 2 of the proof of Theorem 3.1. Hence, the equation $Tx + Cx = f$ is solvable in either case. The second part follows from Theorem 2.4 since $T + C$ is proper on bounded closed subsets and satisfies condition (+). □

Next, we shall prove a more general version of Theorem 3.3.

Theorem 3.4. *Let $T : X \to Y$ be a Fredholm map of index zero that is proper on closed bounded subsets of X and $C_1, C_2 : X \to Y$ be compact such that $T + C_1$ is odd outside some ball $B(0,R)$. Suppose that $H(t,x) = Tx + C_1x + tC_2x - tf$ satisfies condition (+). Then $Tx + C_1x + C_2x = f$ is solvable for each $f \in Y$ with $(T + C_1 + C_2)^{-1}(f)$ compact and the cardinal number* $\operatorname{card}(T + C_1 + C_2)^{-1}(f)$ *is positive and constant on each connected component of $Y \setminus (T + C_1 + C_2)(\Sigma)$.*

Proof. Condition (+) implies that for each $f \in Y$ there is an $r = r_f > R$ with $0 \notin H([0,1] \times \partial B(0,r)$. If p is a base points of T, then by [25, Theorem 7.1],

$$\deg_{T,p}(H(1,.), B(0,r), 0) = \nu \deg_{T,p}(T_1 + C_1, B(0,r), 0) \neq 0$$

where $\nu \in \{-1, 1\}$.

Next, if T has no a base point, then pick $q \in X$ and let A be a continuous linear map from X to Y with finite dimensional ranges such that $T'(q) + A$ is invertible. Then we can rewrite H as $H(t,x) = (T + A)x + (C_1 - A)x + tC_2x - tf$, where $T + A$, $C_1 - A$ and C_2 satisfy all the conditions of the theorem and $T + A$ has a base point. As in the first case, we get that

$$\begin{aligned} \deg_{T+A,q}(H(1,.), B(0,r), 0) &= \deg_{T+A,q}((T + A) + (C_1 - A) + C_2 - f, B(0,r), 0) \\ &= \nu \deg_{T+A,q}((T + A) + (C_1 - A), B(0,r), 0) \neq 0 \end{aligned}$$

where $\nu \in \{-1, 1\}$. Hence, the equation $Tx + C_1x + C_2x = f$ is solvable in either case. The other conclusions follow from Theorem 2.4. ☐

Next, we shall study k-set contractive perturbations of Fredholm maps of index zero. Denote by $\deg_{BCF}$ the degree of Benevieri-Calamai-Furi. When $T + C$ is not locally injective, we have the following extension of Theorem 2.22.

Theorem 3.5 (Generalized First Fredholm Theorem). *Let $T : X \to Y$ be a Fredholm map of index zero and $C, D : X \to Y$ be continuous maps such that $\alpha(D) < \beta(T) - \alpha(C)$ with $|D|$ sufficiently small, where*

$$|D| = \limsup_{\|x\| \to \infty} \|Dx\| / \|x\|^k < \infty.$$

Assume that $T + C$ is asymptotically close to a continuous, closed (in particular, proper) on bounded and closed subsets of X positive k-homogeneous map A, outside some ball in X for all $\lambda \geq 1$ and some $k \geq 1$, $\|x\| \leq M < \infty$ if $Ax = 0$, $|T + C - A|$ sufficiently small, and $\deg_{BCF}(T + C, B(0,r), 0) \neq 0$ for all large r. Then the equation $Tx + Cx + Dx = f$ is solvable for each $f \in Y$ with $(T + C + D)^{-1}(\{f\})$ compact and the cardinal number $\operatorname{card}(T + C + D)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (T + C + D)(\Sigma)$.

Proof. Since $\beta(T + C) \geq \beta(T) - \alpha(C) > 0$, we have that $T + C$ is proper on closed bounded sets and satisfies condition (2.2) for some $R > 0$ by Lemma 2.13 with $c = c - |T - C|$. Moreover, $\alpha(D + C) \leq \alpha(D) + \alpha(C) < \beta(T)$. Let $f \in Y$ be fixed and $\epsilon > 0$ and $R_1 > R$ such that $|D| + \epsilon + \|f\|/R_1^k < c$ and $\|Dx\| \leq (|D| + \epsilon)\|x\|$ for all $\|x\| \geq R_1$. Define the homotopy $H(t,x) = Tx + Cx + tDx - tf$ for $t \in [0,1]$. Then, $H(t,x) \neq 0$ for $(t,x) \in [0,1] \times (X \setminus B(0,R_1)$. If not, then there is a $t \in [0,1]$ and x with $\|x\| \geq R_1$ such that $H(t,x) = 0$. Then

$$c_1\|x_n\|^k \leq \|Tx_n + Cx_n\| = t_n\|Dx_n - f\| \leq (|D| + \epsilon)\|x_n\|^k + \|f\|.$$

Hence, $c_1 < |D| + \epsilon + \|f\|/R_1^k$, in contradiction to our choice of ϵ and R_1. Similarly, arguing by contradiction, we get that $H(t,x)$ satisfies condition (+).

Next, we will show that $H(t,x)$ is an admissible oriented homotopy on $[0,1] \times B(0,R_2)$ for some $R_2 \geq R_1$. Set F(t,x)=tDx. Then for any subset $A \subset B(0,R_2)$, $\alpha(F([0,1] \times A)) = \alpha(\{tDx : t \in [0,1], x \in A\}) = \alpha(\{ty : t \in [0,1], y \in D(A)\}) = \alpha(D(A)) \leq \alpha(D)\alpha(A)$. Hence, $\alpha(F) \leq \alpha(D)$. Moreover, $\alpha(C+F) \leq \alpha(C) + \alpha(F) \leq \alpha(C) + \alpha(D) < \beta(T)$. Thus, we get that $\beta(H) \geq \beta(T) - \alpha(C + F) > 0$. This implies that H is proper on bounded closed subsets of $[0,1] \times X$ with the norm

