Electronic Journal of Differential Equations, Vol. 2009(2009), No. 114, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LOW REGULARITY SOLUTIONS OF THE CHERN-SIMONS-HIGGS EQUATIONS IN THE LORENTZ GAUGE

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Abstract. We prove local well-posedness for the 2 + 1-dimensional Chern-Simons-Higgs equations in the Lorentz gauge with initial data of low regularity. Our result improves earlier results by Huh [10, 11].

1. Introduction

The Chern-Simon-Higgs model was proposed by Jackiw and Weinberg [12] and Hong, Pac and Kim [9] in the context of their studies of vortex solutions in the abelian Chern-Simons theory.

Local well-posedness of low regularity solutions was recently studied in Huh [10, 11] using a null-form estimate for solutions of the linear wave equation due to Foschi and Klainerman [8] as well as Strichartz estimates. Our aim in this paper is to improve the results of [10, 11] in the Lorentz gauge. For this purpose we use estimates in the restriction spaces $X^{s,b}$ introduced by Bourgain, Klainerman and Machedon. A key ingredient in our proof is a modified version of a null-form estimate of Zhou [19] and product rules in $X^{s,b}$ spaces due to D'Ancona, Foschi and Selberg [6, 7] and Klainerman and Selberg [13]. The Higgs field has fractional dimension (see below for details), a common feature of systems involving the Dirac equation, see for example Bournaveas [1, 2], D'Ancona, Foschi and Selberg [6, 7], Machihara [14, 15], Machihara, Nakamura, Nakanishi and Ozawa [16], Selberg and Tesfahun [17], Tesfahun [18].

The Chern-Simon-Higgs equations are the Euler-Lagrange equations corresponding to the Lagrangian density

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + D_\mu\phi\,\overline{D^\mu\phi} - V\big(|\phi|^2\big).$$

Here A_μ is the gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature, $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative, ϕ is the Higgs field, V is a given positive function and κ is a positive coupling constant. Greek indices run through $\{0, 1, 2\}$, Latin indices run through $\{1, 2\}$ and repeated indices are summed. The Minkowski metric is defined

2000 *Mathematics Subject Classification.* 35L15, 35L70, 35Q40.
Key words and phrases. Chern-Simons-Higgs equations; Lorentz gauge; null-form estimates; low regularity solutions.

Submitted February 22, 2009. Published September 12, 2009.

by $(g^{\mu\nu}) = diag(1,-1,-1)$. We define $\epsilon^{\mu\nu\rho} = 0$ if two of the indices coincide and $\epsilon^{\mu\nu\rho} = \pm 1$ according to whether (μ,ν,ρ) is an even or odd permutation of $(0,1,2)$.

We define Klainerman's null forms by

$$Q_{\mu\nu}(u,v) = \partial_\mu u \partial_\nu v - \partial_\nu u \partial_\mu v, \tag{1.1a}$$

$$Q_0(u,v) = g^{\mu\nu}\partial_\mu u \partial_\nu v. \tag{1.1b}$$

Let $I^\mu = 2Im\big(\overline{\phi} D^\mu \phi\big)$. Then the Euler-Lagrange equations are (we set $\kappa = 2$ for simplicity)

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha} I^\alpha, \tag{1.2a}$$

$$D_\mu D^\mu \phi = -\phi V'\big(|\phi|^2\big). \tag{1.2b}$$

The system has the positive conserved energy given by

$$\mathcal{E} = \int_{\mathbb{R}^2} \sum_{\mu=0}^{2} |D_\mu \phi|^2 + V(|\phi|^2)\, dx.$$

We are interested in the so-called 'non-topological' case in which $|\phi| \to 0$ as $|x| \to +\infty$. For the sake of simplicity we follow [10, 11] and set $V = 0$. It will be clear from our proof that for various classes of V's the term $\phi V'(|\phi|^2)$ can easily be handled.

Under the Lorentz gauge condition $\partial^\mu A_\mu = 0$ the Euler-Lagrange equations (1.2) become

$$\partial_0 A_j = \partial_j A_0 + \tfrac{1}{2}\epsilon_{ij} I_i, \tag{1.3a}$$

$$\partial_1 A_2 = \partial_2 A_1 + \tfrac{1}{2} I_0, \tag{1.3b}$$

$$\partial_0 A_0 = \partial_1 A_1 + \partial_2 A_2, \tag{1.3c}$$

$$D_\mu\, D^\mu\, \phi = 0. \tag{1.3d}$$

Alternatively, they can be written as a system of two nonlinear wave equations:

$$\Box A^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\,\mathrm{Im}(\overline{D_\gamma\phi} D_\beta \phi - \overline{D_\beta \phi} D_\gamma \phi) + \frac{1}{2}\epsilon^{\alpha\beta\gamma}(\partial_\beta A_\gamma - \partial_\gamma A_\beta)|\phi|^2, \tag{1.4a}$$

$$\Box \phi = 2iA^\alpha \partial_\alpha \phi + A^\alpha A_\alpha \phi. \tag{1.4b}$$

