Electronic Journal of Differential Equations, Vol. 2009(2009), No. 121, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TWO COMPONENT REGULARITY FOR THE NAVIER-STOKES EQUATIONS

JISHAN FAN, HONGJUN GAO

Abstract. We consider the regularity of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. Let $u := (u_1, u_2, u_3)$ be a weak solution and $\widetilde{u} := (u_1, u_2, 0)$. We prove that u is strong solution if $\nabla\widetilde{u}$ satisfy Serrin's type criterion.

1. Introduction

In this article we study the regularity of the weak solutions of the Navier-Stokes equations:

$$u_t + u \cdot \nabla u + \nabla p - \Delta u = 0, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^3, \tag{1.2}$$

$$u|_{t=0} = u_0, \quad \operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

where $u := (u_1, u_2, u_3)$ represents the velocity and p represents the pressure.

The existence of global weak solutions for any initial data with finite energy is known since the work of Leray [9]. The smoothness of Leray's weak solution is not known. While the existence of a regular solution is still an open problem, there are many interesting sufficient conditions which guarantee that a given weak solution is smooth. A well-known condition states that if $u \in L^r(0, T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{3}{s} = 1$ and $s \in [3, \infty]$, then the solution u is actually regular [4, 5, 6, 12, 13, 14, 15]. A similar condition $\omega = \operatorname{curl} u \in L^r(0, T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{3}{s} = 2$ where $\frac{3}{2} \le s \le \infty$ also implies the regularity as shown by Beião da Veiga [2]. Chae and Choe [3] proved that if $\widetilde{\omega} = (\omega_1, \omega_2, 0) \in L^r(0, T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{3}{s} = 2$ and $\frac{3}{2} \le s < \infty$, then the solution is regular. Kozono and Yatsu [7] showed that if $\widetilde{\omega} \in L^1(0, T; BMO)$, then the solution remains smooth. Zhang and Chen [17] proved that u is regular if $\widetilde{\omega} \in L^1(0, T; \dot{B}^0_{\infty,\infty})$. Bae and Choe [1] proved that u is strong if $\widetilde{u} \in L^r(0, T; L^s(\mathbb{R}^3))$ with $\frac{2}{r} + \frac{3}{s} = 1$ with $s > 3$. In [3], the authors also proved that u is strong if $\nabla\widetilde{u} \in L^r(0, T; L^s)$ with $\frac{2}{r} + \frac{3}{s} \le 1, 2 \le r \le \infty$ and $3 \le s \le \infty$, which is not optimal

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2000 *Mathematics Subject Classification.* 35Q30, 35K15, 76D03.

Key words and phrases. Navier-Stokes equations; regularity criterion; two component; multiplier spaces; Besov spaces.

Submitted August 28, 2009. Published September 29, 2009.
Supported by grants 10871097 from the NSFC, 20080441062 from the China Postdoctoral Research Foundation, and 0802020C from Jiangsu Planned Projects for Postdoctoral Research Foundation.

from the scaling argument. Here we would like to improve the regularity criterion on $\nabla\widetilde{u}$ in such a way that it undergoes the correct scaling.

There have been many efforts to show that analogous conditions on only one component of the velocity or the gradient of velocity imply the regularity of solutions but all the results are partial, see [8], citez2, z3 and the references there in.

We say that a function belongs to the multiplier spaces $M(\dot{H}^r, L^2)$ if it maps, by point-wise multiplication, $\dot{H}^r$ in L^2:

$$\dot{X}_r := M(\dot{H}^r, L^2) := \{f \in \mathcal{S}', \|f\phi\|_{L^2} \le C\|\phi\|_{\dot{H}^r}\}. \tag{1.4}$$

Similarly we can define $\dot{Y}_{1+r} := M(\dot{H}^r, \dot{H}^{-1})$ and $\dot{Y}_2^{(0)}$ denotes the closure of the Schwartz class $\mathcal{S}$ in $\dot{Y}_2$. We denote $\Lambda := (-\Delta)^{\frac{1}{2}}$, then $\dot{Y}_2 = \Lambda^2 BMO$ [10], $\dot{X}_r$ and $\dot{Y}_{1+r}$ have been characterized in [10, 11].

Now we are in a position to state the main result in this paper.