$\|(t,x)\| = \max\{|t|, \|x\|\}$ for $(t,x) \in \mathbb{R} \times X$. Since H satisfies condition (+), it is proper on $[0,1] \times X$. Thus, $H^{-1}(0)$ is compact and contained in $[0,1] \times B(0,R_2)$ for some $R_2 \geq R_1$. Since $B(0,R_2)$ is simply connected, $H(0,.) = T+C : B(0,R_2) \to Y$ is oriented by [3, Proposition 3.7]. Hence, H is oriented by [3, Proposition 3.5] and the homotopy [5, Theorem 6.1] implies that

$$\begin{aligned} \deg_{BCF}(H_1, B(R_1,0), 0) &= \deg_{BCF}(T + C + D - f, B(R_1,0), 0) \\ &= \deg_{BCF}(T + C, B(R_1,0), 0) \neq 0. \end{aligned}$$

Thus, the equation $Tx + Cx + Dx = f$ is solvable. The other conclusions follow from Theorem 2.4 since $H(0,.) = T + C + D$ satisfies condition (+) and is therefore proper. □

Remark 3.6. Theorems 3.1 and 3.5 are valid without the asymptotic assumption if the k-positive homogeneity of $T + C$ is replaced by $\|Tx + Cx\| \geq c\|x\|^k$ for all x outside some ball (see Lemma 2.13), or by condition (+) for $T + C$ if $D = 0$. Moreover, the degree assumption in Theorem 3.5 holds if T and C are odd maps by the generalized Borsuk theorem in Benevieri-Calamai [2]. In particular, we have the following result.

Corollary 3.7. *Let $T : X \to Y$ be a Fredholm map of index zero and $C : X \to Y$ be a continuous map such that $\alpha(C) < \beta(T)$, T and C be odd outside some ball and $T + C$ satisfy condition (+). Then the equation $Tx + Cx = f$ is solvable for each $f \in Y$ with $(T+C)^{-1}(\{f\})$ compact and the cardinal number* $\operatorname{card}(T+C)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the set $Y \setminus (T+C)(\Sigma)$.*

4. Applications to (quasi) linear elliptic nonlinear boundary-value problems

Potential problems with strongly nonlinear boundary-value conditions. Consider the nonlinear boundary-value problem

$$\begin{gathered} \Delta\Phi = 0 \quad \text{in } Q \subset \mathbb{R}^2 \\ -\partial_n \Phi = b(x, \Phi(x)) - f \quad \text{on } \Gamma = \partial Q. \end{gathered} \tag{4.1}$$

where Γ is a simple smooth closed curve, ∂_n is the outer normal derivative on Γ. The nonlinearities appear only in the boundary conditions. Using the Kirchhoff transformation, more general quasilinear equations can be transformed into this form. This kind of equations with various nonlinearities arise in many applications like steady-state heat transfer, electromagnetic problems with variable electrical conductivity of the boundary, heat radiation and heat transfer (cf. [28] and the references therein). Except for [27], the earlier studies assume that the nonlinearities have at most a linear growth and were based on the boundary element method. We shall study (4.1) using the theory in Sections 2-3. We note also that bifurcation problems for quasilinear elliptic systems with nonlinearities in the boundary conditions have recently been discussed by Shi and Wang [29]. Their study is based on the abstract global bifurcation theorem of Pejsachowicz and Rabier [23] for Fredholm maps of index zero.

Assume $b(x,u) = b_0(x,u) + b_1(x,u)$ satisfies

(1) $b_0 : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function; i.e., $b_0(.,u)$ is measurable for all $u \in \mathbb{R}$ and $b_0(x,.)$ is continuous for a.e. $x \in \Gamma$
(2) $b_0(x,)$ is strictly increasing on $\mathbb{R}$

(3) For $p \geq 2$, there exist constants $a_1 > 0$, $a_2 \geq 0$, $c_1 > 0$ and $c_2 \geq 0$ such that
$$|b_0(x,u)| \leq a_1|u|^{p-1} + a_2, \quad b_0(x,u)u \geq c_1|u|^p + c_2.$$
(4) b_1 satisfies the Carathéodory conditions and $|b_1(x,u)| \leq M$ for all $(x,u) \in \Gamma \times \mathbb{R}$ and some $M > 0$.

We shall reformulate (4.1) as a boundary integral equation. Recall that the single layer operator V is defined by
$$Vu(x) = -1/(2\pi)\int_\Gamma u(y)log|x-y|ds_y, \quad x \in \Gamma$$
and the double layer operator K is defined by
$$Ku(x) = 1/(2\pi)\int_\Gamma u(y)\partial_n \log|x-y|ds_y, \quad x \in \Gamma.$$
We shall make the ansatz: Find a boundary distribution u (in some space) such that
$$\Phi(x) = -1/(2\pi)\int_\Gamma u(y)log|x-y|ds_y, \quad x \in Q.$$
Then, by the properties of the normal derivative of the monopole potential [10, 32], we derive the nonlinear boundary integral equation [27]
$$((1/2I - K^*)u + B(Vu) = f.$$
This equation can be written in the form
$$Tu + Cu = f \tag{4.2}$$
where we set $T = 1/2I - K^* + B_0V$, $Cu = B_1V$ with $B_iu = b_i(x,u)$, $i = 0,1$.

Theorem 4.1. *Let* (1)-(4) *hold and* $q = p/(p-1)$. *Then either*

(i) *the BVP* (4.1) *is locally injective in* $L_q(Q)$, *in which case it is uniquely solvable in* $L_q(Q)$ *for each* $h \in L_q(Q)$ *and the solution depends continuously on* h, *or*

(ii) *the BVP* (4.1) *is not locally injective, in which case it is solvable in* $L_q(Q)$ *for each* $h \in L_q(Q)$, *the solution set is compact and the cardinality of the solution set is finite and constant on each connected component of* $L_q(Q) \setminus (T+C)(\Sigma)$, *where* $\Sigma = \{u \in L_q(Q) : (4.1)$ *is not locally uniquely solvable*$\}$.

Proof. We have that $V, K, K^* : L_p(\Gamma) \to L_q(\Gamma)$ are compact maps for each $p, q \in [1,\infty]$ [13] and therefore such are the maps $B_0V, C : L_q(\Gamma) \to L_q(\Gamma)$. By condition (2), it was shown in [27] that T is strictly V-monotone; i.e., for each $u, v \in L_p(\Gamma)$, $u \neq v$,
$$(Tu - Tv, V(u-v))_{L_2(\Gamma)} > 0.$$
This implies that T is injective.