We prescribe initial data in the classical Sobolev spaces $A^\mu(0,x) = a_0^\mu(x) \in H^a$, $\partial_t A^\mu(0,x) = a_1^\mu(x) \in H^{a-1}$, $\phi(0,x) = \phi_0(x) \in H^b$, $\partial_t \phi(0,x) = \phi_1(x) \in H^{b-1}$. Dimensional analysis shows that the critical values of a and b are $a_{cr} = 0$ and $b_{cr} = \frac{1}{2}$. It is well known that in low space dimensions the Cauchy problem may not be locally well posed for a and b close to the critical values due to lack of decay at infinity. Observe also that ϕ has fractional dimension.

From the point of view of scaling it is natural to take $b = a + \frac{1}{2}$. With this choice it was shown in Huh [10] that the Cauchy problem is locally well posed for $a = \frac{3}{4} + \epsilon$ and $b = \frac{5}{4} + \epsilon$. This was improved in Huh [11] to

$$a = \frac{3}{4} + \epsilon\,, \quad b = \frac{9}{8} + \epsilon \tag{1.5}$$

(slightly violating $b = a + \frac{1}{2}$). The proof relies on the null structure of the right hand side of (1.4a). Indeed,

$$\overline{D_\gamma \phi} D_\beta \phi - \overline{D_\beta \phi} D_\gamma \phi = Q_{\gamma\beta}(\overline{\phi}, \phi) + i\left(A_\gamma \partial_\beta(|\phi|^2) - A_\beta \partial_\gamma(|\phi|^2)\right).$$

On the other hand, since in (1.3) the A_μ satisfy first order equations and ϕ satisfies a second order equation it is natural to investigate the case $b = a + 1$. It turns out that this choice allows us to improve on a at the expense of b. It is shown in Huh [11] that we have local well posedness for

$$a = \frac{1}{2}, \quad b = \frac{3}{2}. \tag{1.6}$$

To prove this result Huh uncovered the null structure in the right hand side of equation (1.4b). Indeed, if we introduce B_μ by $\partial_\mu B^\mu = 0$ and $\partial_\mu B_\nu - \partial_\nu B_\mu = \epsilon_{\mu\nu\lambda} A^\lambda$, then the equations take the form:

$$\Box B^\gamma = -\operatorname{Im}\left(\bar{\phi} D^\gamma \phi\right) = -\operatorname{Im}\left(\bar{\phi}\partial^\gamma \phi\right) + i\epsilon^{\mu\nu\gamma}\partial_\mu B_\nu |\phi|^2, \tag{1.7a}$$

$$\Box \phi = i\epsilon^{\alpha\mu\nu} Q_{\mu\alpha}(B_\nu, \phi) + Q_0(B_\mu, B^\mu)\phi + Q_{\mu\nu}(B^\mu, B^\nu)\phi \, . \tag{1.7b}$$

In this article we shall prove the Theorem stated below which corresponds to exponents $a = \frac{1}{4} + \epsilon$ and $b = \frac{5}{4} + \epsilon$. This improves (1.6) by $\frac{1}{4} - \epsilon$ derivatives in both a and b. Compared to (1.5), it improves a by $\frac{1}{2}$ derivatives at the expense of $\frac{1}{8}$ derivatives in b.

Theorem 1.1. *Let $n = 2$ and $\frac{1}{4} < s < \frac{1}{2}$. Consider the Cauchy problem for the system* (1.7) *with initial data in the following Sobolev spaces:*

$$B^\gamma(0) = b_0^\gamma \in H^{s+1}(\mathbb{R}^2), \quad \partial_t B^\gamma(0) = b_1^\gamma \in H^s(\mathbb{R}^2), \tag{1.8a}$$

$$\phi(0) = \phi_0 \in H^{s+1}(\mathbb{R}^2), \quad \partial_t \phi(0) = \phi_1 \in H^s(\mathbb{R}^2). \tag{1.8b}$$

Then there exists a $T > 0$ and a solution (B, ϕ) of (1.7)-(1.8) *in $[0, T] \times \mathbb{R}^2$ with*

$$B, \phi \in C^0([0, T]; H^{s+1}(\mathbb{R}^2)) \cap C^1([0, T]; H^s(\mathbb{R}^2)).$$

The solution is unique in a subspace of $C^0([0, T]; H^{s+1}(\mathbb{R}^2)) \cap C^1([0, T]; H^s(\mathbb{R}^2))$, namely in $\mathcal{H}^{s+1,\theta}$, where $\frac{3}{4} < \theta < s + \frac{1}{2}$ (the definition of $\mathcal{H}^{s+1,\theta}$ is given in the next section).

