Electronic Journal of Differential Equations, Vol. 2009(2009), No. 13, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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SOLVABILITY FOR SECOND-ORDER M-POINT BOUNDARY VALUE PROBLEMS AT RESONANCE ON THE HALF-LINE

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Abstract. In this article, we investigate the existence of positive solutions for second-order m-point boundary-value problems at resonance on the half-line

$$(q(t)x'(t))' = f(t, x(t), x'(t)), \quad \text{a.e. in } (0,\infty),$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad \lim_{t\to\infty} q(t)x'(t) = 0.$$

Some existence results are obtained by using the Mawhin's coincidence theory.

1. Introduction

In this article, we study the existence of positive solutions for the second-order m-point boundary-value problems at resonance on the half-line

$$(q(t)x'(t))' = f(t, x(t), x'(t)), \quad a.e. \text{ in } (0,\infty), \tag{1.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad \lim_{t\to\infty} q(t)x'(t) = 0, \tag{1.2}$$

where $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, $\alpha_i \in \mathbb{R}$ $(1 \le i \le m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $q \in C[0,\infty) \cap C^1(0,\infty)$ with $q > 0$ on $[0,\infty)$ and $\frac{1}{q} \in L_1[0,\infty)$.

In recent years, many authors have studied the existence of positive solutions for some boundary value problems on the half-line (see [6, 7, 12, 13, 14, 15]) or at resonance (see [2, 3, 4, 5, 9, 10]). However, to the best of our knowledge, only one paper [8] studied the existence and uniqueness positive solutions for second-order three-point boundary value problems at resonance on the half-line. There is little research concerning (1.1)-(1.2), so it is worthwhile to investigate the problem.

Inspired by [2, 4, 5], the purpose of our paper is to discuss the existence of positive solutions for the second-order m-point boundary value problem at resonance on the half-line. Our method is based on the coincidence degree theory of Mawhin.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. m-point boundary value problem; resonance; half-line; coincidence degree theory.

Submitted July 8, 2008. Published January 12, 2009.

The remaining part of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to proving the existence of positive solutions for (1.1)-(1.2).

2. Preliminaries and lemmas

Now, we briefly recall some notation and an abstract existence result.

Let X, Z be normed spaces, $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm operator of index zero, and $P : X \to X$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$ and $X = \ker L \oplus \ker P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$, It follows that $L|_{\operatorname{dom} L\cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of the mapping by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_p(I - Q)$.

Definition 2.1. Let $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$ be a mapping. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

Definition 2.2. We say that the map $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $(t, x) \to f(t, z)$ is $L_1[0,\infty)$-Carathéodory, if the following conditions are satisfied

- (i) for each $z \in \mathbb{R}^n$, the mapping $t \to f(t, z)$ is Lebesgue measurable;
- (ii) for a.e. $t \in [0,\infty)$, the mapping $z \to f(t, z)$ is continuous on $\mathbb{R}^n$;
- (iii) for each $r > 0$, there exists $\varphi_r \in L_1[0,\infty)$ such that, for a.e. $t \in [0,\infty)$ and every z such that $|z| \le r$, we have $|f(t, z)| \le \varphi_r(t)$.

Lemma 2.3 ([1]). *Let X be the space of all bounded continuous vector-value functions on $[0,\infty)$ and $M \subset X$. Then M is relatively compact in X if the following conditions hold:*

- (i) *M is bounded in X:*
- (ii) *the functions from M are equicontinuous on any compact interval of $[0,\infty)$;*
- (iii) *the functions from M are equiconvergent, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|\phi(t) - \phi(\infty)| < \epsilon$, for all $t > T$ and all $\phi \in S$.*

Lemma 2.4 ([11]). *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

- (1) *$Lx \ne \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;*
- (2) *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
- (3) *$\deg(JQN|_{\partial\Omega\cap\ker L}, \Omega \cap \ker L, 0) \ne 0$, with $Q : Z \to Z$ is a continuous projection such that $\operatorname{Im} L = \ker Q$ and $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Let $AC[0,\infty)$ denote the space of absolutely continuous functions on the interval $[0,\infty)$. In this paper, the following space X will be basic space to study (1.1)-(1.2), which is denoted by

$$X = \{x \in C^1[0,\infty), x, qx' \in AC[0,\infty) \lim_{t\to\infty} x(t) \text{ and } \lim_{t\to\infty} x'(t) \text{exist}, \ (qx')' \in L_1[0,\infty)\}$$

endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \sup_{t\in[0,\infty)} |x(t)|$.