Next, as in [26] but without the differentiability of b_0, we shall show first that T is surjective in $L_q(Q)$. To that end, we shall show that the problem
$$\Delta\Phi = 0, \quad \partial_n\Phi = B_0(\Phi) - f \tag{4.3}$$
has a solution in $W_2^1(Q)$. For $\Phi \in W_2^1(Q)$, define
$$j(\Phi(x)) = \int_0^{\Phi(x)} b_0(x,s)ds.$$

Since $b_0(x,.)$ is a strictly monotone proper function, j is a strictly convex and lower semicontinuous function whose subgradient is given by $\partial j(u) = B_0(u)$ [1, Theorem 2.3]. Then the above problem is equivalent to the minimization of the functional

$$F(\Phi) = 1/2 \int_Q |\Delta\Phi|^2 + G(\Phi) - \int_\Gamma f\Phi ds_\Gamma$$

over $W_2^1(Q)$, where, for $\Phi \in W_2^1$,

$$G(\Phi) = \int_\Gamma j(\Phi) ds_x, \quad \text{for } j(\Phi) \in L_1(\Gamma)$$

and $G(\Phi) = +\infty$ otherwise. Since $W_2^1(Q)$ is continuously imbedded in $L_p(\Gamma)$, condition (3) implies that for each $\Phi \in W_2^1(Q)$ there exist constants $c > 0$ and $c_1 \in \mathbb{R}$ independent of Φ such that

$$G(\Phi|_\Gamma) \geq c \int_\Gamma |u(x)|^p ds_x + c_1 \int_\Gamma |u(x)| ds_x,$$

where $u = \Phi|_\Gamma$. This implies that $F(\Phi)$ is coercive because

$$\begin{aligned} F(\Phi) &= 1/2 \int_Q |\Delta\Phi|^2 + \int_\Gamma j(\Phi) ds_x - \int_\Gamma f\Phi ds_\Gamma \\ &\geq 1/2 \int_Q |\Delta\Phi|^2 + c(p) \int_\Gamma |\Phi|^p ds_x - \|f\|_{L_p(\Gamma)} \|\Phi\|_{L_p(\Gamma)}. \end{aligned}$$

Hence, there is a unique function $\Phi \in W_2^1(Q)$ that minimizes F.

Next, if $\Phi \in W_2^1(Q)$ is the unique minimizer of F, then by the properties of V there is a unique boundary function $u \in W_2^{-1/2}(\Gamma)$ such that $Vu = \Phi|_\Gamma$. As in [27], we get that $u \in L_q(\Gamma)$ and is a solution of $Tu = f$ for each $f \in L_q(\Gamma)$. Thus, $T : L_q(\Gamma) \to L_q(\Gamma)$ is bijective. Since T is a compact perturbation of the identity and injective, it is an open map by the Shauder invariance of domain theorem. Hence, T is a homeomorphism. Since $b_1(x,.)$ is bounded and C is compact by the compactness of V, the conclusions follow from Corollary 2.8. □

Example 4.2. If $b_0(u) = |u|u^{p-2}$ and $b_1 = 0$, or if $b_0(u) = |u|u^{p-2}$ and $b_1(u) = arctan\ u$ with p even, then part (i) of Theorem 4.1 holds. Note that $b(u)$ is strictly increasing in either case and therefore (4.1) is injective as in the proof of Theorem 4.1. Part (ii) of Theorem 4.1 is valid if e.g., $b_1(u) = a \sin u + b \cos u$.

When the nonlinearities have a linear growth, we have the following result.

Theorem 4.3. *Let $b(x,u) = b_0(x,u) + b_1(x,u)$ be such that $b_0(x,u)$ is a Carathéodory function, $b_0(x,.)$ is strictly increasing on $\mathbb{R}$ and*

1. *There exist constants $a_1 > 0$ and $a_2 \geq 0$ such that*
$$|b_0(x,u)| \leq a_1|u| + a_2$$
2. *b_1 satisfies the Carathéodory conditions and for some positive constants c_1 and c_2 with c_1 sufficiently small*
$$|b_1(x,u)| \leq c_1|u| + c_2 \quad \text{for all } (x,u) \in \Gamma \times \mathbb{R}.$$

Assume that $I - K$ is injective. Then either

- (i) *the BVP (4.1) is locally injective in $L_2(Q)$, in which case it is uniquely solvable in $L_2(Q)$ for each $h \in L_2(Q)$ and the solution depends continuously on h, or*

(ii) *the BVP* (4.1) *is not locally injective, in which case it is solvable in* $L_2(Q)$ *for each* $h \in L_2(Q)$*, the solution set is compact and the cardinality of the solution set is finite and constant on each connected component of* $L_2(Q) \setminus (T+C)(\Sigma)$*, where* $\Sigma = \{u \in L_2(Q) : (4.1)$ *is not locally uniquely solvable*$\}$ *and* $T = I - K + B_0 V$*.*

Proof. The BVP (4.1) is equivalent to the operator equation

$$(I-K)u + B(Vu) = f$$

(cf. [28]). This equation can be written in the form

$$Tu + Cu = f \tag{4.4}$$

where we set $T = I - K + B_0 V$, $Cu = B_1 V$ with $B_i u = b_i(x,u)$, i=0,1.

Since K is compact and $I-K$ is injective, it is a homeomorphism by the Fredholm alternative. As above, T is a compact perturbation of the identity and injective. It satisfies condition (+). Indeed, let $y_n = (I - K + B_0 V)u_n \to y$ in $L_2(\Gamma)$. Then $c\|u_n\| \le \|(I-K)u_n\| = \|y_n - B_0 V u_n\| \le c_1 + a\|V\|\|u_n\| + b$. Since a is sufficiently small, this implies that $\{u_n\}$ is bounded and T satisfies condition (+). Since T is a compact perturbation of the identity, it is proper on bounded closed subsets of $L_2(\Gamma)$. Condition (+) implies that T is proper on $L_2(\Gamma)$. Hence, the range $R(T)$ is closed and therefore T is a homeomorphism by Theorem 2.5(i)(b) applied to $I-K$ and $B_0 V$. By condition 2), it follows as above that that $T + tC$ satisfies condition (+) and the conclusions follow from Theorem 2.5. $\square$

Remark 4.4. Theorem 4.3 is also valid when $b_0 = 0$. The injectivity of $I-K$ has been studied in [10, 32].