Finally, we remark that the problem of global existence is much more difficult. We refer the reader to Chae and Chae [4], Chae and Choe [5] and Huh [10, 11].

2. Bilinear Estimates

In this Section we collect the bilinear estimates we need for the proof of Theorem 1.1. We shall work with the spaces $H^{s,\theta}$ and $\mathcal{H}^{s,\theta}$ defined by

$$H^{s,\theta} = \{u \in \mathcal{S}' : \Lambda^s \Lambda_-^\theta u \in L^2(\mathbb{R}^{2+1})\},$$

$$\mathcal{H}^{s,\theta} = \{u \in H^{s,\theta} : \partial_t u \in H^{s-1,\theta}\}$$

where Λ and Λ_- are defined by

$$\widetilde{\Lambda^s u}(\tau, \xi) = (1 + |\xi|^2)^{s/2} \widetilde{u}(\tau, \xi),$$

$$\widetilde{\Lambda_-^\theta u}(\tau, \xi) = \Big(1 + \frac{(\tau^2 - |\xi|^2)^2}{1 + \tau^2 + |\xi|^2}\Big)^{\theta/2} \widetilde{u}(\tau, \xi).$$

Notice that the weight $\big(1 + \frac{(\tau^2 - |\xi|^2)^2}{1+\tau^2+|\xi|^2}\big)^{\theta/2}$ is equivalent to the weight $w_-(\tau, \xi)^\theta$, where we define

$$w_\pm(\tau, \xi) = 1 + ||\tau| \pm |\xi||.$$

We define the norms

$$\|u\|_{H^{s,\theta}} = \|\langle \xi \rangle^s w_-(\tau, \xi)^\theta \widetilde{u}(\tau, \xi)\|_{L^2(\mathbb{R}^{2+1})},$$

$$\|u\|_{\mathcal{H}^{s,\theta}} = \|u\|_{H^{s,\theta}} + \|\partial_t u\|_{H^{s,\theta}}.$$

The last norm is equivalent to

$$\|\langle\xi\rangle^{s-1}w_+(\tau,\xi)w_-(\tau,\xi)^\theta\widetilde{u}(\tau,\xi)\|_{L^2(\mathbb{R}^{2+1})}.$$

We can now state the null form estimate we are going to use in the proof of Theorem 1.1.

Proposition 2.1. *Let $n=2$, $\frac14<s<\frac12$, $\frac34<\theta<s+\frac12$. Let Q denote any of the null forms defined by (1.1). Then for all sufficiently small positive δ we have*

$$\|Q(\phi,\psi)\|_{H^{s,\theta-1+\delta}}\lesssim\|\phi\|_{\mathcal{H}^{s+1,\theta}}\|\psi\|_{\mathcal{H}^{s+1,\theta}}. \tag{2.1}$$

If $Q=Q_0$ there is a better estimate.

Proposition 2.2. *Let $n=2$, $s>0$ and let θ and δ satisfy*

$$\frac12<\theta\le\min\{1,s+\frac12\},$$
$$0\le\delta\le\min\{1-\theta,s+\frac12-\theta\}.$$

Then

$$\|Q_0(\phi,\psi)\|_{H^{s,\theta-1+\delta}}\lesssim\|\phi\|_{\mathcal{H}^{s+1,\theta}}\|\psi\|_{\mathcal{H}^{s+1,\theta}} \tag{2.2}$$

For a proof of the above proposition, see [13, estimate (7.5)].

For $Q=Q_{ij}$, Q_{0j} estimate (2.1) should be compared (if we set $\theta=s+\frac12$ and $\delta=0$) to the following estimate of Zhou [19]:

$$N_{s,s-\frac12}\left(Q_{\alpha\beta}(\phi,\psi)\right)\lesssim N_{s+1,s+\frac12}(\phi)N_{s+1,s+\frac12}(\psi), \tag{2.3}$$

where $\frac14<s<\frac12$ and

$$N_{s,\theta}(u)=\|w_+(\tau,\xi)^s w_-(\tau,\xi)^\theta\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}. \tag{2.4}$$

The spaces in estimate (2.1) are different, with ϕ and ψ slightly less regular in the sense that $\|u\|_{\mathcal{H}^{s,\theta}}\le N_{s,\theta}(u)$. Moreover we have to account for the extra hyperbolic derivative of order δ on the left hand side.

