Electronic Journal of Differential Equations, Vol. 2009(2009), No. 127, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE SOLUTIONS FOR A SYSTEM OF NONLINEAR BOUNDARY-VALUE PROBLEMS ON TIME SCALES

A. KAMESWARA RAO

ABSTRACT. We determine the values of a parameter λ for which there exist positive solutions to the system of dynamic equations

$$\begin{aligned} u^{\Delta\Delta}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a,b]_{\mathbb{T}}, \\ v^{\Delta\Delta}(t) + \lambda q(t) g(u(\sigma(t))) = 0, \quad t \in [a,b]_{\mathbb{T}}, \end{aligned}$$

with the boundary conditions, $\alpha u(a) - \beta u^{\Delta}(a) = 0$, $\gamma u(\sigma^2(b)) + \delta u^{\Delta}(\sigma(b)) = 0$, $\alpha v(a) - \beta v^{\Delta}(a) = 0$, $\gamma v(\sigma^2(b)) + \delta v^{\Delta}(\sigma(b)) = 0$, where $\mathbb{T}$ is a time scale. To this end we apply a Guo-Krasnosel'skii fixed point theorem.

1. Introduction

Let $\mathbb{T}$ be a time scale with $a, \sigma^2(b) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}} = J \cap \mathbb{T}. \tag{1.1}$$

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of dynamic equations

$$\begin{aligned} u^{\Delta\Delta}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a,b]_{\mathbb{T}}, \\ v^{\Delta\Delta}(t) + \lambda q(t) g(u(\sigma(t))) = 0, \quad t \in [a,b]_{\mathbb{T}}, \end{aligned} \tag{1.2}$$

satisfying the boundary conditions

$$\begin{aligned} \alpha u(a) - \beta u^{\Delta}(a) = 0, \quad \gamma u(\sigma^2(b)) + \delta u^{\Delta}(\sigma(b)) = 0, \\ \alpha v(a) - \beta v^{\Delta}(a) = 0, \quad \gamma v(\sigma^2(b)) + \delta v^{\Delta}(\sigma(b)) = 0\,. \end{aligned} \tag{1.3}$$

We will use the following assumptions:

(A1) $f, g \in C([0,\infty), [0,\infty))$;
(A2) $p, q \in C([a,\sigma(b)]_{\mathbb{T}}, [0,\infty))$, and each function does not vanish identically on any closed subinterval of $[a,\sigma(b)]_{\mathbb{T}}$;
(A3) the following limits exist as real numbers:
$f_0 := \lim_{x\to 0^+} f(x)/x$, $g_0 := \lim_{x\to 0^+} g(x)/x$,
$f_\infty := \lim_{x\to\infty} f(x)/x$, and $g_\infty := \lim_{x\to\infty} g(x)/x$

2000 *Mathematics Subject Classification.* 39A10, 34B15, 34A40.
Key words and phrases. Dynamic equations; eigenvalue intervals; positive solution; cone.

Submitted July 6, 2009. Published October 4, 2009.

There is an ongoing flurry of research activities devoted to positive solutions of dynamic equations on time scales. This work entails an extension of the paper by Chyan and Henderson [7] to eigenvalue problem for system of nonlinear boundary value problems on time scales. Also, in that light, this paper is closely related to the works of Li and Sun [21, 23].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [9, 11, 13, 19, 20] and as applications for which only positive solutions are meaningful [2, 10, 16, 25]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [14, 15, 17, 18].

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant [9]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Green's Function and Bounds

In this section, we state the well-known Guo-Krasnosel'skii fixed point theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$ is the Green's function for

$$\begin{gathered} -y^{\Delta\Delta} = 0, \\ \alpha u(a) - \beta u^{\Delta}(a) = 0, \quad \gamma u(\sigma^2(b)) + \delta u^{\Delta}(\sigma(b)) = 0 \end{gathered} \tag{2.1}$$

is given by

$$G(t,s) = \frac{1}{d}\begin{cases} \{\alpha(t-a)+\beta\}\{\gamma(\sigma^2(b)-\sigma(s))+\delta\} : & a \le t \le s \le \sigma^2(b) \\ \{\alpha(\sigma(s)-a)+\beta\}\{\gamma(\sigma^2(b)-t)+\delta\} : & a \le \sigma(s) \le t \le \sigma^2(b) \end{cases} \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta \ge 0$ and

$$d := \gamma\beta + \alpha\delta + \alpha\gamma(\sigma^2(b) - a) > 0.$$

One can easily check that

$$G(t,s) > 0, \quad (t,s) \in (a, \sigma^2(b))_{\mathbb{T}} \times (a, \sigma(b))_{\mathbb{T}} \tag{2.3}$$