Theorem 1.1. *Let $u_0 \in H^1$. Assume that one of the following four conditions holds:*

$$\nabla\widetilde{u} \in L^{\frac{2}{2-r}}(0,T;\dot{X}_r) \quad \text{for some } r \in [0,1), \tag{1.5}$$

$$\nabla\widetilde{u} \in L^{\frac{2}{1-r}}(0,T;\dot{Y}_{1+r}) \quad \text{for some } r \in [0,1), \tag{1.6}$$

$$\nabla\widetilde{u} \in C([0,T];\dot{Y}_2^{(0)}), \tag{1.7}$$

$$\nabla\widetilde{u} \in L^1(0,T;\dot{B}^0_{\infty,\infty}). \tag{1.8}$$

Then

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2). \tag{1.9}$$

Here and thereafter, $\dot{B}^s_{p,q}$ stands for the homogeneous Besov space, see below for the definition.

Remark 1.2. Since $L^\infty \subsetneq BMO \subsetneq \dot{B}^0_{\infty,\infty}, L^{\frac{3}{r}} \subset L^{\frac{3}{r},\infty} \subset \dot{X}_r$ and $L^{\frac{3}{1+r}} \subset L^{\frac{3}{1+r},\infty} \subset \dot{Y}_{1+r}$, our results improve that given in [3].

2. Preliminaries

We introduce the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathscr{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\xi} f(x)dx,$$

and its inverse Fourier transform $\mathscr{F}^{-1}f = f^\vee$ is defined by

$$f^\vee(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\xi} f(\xi)d\xi.$$

Let us choose a nonnegative radial function $\phi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \le \hat{\phi}(\xi) \le 1, \quad \hat{\phi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 2, \end{cases}$$

and let

$$\psi(x) = \phi(x) - 2^{-3}\phi(\frac{x}{2}), \quad \phi_j(x) = 2^{3j}\phi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and Δ_j are, respectively, defined by

$$S_j f = \phi_j * f, \tag{2.1}$$

$$\Delta_j f = \psi_j * f. \tag{2.2}$$

Observe that $\Delta_j = S_j - S_{j-1}$. Also, if f is an L^2 function, then $S_j f \to 0$ in L^2 as $j \to -\infty$ and $S_j f \to f$ in L^2 as $j \to +\infty$ (this is an easy consequence of Parseval's theorem). By telescoping the series, we thus have the Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{+\infty} \Delta_j f, \tag{2.3}$$

for all $f \in L^2$, where the summation is in the L^2 sense. Notice that

$$\Delta_j f = \sum_{l=j-2}^{j+2} \Delta_l(\Delta_j f) = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f,$$

then from the Young inequality, it follows that

$$\|\Delta_j f\|_{L^q} \le C 2^{3j(\frac{1}{p}-\frac{1}{q})} \|\Delta_j f\|_{L^p}, \tag{2.4}$$

where $1 \le p \le q \le \infty$, C is a constant independent of f, j.

Let $s \in \mathbb{R}, p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the full-dyadic decomposition such as

$$\dot{B}^s_{p,q} = \{f \in \mathcal{Z}'(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{j=-\infty}^{+\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p} \Big)^{1/q}$$

and $\mathcal{Z}'(\mathbb{R}^3)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3) : D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^3\}$. We refer to [16] for more detailed properties.

3. Proof of Theorem 1.1

We set

$$|\nabla u|^2 = \sum_{i,k} |\partial_k u_i|^2, \quad |\nabla^2 u|^2 = \sum_{i,j,k} |\partial_k \partial_j u_i|^2.$$

Differentiating both sides of equation (1.1) with respect to x_k, taking the scalar product with $\partial_k u$, adding over k and, finally, integrating by parts over $\mathbb{R}^n$, we show that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int |\nabla u|^2 dx + \int |\nabla^2 u|^2 dx &= -\int \nabla[(u \cdot \nabla)u] \cdot \nabla u dx \\ &= \sum_{i,j,k} \int \partial_k u_i \cdot \partial_i u_j \cdot \partial_k u_j dx. \end{aligned}$$