Let $Z = L_1[0,\infty)$, and denote the norm in $L_1[0,\infty)$ by $\|\cdot\|_1$.

Define L to be the linear operator from $L \subset X$ to Z with

$$\operatorname{dom} L = \{x \in X : x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \lim_{t\to\infty} q(t)x'(t) = 0\}$$

and $Lx(t) = (q(t)x'(t))'$, $x \in \operatorname{dom} L$, $t \in [0,\infty)$. We define $N : X \to Z$ by setting

$$Nx(t) = f(t, x(t), x'(t)), \quad t \in [0,\infty),$$

then (1.1)-(1.2) can be written

$$Lx = Nx$$

Lemma 2.5. *If* $\sum_{i=1}^{m-2} \alpha_i = 1$ *and* $\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{e^{-s}}{q(s)} ds \neq 0$, *then*

(i) $\ker L = \{x \in \operatorname{dom} L : x(t) = c, c \in \mathbb{R}, t \in [0,\infty)\}$;

(ii) $\operatorname{Im} L = \{y \in Z : \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds = 0\}$;

(iii) $L : \operatorname{dom} L \subset X \to X$ *is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator* $Q : Z \to Z$ *can be defined*

$$(Qy)(t) = h(t) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds, \quad t \in [0,\infty),$$

where

$$h(t) = \frac{e^{-t}}{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{e^{-s}}{q(s)} ds}, \quad t \in [0,\infty).$$

(iv) *The generalized inverse* $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *of* L *can be written by*

$$K_P y(t) = -\int_0^t \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds.$$

(v) $\|K_P y\| \le \max\{\|q^{-1}\|_\infty, \|q^{-1}\|_1\} \|y\|_1$, *for all* $y \in$ ImL.

Proof. By direct calculations, we easily know that (i) and (ii) hold. (iii) For any $y \in Z$, take the prosector

$$(Qy)(t) = h(t) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds, \quad t \in [0,\infty).$$

Let $y_1 = y - Qy$, by direct calculations, we have

$$\begin{aligned} &\sum_{i=1}^{m-2} \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty y_1(\tau) d\tau ds \\ &= \sum_{i=1}^{m-2} \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds \Big(1 - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^\infty h(\tau) d\tau ds\Big) = 0. \end{aligned}$$

So $y_1 \in \operatorname{Im} L$. Hence, $Z = \operatorname{Im} L + \operatorname{Im} Q$, since $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, we obtain

$$Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus, $\dim \ker L = \dim \operatorname{Im} Q = 1$.
Hence, L is a Fredholm operator of index zero.

(iv) Let $P : Z \to Z$ be defined by

$$Px(t) = x(0), \quad t \in [0,\infty).$$

Then the generalized inverse $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L can be written as

$$K_P y(t) = -\int_0^t \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds.$$

In fact, for any $y \in \operatorname{Im}L$, we have

$$LK_P y(t) = (q(t)K_P y'(t))' = y(t).$$

and for $x \in \operatorname{dom} L \cap \ker P$, one has

$$\begin{aligned} K_P Lx(t) = K_P(q(t)x'(t))' &= -\int_0^t \frac{1}{q(s)} \int_s^\infty (q(\tau)x'(\tau))' d\tau ds \\ &= -\int_0^t \frac{1}{q(s)} \Big(\lim_{\sigma\to\infty} q(\sigma)x'(\sigma) - q(s)x'(s) \Big) ds \\ &= \int_0^t x'(s) ds = x(t) - x(0), \end{aligned}$$

in view of $x(0) = 0$ (since $x \in \ker P$), thus,

$$(K_P L)x(t) = x(t), \quad t \in [0, \infty).$$

Hence, $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

(v) From the definition of K_P, we have

$$\|K_P y\|_\infty = \sup_{t\in[0,\infty)} |K_P y| \le \sup_{t\in[0,\infty)} \int_0^t \frac{1}{q(s)} \int_s^\infty |y(\tau)| d\tau ds \le \|q^{-1}\|_1 \|y\|_1,$$

and

$$\|(K_P y)'\|_\infty = \sup_{t\in[0,\infty)} |(K_P y)'| \le \sup_{t\in[0,\infty)} \frac{1}{q(t)} \int_t^\infty |y(s)| ds \le \|q^{-1}\|_\infty \|y\|_1.$$