and

$$G(t,s) \le G(\sigma(s), s) = \frac{[\alpha(\sigma(s)-a)+\beta][\gamma(\sigma^2(b)-\sigma(s))+\delta]}{d} \tag{2.4}$$

for $t \in [a, \sigma^2(b)]_{\mathbb{T}}$, $s \in [a, \sigma(b)]_{\mathbb{T}}$. Let $I = \big[\frac{3a+\sigma^2(b)}{4}, \frac{a+3\sigma^2(b)}{4}\big]_{\mathbb{T}}$. Then

$$G(t,s) \ge kG(\sigma(s), s) = k\frac{[\alpha(\sigma(s)-a)+\beta][\gamma(\sigma^2(b)-\sigma(s))+\delta]}{d} \tag{2.5}$$

for $t \in I$, $s \in [a, \sigma(b)]_{\mathbb{T}}$, where

$$k = \min\Big\{ \frac{\gamma(\sigma^2(b)-a)+4\delta}{4(\gamma(\sigma^2(b)-a)+\delta)}, \frac{\alpha(\sigma^2(b)-a)+4\beta}{4(\alpha(\sigma^2(b)-a)+\beta)} \Big\}. \tag{2.6}$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.2), (1.3) if and only if

$$u(t) = \lambda \int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s, a \le t \le \sigma^2(b),$$
$$v(t) = \lambda \int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s, \quad a \le t \le \sigma^2(b). \tag{2.7}$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.2), (1.3) will be determined via applications of the following fixed point theorem [19].

Theorem 2.1 (Krasnosel'skii). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P} \tag{2.8}$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive Solutions in a Cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (i.e., positive solutions) of (1.2), (1.3). Assume throughout that $[a, \sigma^2(b)]_{\mathbb{T}}$ is such that

$$\xi = \min\Big\{t \in \mathbb{T} : t \ge \frac{3a + \sigma^2(b)}{4}\Big\},$$
$$\omega = \max\Big\{t \in \mathbb{T} : t \le \frac{a + 3\sigma^2(b)}{4}\Big\}; \tag{3.1}$$

both exist and satisfy

$$\frac{3a + \sigma^2(b)}{4} \le \xi < \omega \le \frac{a + 3\sigma^2(b)}{4}. \tag{3.2}$$

Next, let $\tau \in [\xi, \omega]_{\mathbb{T}}$ be defied by

$$\int_\xi^\omega G(\tau,s)p(s)\Delta s = \max_{t \in [\xi,\omega]_{\mathbb{T}}} \int_\xi^\omega G(t,s)p(s)\Delta s. \tag{3.3}$$

Finally, we define

$$l = \min_{s \in [a,\sigma(b)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}, \tag{3.4}$$
$$\gamma = \min\{k, l\}. \tag{3.5}$$

For our construction, let $\mathcal{B} = \{x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{|x(t)| : t \in [a, \sigma^2(b)]_{\mathbb{T}}\}$ and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{x \in \mathcal{B} | x(t) \ge 0 \text{ on } [a, \sigma^2(b)]_{\mathbb{T}}, \quad \text{and} \quad x(t) \ge \gamma\|x\|, \text{ for } t \in [\xi, \omega]_{\mathbb{T}}\Big\}. \tag{3.6}$$

For our first result, define positive numbers L_1 and L_2, by

$$L_1 := \max\Big\{\Big[\gamma\int_\xi^\omega G(\tau,s)p(s)\Delta s f_\infty\Big]^{-1}, \Big[\gamma\int_\xi^\omega G(\tau,s)q(s)\Delta s g_\infty\Big]^{-1}\Big\},$$

$$L_2 := \min\Big\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s f_0\Big]^{-1}, \Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s g_0\Big]^{-1}\Big\}.$$

Theorem 3.1. *Assume that conditions* (A1)–(A3) *are satisfied. Then, for each λ satisfying*

$$L_1 < \lambda < L_2, \tag{3.7}$$

there exists a pair (u,v) satisfying (1.2), (1.3) *such that $u(x) > 0$ and $v(x) > 0$ on $(a,\sigma^2(b))_{\mathbb{T}}$.*

Proof. Let λ be as in (3.7). And let $\epsilon > 0$ be chosen such that

$$\max\Big\{\Big[\gamma\int_\xi^\omega G(\tau,s)p(s)\Delta s(f_\infty-\epsilon)\Big]^{-1}, \Big[\gamma\int_\xi^\omega G(\tau,s)q(s)\Delta s(g_\infty-\epsilon)\Big]^{-1}\Big\} \le \lambda$$

$$\begin{aligned}\lambda \le \min\Big\{&\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s(f_0+\epsilon)\Big]^{-1},\\ &\Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s(g_0+\epsilon)\Big]^{-1}\Big\}.\end{aligned}$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = \lambda\int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s. \tag{3.8}$$

By the remarks in Section 2, we seek suitable fixed points of T in the cone $\mathcal{P}$.