Proof of Proposition 2.1. We only sketch the proof for $Q=Q_{0j}$. The proof for $Q=Q_{ij}$ is similar. Let

$$F(\tau,\xi)=\langle\xi\rangle^s w_+(\tau,\xi)w_-^\theta(\tau,\xi)\widetilde{\phi}(\tau,\xi),$$
$$G(\tau,\xi)=\langle\xi\rangle^s w_+(\tau,\xi)w_-^\theta(\tau,\xi)\widetilde{\psi}(\tau,\xi).$$

Let $H(\tau,\xi)$ be a test function. We may assume $F,G,H\ge0$. We need to show:

$$\begin{aligned}&\int\frac{\langle\xi+\eta\rangle^s w_-^{\theta-1+\delta}(\tau+\lambda,\xi+\eta)|\tau\eta_j-\lambda\xi_j|}{\langle\xi\rangle^s w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}\\&\times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau\,d\lambda\,d\xi\,d\eta\\&\lesssim\|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2}.\end{aligned} \tag{2.5}$$

Using

$$\langle\xi+\eta\rangle^s\le\langle\xi\rangle^s+\langle\eta\rangle^s$$

we see that we need to estimate the following integral (and a symmetric one):

$$\int\frac{w_-^{\theta-1+\delta}(\tau+\lambda,\xi+\eta)|\tau\eta_j-\lambda\xi_j|F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)}{w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}\,d\tau\,d\lambda\,d\xi\,d\eta \tag{2.6}$$

We restrict our attention to the region where $\tau \geq 0$, $\lambda \geq 0$. The proof for all other regions is similar. We use

$$\begin{aligned}\tau\eta - \lambda\xi &= (|\xi|\eta - |\eta|\xi) + (\tau - |\xi|)\eta - (\lambda - |\eta|)\xi \\ &= (|\xi|\eta - |\eta|\xi) + (|\tau| - |\xi|)\eta - (|\lambda| - |\eta|)\xi\end{aligned}$$

to see that, we need to estimate the following three integrals:

$$\begin{aligned}R^+ &= \int \frac{||\xi|\eta - |\eta|\xi|F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda\, d\xi\, d\eta}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}, \\ T^+ &= \int \frac{||\tau| - |\xi|||\eta|F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda\, d\xi\, d\eta}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}, \\ L^+ &= \int \frac{||\lambda| - |\eta|||\xi|F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda\, d\xi\, d\eta}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}.\end{aligned}$$

We start with R^+. We have

$$\begin{aligned}&||\eta|\xi - |\xi|\eta| \\ &\lesssim |\xi|^{1/2}|\eta|^{1/2}\left(|\xi| + |\eta|\right)^{1/2}\left(||\tau+\lambda| - |\xi+\eta|| + ||\tau| - |\xi|| + ||\lambda| - |\eta||\right)^{1/2}.\end{aligned} \tag{2.7}$$

Indeed,

$$\begin{aligned}||\eta|\xi - |\xi|\eta|^2 &= 2|\eta||\xi|\left(|\xi||\eta| - \xi\cdot\eta\right) \\ &= |\eta||\xi|\left(|\xi| + |\eta| + |\xi+\eta|\right)\left(|\xi| + |\eta| - |\xi+\eta|\right).\end{aligned}$$

We have $|\xi| + |\eta| + |\xi+\eta| \leq 2\left(|\xi| + |\eta|\right)$ and

$$\begin{aligned}|\xi| + |\eta| - |\xi+\eta| &= (\tau + \lambda - |\xi+\eta|) - (\lambda - |\eta|) - (\tau - |\xi|) \\ &\leq |\tau + \lambda - |\xi+\eta|| + |\lambda - |\eta|| + |\tau - |\xi||,\end{aligned}$$

therefore (2.7) follows. Following Zhou [19] we use (2.7) to obtain

$$\begin{aligned}||\eta|\xi - |\xi|\eta| &= ||\eta|\xi - |\xi|\eta|^{2s}||\eta|\xi - |\xi|\eta|^{1-2s} \\ &\lesssim ||\eta|\xi - |\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi| + |\eta|\right)^{1/2-s}||\tau+\lambda| - |\xi+\eta||^{1/2-s} \\ &\quad + ||\eta|\xi - |\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi| + |\eta|\right)^{1/2-s}||\tau| - |\xi||^{1/2-s} \\ &\quad + ||\eta|\xi - |\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi| + |\eta|\right)^{1/2-s}||\lambda| - |\eta||^{1/2-s}.\end{aligned}$$