Hence,

$$\|K_P y\| \le \max\{\|q^{-1}\|_1, \|q^{-1}\|_\infty\} \|y\|_1.$$

□

Lemma 2.6. *If f is a Carathéodory function and $\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(s)} ds < \infty$, then N is L-compact.*

Proof. Let $M \subset X$ be bounded with $r = \sup\{\|x\| : x \in M\}$ and consider $K_{P,Q}N(M)$. By $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions with respect to $L_1[0,\infty)$, there exists a Lebesgue integrable function φ_r such that

$$|Nx(t)| = |f(t, x(t), x'(t))| \le \varphi_r(t) \quad \text{a.e.} \quad \text{in } (0, \infty).$$

Then for all $x \in M$, we have

$$\begin{aligned} \|QNx\|_1 &\le \int_0^\infty |QNx(s)| ds \\ &= \int_0^\infty \Big| h(s) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_\varsigma^\infty f(\tau, x(\tau), x'(\tau)) d\tau d\varsigma \Big| ds \\ &\le \int_0^\infty |h(s)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_1} \frac{1}{q(\varsigma)} \int_0^\infty \varphi_r(\tau) d\tau d\varsigma ds \end{aligned}$$

$$\leq \|h\|_1 \|\varphi_r\| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma < \infty.$$

Thus,

$$\begin{aligned}
&\|K_{P,Q}Nx\|_\infty \\
&= \Big| \sup_{t\in[0,\infty)} \int_0^t \frac{1}{q(s)} \int_s^\infty \Big(f(\tau, x(\tau), x'(\tau)) \\
&\quad - h(\tau) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_\varsigma^\infty f(\zeta, x(\zeta), x'(\zeta)) d\zeta d\varsigma \Big) d\tau ds \Big| \\
&\leq \sup_{t\in[0,\infty)} \int_0^t \frac{1}{q(s)} \int_s^\infty \Big| f(\tau, x(\tau), x'(\tau)) \\
&\quad - h(\tau) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_\varsigma^\infty f(\zeta, x(\zeta), x'(\zeta)) d\zeta d\varsigma \Big| d\tau ds \\
&\leq \int_0^\infty \frac{1}{q(s)} \int_0^\infty \Big(\varphi_r(\tau) + |h(\tau)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_0^\infty \varphi_r(\zeta) d\zeta d\varsigma \Big) d\tau ds \\
&\leq \|\varphi_r\|_1 \|q^{-1}\|_1 \Big(1 + \|h\|_1 \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma \Big) < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|(K_{P,Q}Nx)'\|_\infty \\
&= \sup_{t\in[0,\infty)} \Big| \frac{1}{q(t)} \int_t^\infty \Big(f(s, x(s), x'(s)) \\
&\quad - h(s) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_\varsigma^\infty f(\tau, x(\tau), x'(\tau)) d\tau d\varsigma \Big) ds \Big| \\
&\leq \sup_{t\in[0,\infty)} \frac{1}{q(t)} \int_0^\infty \Big(\varphi_r(s) + |h(s)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_0^\infty \varphi_r(\tau) d\tau d\varsigma \Big) ds \\
&\leq \|q^{-1}\|_\infty \|\varphi_r\|_1 \Big(1 + \|h\|_1 \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma \Big) < \infty.
\end{aligned}$$

It follows that $K_{P,Q}N(M)$ is uniformly bounded in X.

Let $x \in M$ and t_1, $t_2 \in [0, T]$ with $T \in (0, \infty)$, we have

$$\begin{aligned}
&|K_{P,Q}Nx(t_2) - K_{P,Q}Nx(t_1)| \\
&= \Big| \int_{t_1}^{t_2} \frac{1}{q(s)} \int_s^\infty \Big(f(\tau, x(\tau), x'(\tau)) \\
&\quad - h(\tau) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_\varsigma^\infty f(\zeta, x(\zeta), x'(\zeta)) d\zeta d\varsigma \Big) d\tau ds \Big| \\
&\leq \int_{t_1}^{t_2} \frac{1}{q(s)} \int_0^\infty \Big(\varphi_r(\tau) + |h(\tau)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_0^\infty \varphi_r(\zeta) d\zeta d\varsigma \Big) d\tau ds
\end{aligned}$$

$$\le \int_{t_1}^{t_2} \frac{1}{q(s)} \|\varphi_r\|_1 \Big(1 + \|h\|_1 \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma\Big) ds \to 0, \quad \text{as } t_1 \to t_2,$$