Notice from (A1), (A2), and (2.3) that, for $u \in \mathcal{P}$, $Tu(t) \ge 0$ on $[a,\sigma^2(b)]_{\mathbb{T}}$. Also, for $u \in \mathcal{P}$, we have from (2.4) that

$$\begin{aligned}Tu(t) &:= \lambda\int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\ &\le \lambda\int_a^{\sigma(b)} G(\sigma(s),s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\end{aligned} \tag{3.9}$$

so that

$$\|Tu\| \le \lambda\int_a^{\sigma(b)} G(\sigma(s),s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s. \tag{3.10}$$

Next, if $u \in \mathcal{P}$, we have from (2.5), (3.5), and (3.8) that

$$\begin{aligned}&\min_{t\in[\xi,\omega]_{\mathbb{T}}} Tu(t)\\ &= \min_{t\in[\xi,\omega]_{\mathbb{T}}} \lambda\int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\ &\ge \lambda\gamma\int_a^{\sigma(b)} G(\sigma(s),s)p(s)f\Big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\ &\ge \gamma\|Tu\|.\end{aligned} \tag{3.11}$$

Consequently, $T : \mathcal{P} \to \mathcal{P}$. In addition, standard arguments shows that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \le (f_0 + \epsilon)x,\ g(x) \le (g_0 + \epsilon)x, \quad 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.4) and choice of ϵ, for $a \le s \le \sigma(b)$, that

$$\begin{aligned}
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r &\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(u(\sigma(r)))\Delta r \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)(g_0 + \epsilon)u(r)\Delta r \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r(g_0 + \epsilon)\|u\| \\
&\le \|u\| = H_1.
\end{aligned}$$

As a consequence, we next have from (2.4) and choice of ϵ, for $a \le t \le \sigma^2(b)$, that

$$\begin{aligned}
Tu(t) &= \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)H_1\Delta s \\
&\le H_1 = \|u\|.
\end{aligned}$$

So, $\|Tu\| \le \|u\|$. If we set$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.12}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \ge (f_\infty - \epsilon)x, \quad g(x) \ge (g_\infty - \epsilon)x, \quad x \ge \overline{H}_2. \tag{3.13}$$

Let $H_2 = \max\{2H_1, \overline{H}_2/\gamma\}$. Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$\min_{t \in [\xi,\omega]_{\mathbb{T}}} u(t) \ge \gamma\|u\| \ge \overline{H}_2. \tag{3.14}$$

Consequently, from (2.5) and choice of ϵ, for $a \le s \le \sigma(b)$, we have that

$$\begin{aligned}
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r &\ge \lambda \int_\xi^{\omega} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \\
&\ge \lambda \int_\xi^{\omega} G(\tau, r)q(r)g(u(\sigma(r)))\Delta r \\
&\ge \lambda \int_\xi^{\omega} G(\tau, r)q(r)(g_\infty - \epsilon)u(r)\Delta r \\
&\ge \gamma\lambda \int_\xi^{\omega} G(\tau, r)q(r)(g_\infty - \epsilon)\Delta r\|u\| \\
&\ge \|u\| = H_2.
\end{aligned} \tag{3.15}$$

And so, we have from (2.5) and choice of ϵ that

$$\begin{aligned} Tu(\tau) &= \lambda \int_a^{\sigma(b)} G(\tau,s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s \\ &\geq \lambda \int_a^{\sigma(b)} G(\tau,s)p(s)(f_\infty - \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r \Delta s \\ &\geq \lambda \int_a^{\sigma(b)} G(\tau,s)p(s)(f_\infty - \epsilon)H_2 \Delta s \\ &\geq \gamma H_2 > H_2 = \|u\|. \end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$. So if we set $\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\}$, then

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.16}$$

Applying Theorem 2.1 to (3.12) and (3.16), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, and with v being defined by

$$v(t) = \lambda \int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s, \tag{3.17}$$

the pair (u,v) is a desired solution of (1.2), (1.3) for the given λ. The proof is complete. □

Prior to our next result, we introduce another hypothesis.

(A4) $g(0) = 0$, and f is an increasing function.