Therefore,

$$R^+ \lesssim R_1^+ + R_2^+ + R_3^+,$$

where

$$\begin{aligned}R_1^+ &= \int \frac{||\eta|\xi - |\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi| + |\eta|\right)^{1/2-s}||\tau+\lambda| - |\xi+\eta||^{\frac{1}{2}-s}}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)} \\ &\quad \times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\, d\tau\, d\lambda\, d\xi\, d\eta \\ &\leq \int \frac{||\eta|\xi - |\xi|\eta|^{2s}\left(|\xi| + |\eta|\right)^{1/2-s}}{w_-^\theta(\tau,\xi)w_-^\theta(\lambda,\eta)|\xi|^{s+1/2}|\eta|^{2s+1/2}} \\ &\quad \times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\, d\tau\, d\lambda\, d\xi\, d\eta\end{aligned}$$

(we have used the fact that $w_-^{s+\frac12-\theta-\delta}(\tau+\lambda,\xi+\eta)\ge 1$. Indeed, $s+\frac12-\theta-\delta>0$ for small δ, because $\theta<s+\frac12$.)

$$
\begin{aligned}
R_2^+ &= \int \frac{||\eta|\xi-|\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi|+|\eta|\right)^{1/2-s}||\tau|-|\xi||^{1/2-s}}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}\\
&\quad\times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\,d\tau\,d\lambda\,d\xi\,d\eta\\
&\le \int \frac{||\eta|\xi-|\xi|\eta|^{2s}\left(|\xi|+|\eta|\right)^{1/2-s}}{w_-^{\theta+s-\frac12}(\tau,\xi)w_-^\theta(\lambda,\eta)|\xi|^{s+1/2}\ |\eta|^{2s+1/2}}\\
&\quad\times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\,d\tau\,d\lambda\,d\xi\,d\eta
\end{aligned}
$$

(we have used the fact that $w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)\ge 1$. Indeed, $1-\theta-\delta\ge 0$ for small δ because $\theta<s+\frac12<1$.)

$$
\begin{aligned}
R_3^+ &= \int \frac{||\eta|\xi-|\xi|\eta|^{2s}|\xi|^{1/2-s}|\eta|^{1/2-s}\left(|\xi|+|\eta|\right)^{1/2-s}||\lambda|-|\eta||^{1/2-s}}{w_-^{1-\theta-\delta}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^\theta(\tau,\xi)\langle\eta\rangle^s w_+(\lambda,\eta)w_-^\theta(\lambda,\eta)}\\
&\quad\times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\,d\tau\,d\lambda\,d\xi\,d\eta\\
&\le \int \frac{||\eta|\xi-|\xi|\eta|^{2s}\left(|\xi|+|\eta|\right)^{1/2-s}}{w_-^\theta(\tau,\xi)w_-^{\theta+s-\frac12}(\lambda,\eta)|\xi|^{s+1/2}|\eta|^{2s+1/2}}\\
&\quad\times F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)\,d\tau\,d\lambda\,d\xi\,d\eta
\end{aligned}
$$

We present the proof for R_2^+. The proofs for R_1^+ and R_3^+ are similar. We change variables $\tau\mapsto u:=|\tau|-|\xi|=\tau-|\xi|$ and $\lambda\mapsto v:=|\lambda|-|\eta|=\lambda-|\eta|$ and we use the notation

$$
f_u(\xi)=F(u+|\xi|,\xi),\;\; g_v(\eta)=G(v+|\eta|,\eta),\;\; H_{u,v}(\tau',\xi')=H(u+v+\tau',\xi')
$$

to get

$$
\begin{aligned}
R_2^+ &= \iint \frac{1}{(1+|u|)^{\theta+s-\frac12}(1+|v|)^\theta}\Big[\iint \frac{||\eta|\xi-|\xi|\eta|^{2s}\left(|\xi|+|\eta|\right)^{1/2-s}}{|\xi|^{s+1/2}|\eta|^{2s+1/2}}\\
&\quad\times f_u(\xi)g_v(\eta)H_{u,v}(|\xi|+|\eta|,\xi+\eta)\,d\xi\,d\eta\Big]\,du\,dv.
\end{aligned}
$$