Semilinear elliptic equations with nonlinear boundary-value conditions. Consider the nonlinear BVP

$$\Delta u = f(x,u,\nabla u) + g \quad \text{in } Q \subset \mathbb{R}^n, \tag{4.5}$$

$$-\partial_n u = b(x,u(x)) - h \quad \text{on } \Gamma = \partial Q,. \tag{4.6}$$

where $Q \subset \mathbb{R}^n$, (n=2 or 3), is a bounded domain with smooth boundary Γ satisfying a scaling assumption $\operatorname{diam}(Q) < 1$ for $n=2$, ∂_n is the outer normal derivative on Γ.

Let $b = b_0 + b_1$. As in [28], assume that

(1) $b_0(x,u)$ is a Carathéodory function such that $\frac{\partial}{\partial u} b_0(x,u)$ is Borel measurable and satisfies

$$0 < c \le \frac{\partial}{\partial u} b_0(x,u) \le C < \infty \quad \text{for almost all } x \in \Gamma \text{ and all } u \in \mathbb{R}$$

(2) b_1 and f are Carathéodory functions such that

$$|b_1(x,u)| \le a(x) + c(1+|u|), \quad |f(x,u,v)| \le d(x) + c(1+|u|+|v|)$$

for almost all $x \in \Gamma$ and $u, v \in \mathbb{R}$, some functions $a(x) \in L_2(\Gamma)$ and $b(x) \in L_2(Q)$ and $c > 0$ sufficiently small.

Define the Nemytskii maps $B_i : L_2(\Gamma) \to L_2(\Gamma)$ by $B_i u(x) = b_i(x,u(x))$, $i = 0,1$, and $F : H^1(Q) \to L_2(Q)$ by $Fu(x) = f(x,u(x),\nabla u(x))$. Denote by $H^s(Q)$ and $H^s(\Gamma)$ the Sobolev spaces of order s in Q and on Γ, respectively. In particular, $H^{-s}(Q) = (\tilde{H}^s(Q))^*$, where $\tilde{H}^s$ is the completion of $C_0^\infty(Q)$ in $H^s(\mathbb{R}^n)$. We also have that [28] for each $0 \le s \le 1$, $B_0 : H^s(\Gamma) \to H^s(\Gamma)$ is bounded. Denote by $(.,.)$ the L_2 inner product. As in [12], inserting (4.5)-(4.6) into Green's formula

$$\int_Q \Delta u.v\,dx + \int_Q \Delta u.\Delta v\,dx - \int_\Gamma \frac{\partial u}{\partial n} v\,ds_\Gamma = 0$$

we obtain the weak formulation of (4.5)-(4.6): for a given $g \in \tilde{H}^{-1}$, find $u \in H^1(Q)$ such that for all $v \in H^1(Q)$,

$$\begin{aligned}(Au, v)_{H^1(Q)} &= (\nabla u, \nabla v)_Q + (B_0 u|_\Gamma, v|_\Gamma)_\Gamma - (B_1 u|_\Gamma, v|_\Gamma)_\Gamma \\ &\quad - (h, v|_\Gamma)_\Gamma - (Fu, v)_Q - (g, v)_Q = 0.\end{aligned} \tag{4.7}$$

Define $T, C : H^1(Q) \to H^1(Q)$ by

$$(Tu, v)_{H^1(Q)} = (\nabla u, \nabla v)_Q + (B_0 u|_\Gamma, v|_\Gamma)_\Gamma,$$
$$(Cu, v)_{H^1(Q)} = -(B_1 u|_\Gamma, v|_\Gamma)_\Gamma - (h, v|_\Gamma)_\Gamma - (Fu, v)_Q - (g, v)_Q.$$

Theorem 4.5. *Let* (1)-(2) *hold. Then either*

(i) *The problem* (4.5)-(4.6) *is locally injective in* $H^1(Q)$*, in which case it is uniquely solvable in* $H^1(Q)$ *for each* $g \in \tilde{H}^{-1}(Q)$ *and* $h \in H^{-1/2}(\Gamma)$ *and the solution depends continuously on* (g, h)*, or*

(ii) *the problem* (4.5)-(4.6) *is not locally injective in* $H^1(Q)$*, in which case it is solvable in* $H^1(Q)$ *for each* $(g, h) \in \tilde{H}^{-1}(Q) \times H^{-1/2}(\Gamma)$*, the solution set is compact and the cardinality of the solution set is finite and constant on each connected component of* $H^1(Q) \setminus (T+C)(\Sigma)$*, where* $\Sigma = \{u \in H^1(Q) :$ (4.5)-(4.6) *is not locally uniquely solvable*$\}$.

Proof. By (1), B_0 satisfies the Lipschitz condition and is l-strongly monotone in $L_2(\Gamma)$. This implies that T also satisfies the Lipschitz condition and that

$$(Tu - Tv, u - v)_{H^1(Q)} \ge \|\nabla(u - v)\|^2_{L_2(Q)} + l\|u - v\|_{L_2(\Gamma)} \ge k\|u - v\|^2_{H^1(Q)}.$$

Hence, T is k-strongly monotone and is therefore a homeomorphism in $H^1(Q)$ with $\|Tu - Tv\| \ge k\|u - v\|$. Moreover, $C : H^1(Q) \to H^1(Q)$ is compact since $F : H^1(Q) \to L_2(Q)$ is continuous and the embedding of $L_2(Q)$ into $\tilde{H}^{-1}(Q)$ is compact, and $B_1 : L_2(\Gamma) \to L_2(\Gamma)$ is continuous and the embeddings $H^{1/2}(\Gamma) \to L_2(\Gamma) \to H^{-1/2}(\Gamma)$ are also compact. Since c is sufficiently small, $\|Tu_n + Cu_n\| \to \infty$ as $\|u_n\| \to \infty$. Hence, being proper on bounded closed subsets, $T + C$ is proper. Moreover, $T + tC$ satisfies condition (+) and the conclusions of the theorem follow from Theorem 2.5. □

Remark 4.6. The solvability of problem (4.5)-(4.6) with sublinear nonlinearities was proved by Efendiev, Schmitz and Wendland [12] using a degree theory for compact perturbations of strongly monotone maps.