Following [1], we consider separately the three cases $i \ne 3$; $i = 3$ and $j \ne 3$; $i = j = 3$. We only need to deal with the case $i = j = 3$. Since $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, it readily follows that

$$\begin{aligned}\int \partial_k u_i \cdot \partial_i u_j \cdot \partial_k u_j dx &= -\int \partial_k u_3 \cdot (\partial_1 u_1 + \partial_2 u_2) \cdot \partial_k u_3 dx \\ &\le 2\int |\nabla \widetilde{u}| \cdot |\nabla u|^2 dx.\end{aligned}$$

And thus we get

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int |\nabla^2 u|^2 dx \le 2\int |\nabla \widetilde{u}| \cdot |\nabla u|^2 dx =: I. \tag{3.1}$$

Now we assume that (1.5) holds. Then

$$\begin{aligned}I &\le 2\|\nabla u\|_{L^2} \cdot \|\ |\nabla \widetilde{u}| \cdot |\nabla u|\ \|_{L^2} \\ &\le 2\|\nabla u\|_{L^2} \cdot \|\nabla \widetilde{u}\|_{\dot{X}_r} \|\nabla u\|_{\dot{H}^r} \\ &\le C\|\nabla u\|_{L^2} \cdot \|\nabla \widetilde{u}\|_{\dot{X}_r} \|\nabla u\|_{L^2}^{1-r} \|\nabla^2 u\|_{L^2}^r\end{aligned}$$

by the interpolation inequality

$$\|w\|_{\dot{H}^r} \le C\|w\|_{L^2}^{1-r}\|\nabla w\|_{L^2}^r, \tag{3.2}$$

whence

$$I \le \epsilon\|\nabla^2 u\|_{L^2}^2 + C\|\nabla \widetilde{u}\|_{\dot{X}_r}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2,$$

for any $\epsilon > 0$ by Young's inequality. Inserting the above estimates into (3.1) and taking ϵ small enough and the Gronwall's inequality yield (1.9).

Next we assume that (1.6) holds. Then

$$\begin{aligned}I &\le 2\|\nabla u\|_{\dot{H}^1}\|\ |\nabla \widetilde{u}| \cdot |\nabla u|\ \|_{\dot{H}^{-1}} \\ &\le C\|\nabla u\|_{\dot{H}^1}\|\nabla \widetilde{u}\|_{\dot{Y}_{1+r}}\|\nabla u\|_{\dot{H}^r} \\ &\le C\|\nabla^2 u\|_{L^2}\|\nabla \widetilde{u}\|_{\dot{Y}_{1+r}}\|\nabla u\|_{L^2}^{1-r}\|\nabla^2 u\|_{L^2}^r \quad (\text{by } (3.2)) \\ &\le \epsilon\|\nabla u\|_{L^2}^2 + C\|\nabla \widetilde{u}\|_{\dot{Y}_{1+r}}^{\frac{2}{1-r}}\|\nabla u\|_{L^2}^2\end{aligned}$$

for any $\epsilon > 0$ by Young's inequality. Inserting the above estimates into (3.1) and taking ϵ small enough and the Gronwall's inequality give (1.9).

Now we assume that (1.7) holds. For any $\epsilon > 0$, then there exist α and β such that $\nabla \widetilde{u} = \alpha + \beta, \|\alpha\|_{L^\infty(0,T;Y_2)} \le \epsilon$ and $\beta \in L^\infty((0,T) \times \mathbb{R}^3)$,

$$\begin{aligned}I &\le 2\int |\alpha| \cdot |\nabla u|^2 dx + 2\|\beta\|_{L^\infty}\|\nabla u\|_{L^2}^2 \\ &\le 2\|\nabla u\|_{\dot{H}^1}\||\ \alpha| \cdot \nabla u\|_{\dot{H}^{-1}} + C\|\nabla u\|_{L^2}^2 \\ &\le 2\|\nabla u\|_{\dot{H}^1}\|\alpha\|_{\dot{Y}_2}\|\nabla u\|_{\dot{H}^1} + C\|\nabla u\|_{L^2}^2 \\ &\le 2\epsilon\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2.\end{aligned}$$

Inserting the above estimates into (3.1) and taking ϵ small enough and then the Gronwall's inequality show (1.9).