We now define positive numbers L_3 and L_4 by

$$L_3 := \max\Big\{\Big[\gamma \int_\xi^\omega G(\tau,s)p(s)\Delta s f_0\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau,s)q(s)\Delta s g_0\Big]^{-1}\Big\},$$

$$L_4 := \min\Big\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s f_\infty\Big]^{-1}, \Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s g_\infty\Big]^{-1}\Big\}.$$

Theorem 3.2. *Assume that conditions* (A1)–(A4) *are satisfied. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{3.18}$$

there exists a pair (u,v) *satisfying* (1.2), (1.3) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma^2(b))_{\mathbb{T}}$.

Proof. Let λ be as in (3.18). And let $\epsilon > 0$ be chosen such that

$$\max\Big\{\Big[\gamma \int_\xi^\omega G(\tau,s)p(s)\Delta s(f_0 - \epsilon)\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau,s)q(s)\Delta s(g_0 - \epsilon)\Big]^{-1}\Big\} \leq \lambda,$$

$$\begin{aligned} \lambda \leq \min\Big\{&\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s(f_\infty + \epsilon)\Big]^{-1}, \\ &\Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s(g_\infty + \epsilon)\Big]^{-1}\Big\}. \end{aligned}$$

Let T be the cone preserving, completely continuous operator that was defined by (3.8). From the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \geq (f_0 - \epsilon)x, \quad g(x) \geq (g_0 - \epsilon)x, \quad 0 < x \leq H_1 \tag{3.19}$$

Now, $g(0) = 0$, and so there exists $0 < H_2 < H_1$ such that

$$\lambda g(x) \le \frac{H_1}{\int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s}, \quad 0 \le x \le H_2. \tag{3.20}$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_2$. Then, for $a \le s \le \sigma(b)$, we have

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))\Delta r \le \frac{\int_a^{\sigma(b)} G(\sigma(s), r)q(r)H_1\Delta r}{\int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s} \le H_1. \tag{3.21}$$

Then

$$\begin{aligned}
Tu(\tau) &= \lambda \int_a^{\sigma(b)} G(\tau, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s \\
&\ge \lambda \int_\xi^\omega G(\tau, s)p(s)(f_0 - \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\ge \lambda \int_\xi^\omega G(\tau, s)p(s)(f_0 - \epsilon)\lambda \int_\xi^\omega G(\tau, r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\ge \lambda \int_\xi^\omega G(\tau, s)p(s)(f_0 - \epsilon)\lambda\gamma \int_\xi^\omega G(\tau, r)q(r)(g_0 - \epsilon)\|u\|\Delta r\Delta s \\
&\ge \lambda \int_\xi^\omega G(\tau, s)p(s)(f_0 - \epsilon)\|u\|\Delta s \\
&\ge \lambda\gamma \int_\xi^\omega G(\tau, s)p(s)(f_0 - \epsilon)\|u\|\Delta s \ge \|u\|.
\end{aligned} \tag{3.22}$$

So, $\|Tu\| \ge \|u\|$. If we put $\Omega_1 = \{x \in \mathcal{B} | \; \|x\| < H_2\}$, then

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.23}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_1$ such that

$$f(x) \le (f_\infty - \epsilon)x, \quad g(x) \le (g_\infty - \epsilon)x, \quad x \ge \overline{H}_1 \tag{3.24}$$

There are two cases: (i) g is bounded, and (ii) g is unbounded.

For case (i), suppose $N > 0$ is such that $g(x) \le N$ for all $0 < x < \infty$. Then, for $a \le s \le \sigma(b)$ and $u \in \mathcal{P}$,

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \le N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r. \tag{3.25}$$

Let

$$M = \max\Big\{f(x) | 0 \le x \le N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r\Big\}, \tag{3.26}$$

and let

$$H_3 > \max\Big\{2H_2, M\lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s\Big\}. \tag{3.27}$$

Then, for $u \in \mathcal{P}$ with $\|u\| = H_3$,

$$Tu(t) \le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)M\Delta s \le H_3 = \|u\| \tag{3.28}$$

so that $\|Tu\| \le \|u\|$. If $\Omega_2 = \{x \in \mathcal{B} | \; \|x\| < H_3\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.29}$$

For case (ii), there exists $H_3 > \max\{2H_2, \overline{H}_1\}$ such that $g(x) \le g(H_3)$, for $0 < x \le H_3$. Similarly, there exists $H_4 > \max\{H_3, \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(H_3)\Delta r\}$ such that $f(x) \le f(H_4)$, for $0 < x \le H_4$. Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, we have by $(A4)$ that

$$\begin{aligned} Tu(t) &\le \lambda \int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(r),r)q(r)g(H_3)\Delta r\Big)\Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(H_4)\Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s(f_\infty + \epsilon)H_4 \\ &\le H_4 = \|u\|, \end{aligned} \tag{3.30}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let $\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_4\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2.$$

In either case, application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which in turn yields a pair (u, v) satisfying (1.2), (1.3) for the chosen value of λ. The proof is complete. □

References

[1] R. P. Agarwal and D. O'Regan, Triple solutions to boundary value problems on time scales, *Appl. Math. Lett.*, **13**(2000), No. 4, 7-11.