and

$$\begin{aligned}
&|(K_{P,Q}Nx)'(t_2) - (K_{P,Q}Nx)'(t_1)| \\
&= \Big| \frac{1}{q(t_2)} \int_{t_2}^{\infty} \Big(f(s, x(s), x'(s)) - h(s) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} f(\tau, x(\tau), x'(\tau)) d\tau d\varsigma \Big) ds \\
&\quad - \frac{1}{q(t_1)} \int_{t_1}^{\infty} \Big(f(s, x(s), x'(s)) \\
&\quad - h(s) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} f(\tau, x(\tau), x'(\tau)) d\tau d\varsigma \Big) ds \Big| \\
&\le \Big| \frac{1}{q(t_2)} - \frac{1}{q(t_1)} \Big| \int_{t_2}^{\infty} \Big(|f(s, x(s), x'(s))| \\
&\quad + |h(s)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} |f(\tau, x(\tau), x'(\tau))| d\tau d\varsigma \Big) ds \\
&\quad + \frac{1}{q(t_1)} \int_{t_1}^{t_2} \Big(|f(s, x(s), x'(s))| \\
&\quad + |h(s)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} |f(\tau, x(\tau), x'(\tau))| d\tau d\varsigma \Big) ds \\
&\le \|q^{-1}\|_{\infty}^2 |q(t_1) - q(t_2)| \|\varphi_r\|_1 \Big(1 + \|h\|_1 \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma\Big) \\
&\quad + \|q^{-1}\|_{\infty} \int_{t_1}^{t_2} \Big(\varphi_r(s)) + |h(s)| \sum_{i=1}^{m-2} |\alpha_i| \int_0^{\xi_i} \frac{1}{q(\varsigma)} d\varsigma \|\varphi_r\|_1 \Big) ds \to 0, \quad \text{as } t_1 \to t_2.
\end{aligned}$$

So $K_{P,Q}N(E)$ is equicontinuous on every compact subset of $[0, \infty)$.

We introduce the following notation:

$$\begin{aligned}
&K_{P,Q}Nx(\infty) = \lim_{t \to \infty} K_{P,Q}Nx(t) \\
&= \int_0^{\infty} \frac{1}{q(s)} \int_s^{\infty} \Big(f(\tau, x(\tau), x'(\tau)) \\
&\quad - h(\tau) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} f(\zeta, x(\zeta), x'(\zeta)) d\zeta d\varsigma \Big) d\tau ds,
\end{aligned}$$

and

$$\begin{aligned}
&(K_{P,Q}Nx)'(\infty) = \lim_{t \to \infty} (K_{P,Q}Nx)'(t) \\
&= \lim_{t \to \infty} \frac{1}{q(t)} \int_t^{\infty} \Big(f(s, x(s), x'(s)) \\
&\quad - h(s) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_I} \frac{1}{q(\varsigma)} \int_{\varsigma}^{\infty} f(\tau, x(\tau), x'(\tau)) d\tau d\varsigma \Big) ds = 0.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&|K_{P,Q}Nx(t)-K_{P,Q}Nx(\infty)| \\
&=\Big|\int_t^\infty \frac{1}{q(s)}\int_s^\infty \Big(f(\tau,x(\tau),x'(\tau)) \\
&\quad -h(\tau)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{q(\varsigma)}\int_\varsigma^\infty f(\zeta,x(\zeta),x'(\zeta))d\zeta d\varsigma\Big)d\tau ds\Big| \\
&\le \int_t^\infty \frac{1}{q(s)}\int_s^\infty \Big(\varphi_r(\tau)+|h(\tau)|\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\xi_i}\frac{1}{q(\varsigma)}\int_\varsigma^\infty \varphi_r(\zeta)d\zeta d\varsigma\Big)d\tau ds \\
&\le \int_t^\infty \frac{1}{q(s)}\|\varphi\|_1\Big(1+\|h\|_1\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\xi_i}\frac{1}{q(\tau)}d\tau\Big)ds\to 0, \quad \text{uniformly as } t\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&|(K_{P,Q}Nx)'(t)-(K_{P,Q}Nx)'(\infty)| \\
&=\Big|\frac{1}{q(t)}\int_t^\infty \Big(f(s,x(s),x'(s))-h(s)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_I}\frac{1}{q(\varsigma)}\int_\varsigma^\infty f(\tau,x(\tau),x'(\tau))d\tau d\varsigma\Big)ds\Big| \\
&\le \frac{1}{q(t)}\int_t^\infty \Big(\varphi_r(s)+|h(s)|\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\xi_I}\frac{1}{q(\varsigma)}d\varsigma\|\varphi_r\|_1\Big)ds\to 0,
\end{aligned}
$$