We have $||\eta|\xi-|\xi|\eta|^2=2|\xi||\eta|\left(|\xi||\eta|-\xi\cdot\eta\right)$ therefore

$$
\begin{aligned}
[\cdots] &\lesssim \iint \frac{\left(|\xi||\eta|-\xi\cdot\eta\right)^s\left(|\xi|+|\eta|\right)^{1/2-s}}{|\xi|^{1/2}\ |\eta|^{s+1/2}} f_u(\xi)g_v(\eta)H_{u,v}(|\xi|+|\eta|,\xi+\eta)d\xi d\eta\\
&\le \Big(\iint f_u(\xi)^2 g_v(\eta)^2 d\xi d\eta\Big)^{1/2}K^{1/2}\\
&= \|f_u\|_{L^2(\mathbb{R}^2)}\|g_v\|_{L^2(\mathbb{R}^2)}K^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
K &= \iint \frac{\left(|\xi||\eta|-\xi\cdot\eta\right)^{2s}\left(|\xi|+|\eta|\right)^{1-2s}}{|\xi|\ |\eta|^{2s+1}}H_{u,v}(|\xi|+|\eta|,\xi+\eta)^2\,d\xi\,d\eta\\
&= \iint \frac{\left(|\xi'-\eta||\eta|-(\xi'-\eta)\cdot\eta\right)^{2s}\left(|\xi'-\eta|+|\eta|\right)^{1-2s}}{|\xi'-\eta|\ |\eta|^{2s+1}}\\
&\quad\times H_{u,v}(|\xi'-\eta|+|\eta|,\xi')^2\,d\xi'\,d\eta.
\end{aligned}
$$

We use polar coordinates $\eta = \rho\omega$ to get

$$K \lesssim \iiint \frac{(|\xi' - \rho\omega| + \rho - \xi' \cdot \omega)^{2s} (|\xi' - \rho\omega| + \rho)^{1-2s}}{|\xi' - \rho\omega|} \times H_{u,v}(|\xi' - \rho\omega| + \rho, \xi')^2 \, d\xi' \, d\rho \, d\omega.$$

For fixed ξ' and ω, we change variables $\rho \mapsto \tau' := |\xi' - \rho\omega| + \rho$ to get

$$K \lesssim \iint \Big[\tau'^{1-2s} \int_{S^1} \frac{1}{(\tau' - \xi' \cdot \omega)^{1-2s}} d\omega \Big] H(\tau', \xi')^2 d\xi' d\tau'.$$

From [19, estimate (3.22)] we know that

$$\tau'^{1-2s} \int_{S^1} \frac{1}{(\tau' - \xi' \cdot \omega)^{1-2s}} \, d\omega \lesssim 1;$$

therefore $K \lesssim \|H\|^2_{\tilde{A}}$. Putting everything together we get:

$$R_2^+ \lesssim \Big(\int \frac{\|f_u\|_{L^2(\mathbb{R}^2)}}{(1+|u|)^{\theta + s - \frac{1}{2}}} du \Big) \Big(\int \frac{\|g_v\|_{L^2(\mathbb{R}^2)}}{(1+|v|)^{\theta}} dv \Big) \|H\| \,.$$

Since $2\theta + 2s - 1 > 2 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} - 1 = 1$ and $2\theta > 2 \cdot \frac{3}{4} > 1$ we can use the Cauchy-Schwarz inequality to conclude:

$$R_2^+ \lesssim \| \|f_u\|_{L^2(\mathbb{R}^2)} \|_{L^2_u} \| \|g_v\|_{L^2(\mathbb{R}^2)} \|_{L^2_v} \|H\| = \|F\| \|G\|_{\tilde{A}} \|H\|_{\tilde{A}}.$$

This completes the estimates for R_2^+.

Next we estimate T^+. We use $||\tau| - |\xi|| \le w_+(\tau,\xi)^{1-\theta} w_-(\tau,\xi)^{\theta}$ to get

$$\begin{aligned} T^+ &= \int \frac{||\tau| - |\xi|| |\eta| F(\tau,\xi) G(\lambda,\eta) H(\tau+\lambda, \xi+\eta) d\tau d\lambda \, d\xi \, d\eta}{w_-^{1-\theta-\delta}(\tau+\lambda, \xi+\eta) w_+(\tau,\xi) w_-^{\theta}(\tau,\xi) \langle\eta\rangle^s w_+(\lambda,\eta) w_-^{\theta}(\lambda,\eta)} \\ &\le \int \frac{F(\tau,\xi) G(\lambda,\eta) H(\tau+\lambda,\xi+\eta)}{\langle\xi\rangle^{\theta} \langle\eta\rangle^s w_-^{\theta}(\lambda,\eta)} d\tau d\lambda \, d\xi \, d\eta. \end{aligned}$$