[2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, The Netherlands, 1999.

[3] D. R. Anderson, Eigenvalue intervals for a second-order mixed-conditions problem on time scale, *Int. J. Nonlinear Diff. Eqns.*, **7**(2002), 97-104.

[4] D. R. Anderson, Eigenvalue intervals for a two-point boundary value problem on a measure chain, *J. Comp. Appl. Math.*, **141**(2002), No. 1-2, 57-64.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: *An Introduction with Applications*, Birkhauser, Boston, Mass, USA, 2001.

[6] C. J. Chyan, J. M. Davis, J.Henderson, and W. K. C. Yin, Eigenvalue comparisons for differential equations on a measure chain, *Elec. J. Diff. Eqns.*, **1998**(1998), No. 35, 1-7.

[7] C. J. Chyan and J. Henderson, Eigenvalue problems for nonlinear differential equations on a measure chain, *J. Math. Anal. Appl.*, **245**(2000), No. 2, 547-559.

[8] L. H. Erbe and A. Peterson, Positive solutions for a nonlinear differential equation on ameasure chain, *Math. Comp. Model.*, **32**(2000), No. 5-6, 571-585.

[9] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), No. 3, 743-748.

[10] J. R. Graef and B. Yang, Boundary value problems for second order nonlinear ordinary differential equations, *Comm. Applied Anal.*, **6**(2002), No. 2, 273-288.

[11] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of Notes and Reports in Mathematics in Science and Engineering, Academic Press, Boston, Mass, USA, 1988.

[12] Z. He, Double positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales, *Applicable Anal.*, **84**(2005), No. 4, 377-390.

[13] J. Henderson and H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.*, **208**(1997), No. 1, 252-259.

[14] J. Henderson and H.Wang, Nonlinear eigenvalue problems for quasilinear systems, *Comp. Math. Appl.*, **49**(2005), No. 11-12, 1941-1949.

[15] J. Henderson and H. Wang, An eigenvalue problem for quasilinear systems, *Rocky. Mount. J. Math.*, **37**(2007), No. 1, 215-228.

[16] L. Hu and L. Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations, *J. Math. Anal. Appl.*, **335**(2007), No. 2, 1052-1060.
[17] G. Infante, Eigenvalues of some non-local boundary-value problems, *Proc. Edinburgh Math. Soc.*, **46**(2003), No. 1, 75-86.
[18] G. Infante and J. R. L. Webb, Loss of positivity in a nonlinear scalar heat equation, *Nonlinear Diff. Eqns. Appl.*, **13**(2006), No. 2, 249-261.
[19] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff Ltd, Groningen, The Netherlands (1964).
[20] H. J. Kuiper, On positive solutions of nonlinear elliptic eigenvalue problems, *Rend. Circ. Mat. Palermo.*, **20**(1971), 113-138.
[21] W. T. Li and H. R. Sun, Multiple positive solutions for nonlinear dynamical systems on a measure chain, *J. Comp. Appl. Math.*, **162**(2004), No. 2, 421-430.
[22] R. Ma, Multiple nonnegative solutions of second-order systems of boundary value problems, *Nonlinear Anal.* (T. M. A), **42**(2000), No. 6, 1003-1010.
[23] H. R. Sun, Existence of positive solutions to second-order time scale systems, *Comp. Math. Appl.*, **49**(2005), No. 1, 131-145.
[24] H. Wang, On the number of positive solutions of nonlinear systems, *J. Math. Anal. Appl.*, **281**(2003), No. 1, 287-306.
[25] J. R. L. Webb, Positive solutions of some three point boundary value problems via fixed point index theory, *Nonlinear Anal.* (T. M. A), **47**(2001), No. 7, 4319-4332.
[26] Y. Zhou and Y.Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, *J. Math. Anal. Appl.*, **320**(2006), No. 2, 578-590.

A. Kameswara Rao
Department of Applied Mathematics, Andhra University, Visakhapatnam 530 003, India
E-mail address: kamesh_1724@yahoo.com