Finally we assume that (1.8) holds. Then using the Littlewood-Paley decomposition (2.3), we decompose $\nabla \widetilde{u}$ as follows:

$$\nabla \widetilde{u} = \sum_{\ell=-\infty}^{+\infty} \Delta_\ell(\nabla \widetilde{u}) = \sum_{\ell<-N} \Delta_\ell(\nabla \widetilde{u}) + \sum_{\ell=-N}^{N} \Delta_\ell(\nabla \widetilde{u}) + \sum_{\ell>N} \Delta_\ell(\nabla \widetilde{u}).$$

Here N is a positive integer to be chosen later. Substituting this into I, we have

$$\begin{aligned} I &\le 2\sum_{\ell<-N}\int |\Delta_\ell(\nabla\widetilde{u})|\ |\nabla u|^2dx + 2\sum_{\ell=-N}^{N}\int |\Delta_\ell(\nabla\widetilde{u})|\ |\nabla u|^2dx \\ &\quad + 2\sum_{\ell>N}\int |\Delta_\ell(\nabla\widetilde{u})|\ |\nabla u|^2dx \\ &=: I_1+I_2+I_3. \end{aligned} \tag{3.3}$$

For I_1, from the Hölder inequality and (2.4), it follows that

$$\begin{aligned} I_1 &\le 2\sum_{\ell<-N}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^\infty}\|\nabla u\|_{L^2}^2 \\ &\le 2\|\nabla u\|_{L^2}^2\sum_{\ell<-N}2^{\frac{3}{2}\ell}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^2} \\ &\le C2^{-\frac{3}{2}N}\|\nabla u\|_{L^2}^2\|\nabla\widetilde{u}\|_{L^2} \le C2^{-\frac{3}{2}N}\|\nabla u\|_{L^2}^3. \end{aligned}$$

For I_2, from the Hölder inequality, it follows that

$$\begin{aligned} I_2 &\le 2\|\nabla u\|_{L^2}^2\sum_{\ell=-N}^{N}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^\infty} \\ &\le C\|\nabla u\|_{L^2}^2\cdot N\|\nabla\widetilde{u}\|_{\dot B^0_{\infty,\infty}} \\ &= CN\|\nabla u\|_{L^2}^2\|\nabla\widetilde{u}\|_{\dot B^0_{\infty,\infty}}. \end{aligned}$$

For I_3, from the Hölder inequality and (2.4), it follows that

$$\begin{aligned} I_3 &\le 2\|\nabla u\|_{L^3}^2\sum_{\ell>N}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^3} \\ &\le C\|\nabla u\|_{L^3}^2\sum_{\ell>N}2^{\frac{\ell}{2}}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^2} \\ &\le C\|\nabla u\|_{L^3}^2\Big(\sum_{\ell>N}2^{-\ell}\Big)^{1/2}\Big(\sum_{\ell>N}2^{2\ell}\|\Delta_\ell(\nabla\widetilde{u})\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\nabla u\|_{L^3}^2 2^{-\frac{N}{2}}\|\nabla^2\widetilde{u}\|_{L^2} \\ &\le C2^{-\frac{N}{2}}\|\nabla u\|_{L^3}^2\|\nabla^2 u\|_{L^2} \\ &\le C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2 \end{aligned}$$

by the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^3}^2 \le C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}.$$

Inserting the above estimates into (3.3), we find that

$$I \le C2^{-\frac{3}{2}N}\|\nabla u\|_{L^2}^3 + CN\|\nabla\widetilde{u}\|_{\dot B^0_{\infty,\infty}}\|\nabla u\|_{L^2}^2 + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2.$$

Now we choose N so that $C2^{-\frac{N}{2}}\|\nabla u\|_{L^2} \le \frac{1}{2}$; i.e.,

$$N \ge 2 + \frac{2\log^+(2C\|\nabla u\|_{L^2})}{\log 2}.$$

Then

$$I \le C + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\log(e+\|\nabla u\|_{L^2}) + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2.$$

Insetting the above estimates into (3.1) and the Gronwall's inequality give (1.9). This completes the proof.

Acknowledgements. The authors are indebted to Professor H. O. Bae who kindly sent us the preprint [1].

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Jishan Fan
School of Mathematical Science, Nanjing Normal University, Nanjing, 210097, China
Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
E-mail address: `fanjishan@njfu.com.cn`

Hongjun Gao
School of Mathematical Science, Nanjing Normal University, Nanjing, 210097, China
E-mail address: `gaohj@njnu.edu.cn`