Semilinear elliptic equations with strong nonlinearities. Consider the problem

$$\begin{gathered}\Delta u + \lambda_1 u - f(u) + g(u) = h, \quad (h \in L_2) \\ u\big|_{\partial Q} = 0,\end{gathered} \tag{4.8}$$

where Q is a bounded domain in $\mathbb{R}^n$ and λ_1 is the smallest positive eigenvalue of Δ on Q.

Assume that f and $g = g_1 + g_2$ are Carathéodory functions such that

(1) For $p \geq 2$ if $n = 2$ and $p \in [2, 2n/(n-2))$ if $n \geq 3$, there exist constants $a_1 > 0$, $a_2 \geq 0$, $c_1 > 0$ and $c_2 \geq 0$ such that
$$|f(u)| \leq a_1|u|^p + a_2, \quad f(u)u \geq c_1|u|^{p+1} + c_2.$$
(2) f is differentiable
(3) $|g_1(u)| \leq b_1|u|^p + b_2$ with $b_1 \leq a_1$ and $g_1(u)u \geq c_1|u|^{p+1} + c_2$.
(4) $|g_2(u)| \leq c_1|u|^p + c_2$ for all u with c_1 sufficiently small.

Define $X = \{u \in W_2^1(Q) : u = 0 \text{ on } \partial Q\}$. Note that X is compactly embedded into $L_p(Q)$ for each p as in (1) by the Sobolev embedding theorem. We shall look at weak solutions of (4.8); i.e., $u \in X$ such that $Tu + Cu = h$, where
$$(Tu, v)_{1,2} = (\nabla u, \nabla v) - \lambda_1(u, v) + (f(u), v), \quad (C_i u, v) = (g_i(u), v), \quad i = 1, 2$$
$C = C_1 + C_2$ and $(.,.)$ is the L_2 inner product. In X, the derivative of T is $T'(u)v = \Delta v - \lambda_1 v - f'(u)v$. Since $T'(u)$ is a selfadjoint elliptic map in X, T is Fredholm of index zero in X.

Next, we shall show that $\|Tu + C_1u\| \to \infty$ if $\|u\| \to \infty$ in X. Suppose that $\|u_n\|_{2,1} = \|\nabla u_n\|_2^2 + \|u_n\|_2^2 \to \infty$. Thus
$$\begin{aligned}((T + C_1)u_n, u_n) &= \|\nabla u_n\|_2^2 - \lambda_1\|u_n\|_2^2 + (f(u_n) + g_1(u_n), u_n) \\ &\geq \|\nabla u_n\|_2^2 - \lambda_1\|u_n\|_2^2 + c\|u_n\|_{p+1}^{p+1} - c'\|u\|_2.\end{aligned}$$
If $\|\nabla u_n\|_2^2 \to \infty$ and $\|u_n\|_2^2 \leq k$, then
$$((T + C_1)u_n, u_n) \geq \|\nabla u_n\|_2^2 - \lambda_1\|u_n\|_2^2 + c\|u_n\|_2^{p+1} - c'\|u_n\|_2 \geq \|\nabla u_n\|_2^2 - k_1$$
since $\|\nabla u\|_2^2 \geq c_1\|u\|_2^2$. If also $\|u_n\|_2^2 \to \infty$, then
$$((T + C_1)u_n, u_n) \geq \|\nabla u_n\|_2^2 + (k_2\|u_n\|_2^{p-1} - \lambda_1 - c'/\|u_n\|_2)\|u_n\|_2^2$$
since $L_{p+1} \subset L_2$. Thus, in either case $((T + C_1)u_n, u_n)/\|u_n\|_{2,1}^2 \geq k_3 > 0$ as $\|u_n\|_{2,1}^2 \to \infty$. Hence, $\|(T+C_1)u_n\| \to \infty$ as $\|u_n\|_{2,1}^2 \to \infty$ by the Cauchy-Schwartz inequality. In a similar way, we can show that $\|Tu_n\| \to \infty$ as $\|u\| \to \infty$ in X.

To show that T is proper, let $K \subset X$ be compact and note that $T^{-1}(K)$ is bounded by the coercivity of T. Let $\{u_n\} \subset T^{-1}(K)$. We may assume that it converges weakly to u and $Tu_n \to v$ in X. Since X is compactly embedded in L_p for each p as in (1), we have that $u_n \to u$ in L_2 and L_p and $(Tu_n, u_n) \to (v, u)$. Since
$$(Tu_n, u_n) = \|\nabla u_n\|_2^2 - \lambda_1\|u_n\|_2^2 + (T_1u_n, u_n)$$
and $T_1 u = f(u)$ is compact in X, we get that $\|\nabla u_n\|_2$ converges. Hence, we have that $\{u_n\}$ converges weakly in X and $\|u_n\|_{1,2}$ converges. Since X is a Hilbert space, $\{u_n\}$ converges in X. Hence, T is proper.

Theorem 4.7. *Let* (1)-(4) *hold. Then either*

(i) *BVP* (4.8) *is locally injective in X, in which case it is uniquely solvable in X for each $h \in L_2(Q)$ and the solution depends continuously on h, or*
(ii) *BVP* (4.8) *is not locally injective, in which case it is solvable in X for each $h \in L_2(Q)$ and its solution set is compact. Moreover, the cardinality of the solution set is finite and constant on each connected component of $X \setminus (T + C)(\Sigma)$, where $\Sigma = \{u \in X : (4.8)$ is not locally uniquely solvable$\}$.*

Proof. Since c_1 is sufficiently small, $\|C_2u\| \leq a\|Tu + C_1u\| + b$ for all u and $a < 1$. This and $\|Tu + C_1u\| \to \infty$ as $\|u\| \to \infty$ imply that $\|Tu + C_1u + tC_2u\| \to \infty$ as $\|u\| \to \infty$. Since T is proper and C is compact, it follows that $T + C$ is proper. Indeed, let $K \subset Y$ be compact and $x_n \in (T+C)^{-1}(K)$. Then $y_n = (T+C)x_n \to y$ since $y_n \in K$ and $\{x_n\}$ is bounded by the preceding remark. Since $\{y_n - Cx_n\}$ is compact, T is proper and $Tx_n = y_n - Cx_n$ is compact we get that a subsequence $x_{n_k} \to x$. By the continuity of $T + C$, $Tx + Cx = y$ in Y. Hence, $T + C$ is proper and the conclusions of the theorem follow from Theorem 2.9. □

Theorem 4.8. *Let* (1)-(2) *hold with* $f'(0) = 0$, $f'(u) > 0$ *and* $|g(u)| \leq M$ *for some* $M > 0$ *and all* u. *Then either*

(i) *BVP* (4.8) *is locally injective in* X, *in which case it is uniquely solvable in* X *for each* $h \in L_2(Q)$ *and the solution depends continuously on* h, *or*
(ii) *BVP* (4.8) *is not locally injective, in which case it is solvable in* X *for each* $h \in L_2(Q)$ *and the solution set is compact. Moreover, the cardinality of the solution set is finite and constant on each connected component of* $X \setminus (T+C)(\Sigma)$, *where* $\Sigma = \{u \in X : (4.8)$ *is not locally uniquely solvable*$\}$.