Changing variables $\tau \mapsto u := |\tau| - |\xi| = \tau - |\xi|$ and $\lambda \mapsto v := |\lambda| - |\eta| = \lambda - |\eta|$ we have

$$T^+ \lesssim \iint \frac{1}{\langle\xi\rangle^{\theta} \langle\eta\rangle^s} \Big[\iint \frac{F(u+|\xi|, \xi) G(v+|\eta|, \eta) H(u+v+|\xi|+|\eta|, \xi+\eta)}{(1+|v|)^{\theta}} \, du \, dv \Big] \, d\xi \, d\eta.$$

For fixed ξ and η we apply [19, Lemma A] in the (u,v)-variables to get

$$\begin{aligned} T^+ &\lesssim \iint \frac{1}{\langle\xi\rangle^{\theta} \langle\eta\rangle^s} \|F(u+|\xi|,\xi)\|_{L^2_u} \|G(v+|\eta|,\eta)\|_{L^2_v} \\ &\quad \times \|H(w+|\xi|+|\eta|, \xi+\eta)\|_{L^2_w} d\xi d\eta \\ &= \iint \frac{1}{\langle\xi\rangle^{\theta} \langle\eta\rangle^s} \|F(\cdot,\xi)\|_{L^2(\mathbb{R})} \|G(\cdot,\eta)\|_{L^2(\mathbb{R})} \|H(\cdot,\xi+\eta)\|_{L^2(\mathbb{R})} \, d\xi \, d\eta. \end{aligned}$$

Now we do the same in the (ξ,η)-variables to get

$$\begin{aligned} T^+ &\lesssim \| \|F(\cdot,\xi)\|_{L^2(\mathbb{R})} \|_{L^2_\xi} \| \|G(\cdot,\eta)\|_{L^2(\mathbb{R})} \|_{L^2_\eta} \| \|H(\cdot,\xi')\|_{L^2(\mathbb{R})} \|_{L^2_{\xi'}} \\ &= \|F\|_{\tilde{A}} \|G\|_{\tilde{A}} \|H\|_{\tilde{A}}. \end{aligned}$$

The proof for L^+ is similar. □

We are also going to need the following 'product rules' in $H^{s,\theta}$ spaces.

Proposition 2.3. *Let $n = 2$. Then*

$$\|uv\|_{H^{-c,-\gamma}} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}}, \tag{2.8}$$

provided that

$$a + b + c > 1 \tag{2.9}$$

$$a + b \geq 0, \quad b + c \geq 0, \quad a + c \geq 0 \tag{2.10}$$

$$\alpha + \beta + \gamma > 1/2 \tag{2.11}$$

$$\alpha, \beta, \gamma \geq 0. \tag{2.12}$$

Proof. If $a, b, c \geq 0$, the result is contained in [13, Proposition A1]. If not, observe that, due to (2.10), at most one of the a, b, c is negative. We deal with the case $c < 0$, $a, b \geq 0$. All other cases are similar. Observe that

$$\begin{aligned}
&\langle \xi \rangle^{-c} \langle |\tau| - |\xi| \rangle^{-\gamma} |\widetilde{uv}(\tau, \xi)| \\
&\lesssim \langle |\tau| - |\xi| \rangle^{-\gamma} \iint \langle \xi - \eta \rangle^{-c} |\widetilde{u}(\tau - \lambda, \xi - \eta)| |\widetilde{v}(\lambda, \eta)| d\lambda d\eta \\
&\quad + \langle |\tau| - |\xi| \rangle^{-\gamma} \iint |\widetilde{u}(\tau - \lambda, \xi - \eta)| \langle \eta \rangle^{-c} |\widetilde{v}(\lambda, \eta)| d\lambda d\eta,
\end{aligned}$$

therefore

$$\|uv\|_{H^{-c,-\gamma}} \lesssim \|Uv'\|_{H^{0,-\gamma}} + \|u'V\|_{H^{0,-\gamma}},$$

where

$$\begin{aligned}
\widetilde{U}(\tau, \xi) &= \langle \xi \rangle^{-c} |\widetilde{u}(\tau, \xi)|, \\
\widetilde{u'}(\tau, \xi) &= |\widetilde{u}(\tau, \xi)|, \\
\widetilde{V}(\tau, \xi) &= \langle \xi \rangle^{-c} |\widetilde{v}(\tau, \xi)|, \\
\widetilde{v'}(\tau, \xi) &= |\widetilde{v}(\tau, \xi)|.
\end{aligned}$$