Proof. T is a proper Fredholm map of index zero by the above remarks. Next, we shall show that the singular set of T consists only of 0. Suppose that $T'(u)v = 0$. Using the variational characterization of λ_1, we get that $u = 0$ since $0 = (T'(u)v, v) = \int_Q |\nabla v|^2 - \lambda_1 v^2 + \int_Q f'(u)v^2 > 0$. Hence, since $f'(0) = 0$, $T'(0)v = 0$, i.e., $\Delta v + \lambda_1 v = 0$ and $v = 0$ on the boundary of Q. Thus, this problem has nontrivial solutions and therefore the singular set of T consists only of 0. Since T is Fredholm of index zero and 0 is an isolated singular point, T is a local homeomorphism and therefore a homeomorphism by its properness and the Banach-Mazur theorem. Set $Cu = g(u)$. Since $\|Cu\| \leq M_1 < \infty$ for all u, the conclusions of the theorem follow from Corollary 2.11, since $T + C$ is proper by the compactness of C as shown in Theorem 4.7. □

Theorem 4.9. *Let* (1)-(4) *hold and* f *and* g_1 *be odd. Then* (4.8) *is solvable in* X *for each* $h \in L_2(Q)$ *and its solution set is compact. Moreover, the cardinality of the solution set is finite and constant on each connected component of* $X \setminus (T+C)(\Sigma)$, *where* $\Sigma = \{u \in X : (4.8)$ *is not locally uniquely solvable*$\}$.

Proof. Note that $T + C_1$ is odd. As in the proof of Theorem 4.7, we have that $\|Tu + C_1u + tC_2u\| \to \infty$ as $\|u\| \to \infty$. Hence, the conclusions of the theorem follow from Theorem 3.4. □

5. Solvability of quasilinear elliptic BVP's with asymptotically positive homogeneous nonlinearities

Let $Q \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary and consider the boundary value problem in a divergent form

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, D^m u) + k \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha B_\alpha(x, u, \dots, D^m u) = f \tag{5.1}$$

Let X be the closed subspace of $W_p^m(Q)$ corresponding to the Dirichlet conditions. Define the maps $T, D : X \to X^*$ by

$$(Tu, v) = \sum_{|\alpha| \le m} \int_Q A_\alpha(x, u, \dots, D^m u) D^\alpha v dx,$$
$$(Du, v) = \sum_{|\alpha| \le m} \int_Q B_\alpha(x, u, \dots, D^m u) D^\alpha v dx.$$

Then weak solutions of (5.1) are solutions of the operator equation

$$Tu + Du = f,\ u \in X. \tag{5.2}$$

We impose the following conditions on A_α.

- (A1) For each α, let $A_\alpha : Q \times \mathbb{R}^{s_m} \to \mathbb{R}$ be such that $A_\alpha(x, \xi)$ is measurable in x and for each fixed ξ, and has continuous derivatives in ξ for a.e. x;
- (A2) Assume that for $p > 2$, $x \in Q$, $\xi \in \mathbb{R}^{s_m}$, $\eta \in \mathbb{R}^{s_m - s_{m-1}}$, $|\alpha|, |\beta| \le m$ the A_α's satisfy
$$|A_\alpha(x, \xi)| \le g_0(|\xi_0|)(h(x) + \sum_{m-n/p \le \gamma| \le m} |\xi_\gamma|^{p-1});$$
- (A3) $|A_{\alpha,\beta}(x, \xi)| \le g_1(|\xi_0|)(b(x) + \sum_{m-n/p \le \gamma| \le m} |\xi_\gamma|^{p-2})$,
- (A4) $\sum_{|\alpha|=|\beta|=m} A_{\alpha,\beta}(x, \xi)\eta_\alpha\eta_\beta \ \ge \ g_2(1 + \sum_{|\gamma|=m} |\xi_\gamma|)^{p-2} \ \ \sum_{|\alpha|=m} \eta_\alpha^2$, where $A_{\alpha,\beta}(x, \xi) = \partial/\partial\xi_\beta A_\beta(x, \xi)$, $h, b \in L_q(Q)$, g_0, g_1 are continuous positive nondecreasing functions and $g_2 > 0$ is a constant.

For $|\alpha| \le m$, there are Carathéodory functions a_α such that

- (a1) $a_\alpha(x, t\xi) = t^{p-1} a_\alpha(x, \xi)$ for all $t > 0$, $\xi \in \mathbb{R}^{s_m}$
- (a2) $|1/t^{p-1} A_\alpha(x, t\xi) - a_\alpha(x, \xi)| \le c(t)(1 + |\xi|)^{p-1}$

for each $t > 0$, $x \in Q$, and $\xi \in \mathbb{R}^{s_m}$, where $0 < \lim c(t)$ is sufficiently small as $t \to \infty$.

Proposition 5.1. *Assume* (A1)-(A4). *Then $T : X \to X^*$ is Fredholm of index zero and is proper on bounded closed subsets of X.*

Proof. The map $T : X \to X^*$ is continuous and of type (S_+) [30] and, as shown before, it is proper on bounded closed subsets of X. By [30, Lemma 3.1], the Fréchet derivative $T'(u)$ of T at $u \in X$ is given by

$$(T'(u)v, w) = \sum_{|\alpha|,|\beta| \le m} \int_Q A_{\alpha,\beta}(x, u, \dots, D^m u) D^\beta v D^\alpha w dx \tag{5.3}$$

Next, we shall show that $T'(u)$ is Fredholm of index zero for each $u \in X$. First, we shall show that $T'(u)$ satisfies condition (S_+). We can write it in the form $T'(u) = T_1'(u) + T_2'(u)$, where

$$(T_1'(u)v, w) = \sum_{|\alpha|=|\beta|=m} \int_Q A_{\alpha,\beta}(x, u, \dots, D^m u) D^\beta v D^\alpha w dx, \tag{5.4}$$
$$(T_2'(u)v, w) = \sum_{|\alpha|,|\beta| \le m,\, |\alpha+\beta| < 2m} \int_Q A_{\alpha,\beta}(x, u, \dots, D^m u) D^\beta v D^\alpha w dx. \tag{5.5}$$

It is easy to see that $|(T_2'(u)v, w)| \leq c\|v\|_{m,p}\ \|w\|_{m-1,p}$ for some constant $c > 0$. Hence, $T_2'(u) : X \to X^*$ is compact since the embedding of W_p^m into W_q^k is compact for $0 \leq k \leq m-1$ if $1/q > 1/p - (m-k)/n > 0$, or if $q < \infty$ and $1/p = (m-k)/n$.

Next, we shall show that $T_1'(u) : X \to X^*$ is of type (S_+). Let $v_n \in X$ be such that $v_n \rightharpoonup v$ in X and

$$\limsup_{n\to\infty} (T_1'(u)v_n, v_n - v) \leq 0.$$

It follows that $D^\alpha v_n \to D^\alpha v$ in L_p for each $|\alpha| < m$ by the Sobolev embedding theorem. Next, we shall show that $D^\alpha v_n \to D^\alpha v$ in L_p for each $|\alpha| = m$. Since, X is separable, there are finite dimensional subspaces $\{X_n\}$ in X whose union is dense in it. Since $\operatorname{dist}(v, X_n) \to 0$ for each $v \in X$, there is a $w_n \in X_n$ such that $w_n \to v$ in X as $n \to \infty$. Then

$$\begin{aligned}
&\limsup_n (T_1'(u)v_n - T_1'(u)w_n, v_n - w_n)\\
&\leq \limsup_n (T_1'(u)v_n, v_n - v - (w_n - v)) - \liminf_n (T_1'(u)w_n, v_n - w_n)\\
&\leq \limsup_n (T_1'(u)v_n, v_n - v) - \lim_n (T_1'(u)v_n, w_n - v) - \liminf_n (T_1'(u)w_n, v_n - w_n)\\
&\leq 0.
\end{aligned} \tag{5.6}$$

However,

$$\begin{aligned}
&(T_1'(u)(v_n - w_n), v_n - w_n)\\
&= \sum_{|\alpha|=|\beta|=m} \int_Q A_{\alpha\beta}(x, u, \dots D^m u) D^\alpha(v_n - w_n) D^\beta(v_n - w_n) dx\\
&\geq g_2 \int_Q \sum_{|\alpha|=m} D^\alpha(v_n - w_n)^2 dx\\
&= g_2 \sum_{|\alpha|=m} \|D^\alpha(v_n - w_n)\|^2
\end{aligned} \tag{5.7}$$

This and (5.6) imply that $D^\alpha(v_n - w_n) \to 0$ in L_p for each $|\alpha| = m$. Hence, $D^\alpha v_n \to D^\alpha v$ in L_p for each $|\alpha| = m$, and therefore $v_n \to v$ in X. This shows that $T_1'(u)$ is continuous and of type (S_+) as is $T'(u) = T_1'(u) + T_2'(u)$. Hence, as shown before, $T'(u)$ is proper on bounded closed subsets of X. By Yood's criterion, the index of $T'(u) = \dim N(T'(u)) - \operatorname{codim} R(T'(u) \geq 0$. Moreover, $T'(u)^* = T_1'(u)^* + T_2'(u)^*$ with $T_2'(u)^*$ compact. Hence, using (A4), as above, we get that $T_1'(u)^*$ is continuous and of type (S_+). Thus, the index $i(T_1'(u)^*) \geq 0$ and $i(T_1'(u)^*) = -i(T_1'(u)) \leq 0$. It follows that $i(T_1'(u)) = 0$. It is left to show that $T'(u)$ is a continuous map in u. Let $u_n \to u$. Then

$$\begin{aligned}
&(T'(u_n)v - T'(u)v, w)\\
&= \sum_{|\alpha|,|\beta|\leq m} \int_Q [A_{\alpha\beta}(x, u_n, \dots, D^m u_n) - A_{\alpha\beta}(x, u, \dots, D^m u) D^\beta v D^\alpha w\, dx.
\end{aligned}$$

The Nemytskii map $Nu = A_{\alpha\beta}(x, u, \dots, D^m u)$ is continuous from X to $L_{p'}(Q)$, $1/p + 1/p' = 1$. Hence, $T'(u_n) \to T'(u)$ using also the Sobolev embedding theorem. This completes the proof that T is a Fredholm map of index zero and is proper on bounded closed subsets of X. □

Remark 5.2. *We can put T_2 together with D and require the differentiability (Fredholmness) of only T_1.*

We assume that the $B'_\alpha s$ satisfy

(B1) For each $|\alpha| \leq m$, $B_\alpha(x,\xi)$ is a Caratheodory function and, for $p > 2$ there exist a constant $c > 0$ and $h_\alpha(x) \in L_q(Q)$, $1/p + 1/q = 1$, such that
$$|B_\alpha(x,\xi)| \leq c(h_\alpha(x) + |\xi|^{p-1})$$

(B2) There is a sufficiently small $k_1 > 0$ such that
$$\sum_{|\alpha|=m} |B_\alpha(x,\eta,\xi_\alpha) - B_\alpha(x,\eta,\xi'_\alpha)| \leq k_1 \sum_{|\alpha|=m} |\xi_\alpha - \xi_{\alpha'}|$$
for each a.e. $x \in Q$, $\eta \in \mathbb{R}^?$ and $\xi_\alpha, \xi_{\alpha'} \in \mathbb{R}^?$.

Note that if B_α's are differentiable for $|\alpha| = m$ and $B_{\alpha\alpha}(x,\xi) = \partial/\partial\xi_\alpha B_\alpha(x,\xi)$ are sufficiently small, then (B2) holds.