Since $a + c \geq 0$, we have

$$\|Uv'\|_{H^{0,-\gamma}} \lesssim \|U\|_{H^{a+c,\alpha}} \|v'\|_{H^{b,\beta}} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}}.$$

Since $b + c \geq 0$, we have

$$\|u'V\|_{H^{0,-\gamma}} \lesssim \|u'\|_{H^{a,\alpha}} \|V\|_{H^{b+c,\beta}} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}}.$$

The result follows. □

Proposition 2.4. *Let $n = 2$. If $s > 1$ and $\frac{1}{2} < \theta \leq s - \frac{1}{2}$, then $H^{s,\theta}$ is an algebra.*

For the proof of the above proposition, see [13, Theorem 7.3].

Proposition 2.5. *Let $n = 2$, $s > 1$, $\frac{1}{2} < \theta \leq s - \frac{1}{2}$. Assume that*

$$-\theta \leq \alpha \leq 0 \quad -s \leq a < s + \alpha. \tag{2.13}$$

Then

$$H^{a,\alpha} \cdot H^{s,\theta} \hookrightarrow H^{a,\alpha}. \tag{2.14}$$

The proof can be found in [13, Theorem 7.2].

Proof of Theorem 1.1. Theorem 1.1 follows by well known methods from the following a-priori estimates (together with the corresponding estimates for differences): For any space-time functions $B, B', \phi, \phi' \in \mathcal{H}^{s+1,\theta}$ and any $\gamma, \mu \in \{0,1,2\}$ we have:

$$\|\phi'\,\partial^\gamma \phi\|_{H^{s,\theta-1+\delta}} \lesssim \|\phi'\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}}, \tag{2.15}$$

$$\|(\partial^\mu B)\,\phi\,\phi'\|_{H^{s,\theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}} \|\phi'\|_{\mathcal{H}^{s+1,\theta}}, \tag{2.16}$$

$$\|Q_{\mu\alpha}(B,\phi)\|_{H^{s,\theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}}, \tag{2.17}$$

$$\|Q_0(B,B')\phi\|_{H^{s,\theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|B'\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}}, \tag{2.18}$$

$$\|Q_{\mu\nu}(B,B')\phi\|_{H^{s,\theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|B'\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}}. \tag{2.19}$$

Here $\frac{1}{4} < s < \frac{1}{2}$, $\frac{3}{4} < \theta < s + \frac{1}{2}$ and δ is a sufficiently small positive number.

To prove (2.15) we use Proposition 2.3 to get:

$$\|\phi'\partial^\gamma \phi\|_{H^{s,\theta-1+\delta}} \lesssim \|\phi'\|_{H^{s+1,\theta}} \|\partial^\gamma \phi\|_{H^{s,\theta}} \lesssim \|\phi'\|_{\mathcal{H}^{s+1,\theta}} \|\phi\|_{\mathcal{H}^{s+1,\theta}}.$$

Similarly, for (2.16) we have

$$\|(\partial^\mu B)\,\phi\,\phi'\|_{H^{s,\theta-1+\delta}} \lesssim \|\partial^\mu B\|_{H^{s,\theta}} \|\phi\,\phi'\|_{H^{s+1,\theta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|\phi\,\phi'\|_{H^{s+1,\theta}}.$$

By Proposition 2.4 and our assumptions on s and θ it follows that the space $H^{s+1,\theta}$ is an algebra. Therefore,

$$\|\phi\,\phi'\|_{H^{s+1,\theta}} \lesssim \|\phi\|_{H^{s+1,\theta}} \|\phi'\|_{H^{s+1,\theta}},$$

and estimate (2.16) follows.

Estimate (2.17) follows from Proposition (2.1). Finally, we consider estimates (2.18) and (2.19). We use the letter Q to denote any of the null forms Q_0, $Q_{\mu\nu}$. We have

$$\|Q(B,B')\,\phi\|_{H^{s,\theta-1+\delta}} \lesssim \|Q(B,B')\|_{H^{s,\theta-1+\delta}} \|\phi\|_{H^{s+1,\theta}}. \tag{2.20}$$

This follows from Proposition 2.5 with s replaced by $s+1$ and α replaced by $\theta-1+\delta$. Next, by (2.1),

$$\|Q(B,B')\|_{H^{s,\theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1,\theta}} \|B'\|_{\mathcal{H}^{s+1,\theta}} \tag{2.21}$$

therefore (2.18) and (2.19) follow.

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