In view of Proposition 5.1, the results in the form of Theorems 4.1–4.5 are valid for (5.1) as well as the corresponding ones involving maps that are asymptotically close to positively k-homogeneous maps. A sample of such a theorem is given next.

Consider also the equation
$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha a_\alpha(x,u,\dots,D^m) = f \tag{5.8}$$
in X. Define the map $A : X \to X^*$ by
$$(Au,v) = \sum_{|\alpha|\leq m} \int_Q a_\alpha(x,u,\dots,D^m u) D^\alpha v dx. \tag{5.9}$$
Then weak solutions of (5.8) are solutions of the operator equation
$$Au = f, \quad u \in X. \tag{5.10}$$

Theorem 5.3. *Assume that* (A1)–(A4), (a1)–(a2), (B1)–(B2) *hold and that for all large r,* $\deg_{BCF}(T, B(0,r), 0) \neq 0$. *Suppose that A is of type (S_+) and $Au = 0$ has no a nontrivial solution. Then* (5.1) *is solvable for each $f \in X^*$, has a compact set of solutions whose cardinal number is constant, finite and positive on each connected component of the set $X^* \setminus (T + D)(\Sigma)$, where $\Sigma = \{u \in X : T + D$ is not locally invertible at $u\}$.*

Proof. In view of our discussion above, T is a Fredholm map of index zero that is proper on bounded closed subsets of X. It is proper on X since it satisfies condition (+), and therefore $\beta(T) > 0$. We need to show that $\alpha(D) < \beta(T)$. We note that the boundedness and continuity of D follow from (A1)–(A2), the Sobolev embedding theorem and the continuity of the Nemytskii maps in L_p spaces. We can write $D = D_1 + D_2$, where
$$(D_1 u, v) = \sum_{|\alpha|=m} \int_Q B_\alpha(x,u,\dots,D^m u) D^\alpha v dx,$$
$$(D_2 u, v) = \sum_{|\alpha|<m} \int_Q B_\alpha(x,u,\dots,D^m u) D^\alpha v dx,$$
The map $D_1 : X \to X^*$ is k_1-set contractive by (B2), while $D_2 : X \to X^*$ is compact by the Sobolev embedding theorem. Hence, $\alpha(D) = \alpha(D_1) \leq k_1 < \beta(T)$

since k_1 is sufficiently small. Finally, condition (a2) implies that T is asymptotically close to the $(p-1)-$ positive homogeneous map A given by (5.9). Hence, Theorem 3.5 applies with $C=0$. $\square$

Example 5.4. Let $s>0$, k be sufficiently small, and look at

$$-\Delta u-\mu u\frac{|u|^s}{1+|u|^s}+kF(x,u,\nabla u)=f \tag{5.11}$$

and

$$-\Delta u-\mu u=0 \tag{5.12}$$

Let $A_0(x,\xi_0,\xi_1,\dots,\xi_n)=\xi_0\frac{|\xi_0|^s}{1+|\xi_0|^s}$ and $A_i(x,\xi_0,\xi_1,\dots,\xi_n)=a_i(x,\xi_0,\xi_1,\dots,\xi_n)=\xi_i$ and $a_0(x,\xi_0,\xi_1,\dots,\xi_n)=\xi_0$. Then A_0,A_i,a_0 and a_i satisfy (a1)-(a2). Let F satisfy (B1). Then (5.11) has a solution $u\in W_2^1(Q)$, $u=0$ on ∂Q, for each $f\in L_2(Q)$ if μ is not an eigenvalue of (5.12).

Example 5.5. Let $p>2$, k be sufficiently small, and look at

$$-\sum_{i=1}^n\frac{\partial}{\partial x_i}[(1+\sum_{j=1}^n|\frac{\partial u}{\partial x_j}|^2)^{p/2-1}\frac{\partial u}{\partial x_i}]+\mu(1+|u|^2)^{p/2-1}u+kDu=f \tag{5.13}$$

and

$$-\sum_{i=1}^n\frac{\partial}{\partial x_i}(\sum_{j=1}^n|\frac{\partial u}{\partial x_j}|^2)^{p/2-1}\frac{\partial u}{\partial x_i})+\mu(|u|^2)^{p/2-1}u=0. \tag{5.14}$$

where $D:W_p^1\to W_p^1$ is k-set contractive; e.g., $Du=F(x,u,\nabla u)$ in which case it is compact, or $Du=\sum_{i=1}^n\partial/\partial x_i c_i(x,u,\nabla u)$ with the c_i k_i-contractive in ∇u with k_i small. Let $A_0(x,\xi_0,\xi_1,\dots\xi_n)=(1+\xi_0^2)^{p/2-1}\xi_0$, $A_i(x,\xi_0,\xi_1,\dots,\xi_n)=(1+\sum_{j=1}^n\xi_j^2)^{p/2-1}\xi_i$, $a_0(x,\xi_0,\xi_1,\dots\xi_n)=(\xi_0^2)^{p/2-1}\xi_0$ and $a_i(x,\xi_0,\xi_1,\dots,\xi_n)=\sum_{j=1}^n(\xi_j^2)^{p/2-1}\xi_i$. Then the matrix $(A_{ij}(x,\xi_0,\xi_1,\dots\xi_n))$ is symmetric. Let $n=2$ for simplicity. Then the eigenvalues of the matrix are $\lambda_1=(p/2-1)(1+\xi_1^2+\xi_2^2)^{p/2-1}$ and $\lambda_2=\lambda_1+(p-2)(1+\xi_1^2+\xi_2^2)^{p/2-2}(\xi_1^2+\xi_2^2)$. Hence, A_0,A_i,a_0 and a_i satisfy conditions (A1)–(A4) and (a1)–(a2), respectively. Then (5.13) has a compact set of solution $u\in W_p^1(Q)$, $u=0$ on ∂Q, for each $f\in X^*$, if μ is not an eigenvalue of (5.14) and $n=2$. The solution set is finite for all f as in Theorem 5.3 since the corresponding map T is odd.

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Petronije S. Milojević
Department of Mathematical Sciences and CAMS, New Jersey Institute of Technology, Newark, NJ 07102, USA
E-mail address: milojevi@adm.njit.edu