uniformly as $t\to\infty$. Therefore, $K_{P,Q}N(M)$ is equiconvergent. It follows from Lemma 2.3 that $K_{P,Q}N(M)$ is relatively compact for each bounded $M\in X$. The continuity of $K_{P,Q}N(M)$ follows from the Lebesgue Dominated Theorem. We can easily see that QN is continuous and $QN(M)$ is relatively compact. Thus, by Definition 2.1, we have that the mapping $N:X\to Z$ is L-completely continuous. □

3. Main results

Theorem 3.1. *Let $f:[0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ be a Carathéodory function, in addition, assume that*

(H_0) $\sum_{i=1}^{m-2}\alpha_i=1$, $\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\xi_i}\frac{1}{q(s)}ds<\infty$ *and* $\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{e^{-s}}{q(s)}ds\neq 0$;

(H1) *There exists a constant $M>0$, such that for all $x\in\operatorname{dom}L\setminus\ker L$ if $|x(t)|>M$, $t\in[0,\infty)$, then*

$$
h(t)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{q(s)}\int_s^\infty f(\tau,x(\tau),x'(\tau))d\tau ds\neq 0 \tag{3.1}
$$

(H2) *There exist $\beta,\gamma,\delta,\rho:[0,\infty)\to[0,\infty)$, $\beta,\gamma,\delta,\rho\in L_1[0,\infty)$, and constant $\theta\in[0,1)$, such that for all $(x_1,x_2)\in\mathbb{R}^2$, $t\in[0,\infty)$ satisfying one of the following inequalities*

$$
|f(t,x_1,x_2)|\le\beta(t)|x_1|+\gamma(t)|x'|+\delta(t)|x_2|^\theta+\rho(t), \tag{3.2}
$$

$$
|f(t,x_1,x_2)|\le\beta(t)|x_1|+\gamma(t)|x'|+\delta(t)|x_1|^\theta+\rho(t), \tag{3.3}
$$

(H3) *There exists a constant $N^* > 0$, such that for all $c \in \mathbb{R}$, if $|c| > N^*$, then, either*
$$c \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^{\infty} f(\tau, c, 0) d\tau ds < 0, \tag{3.4}$$
or
$$c \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{q(s)} \int_s^{\infty} f(\tau, c, 0) d\tau ds > 0. \tag{3.5}$$
Then (1.1)-(1.2) has at least one solution if
$$\max\{2\|q^{-1}\|_1, \|q^{-1}\|_1 + \|q^{-1}\|_\infty\}(\|\beta\|_1 + \|\gamma\|_1) < 1.$$

Proof. Set
$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx, \lambda \in [0, 1]\}.$$
For $x \in \Omega_1$, since $Lx = \lambda Nx$, thus, $\lambda \neq 0$, $Nx \in \operatorname{Im} L = \ker Q$, hence,
$$h(t) \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(\tau, x(\tau), x'(\tau)) d\tau ds = 0.$$
Thus, by (H1), there exists $t_0 \in [0, \infty)$, such that $|x(t_0)| \leq M$. In view of
$$|x(0)| = |x(t_0) - \int_0^{t_0} x'(s) ds| \leq M + \|x'\|_1.$$
In addition,
$$x'(t) = -\frac{1}{q(t)} \int_t^{\infty} (q(s)x'(s))' ds = -\int_t^{\infty} Lx(s) ds,$$
which implies
$$\|x'\|_\infty = \sup_{t \in [0,\infty)} \big| -\frac{1}{q(t)} \int_t^{\infty} Lx(s) ds \big| \leq \|q^{-1}\|_\infty \|Lx\|_1 \leq \|q^{-1}\|_\infty \|Nx\|_1,$$
and
$$\|x'\|_1 = \int_0^{\infty} \Big| -\frac{1}{q(\tau)} \int_\tau^{\infty} Lx(s) ds \Big| d\tau \leq \|q^{-1}\|_1 \|Lx\|_1 \leq \|q^{-1}\|_1 \|Nx\|_1.$$
Thus,
$$|x(0)| \leq M + \|q^{-1}\|_1 \|Nx\|_1. \tag{3.6}$$
Again for all $x \in \Omega_1$, $(I - P)x \in \operatorname{dom} L \cap \ker P$, $LPx = 0$, thus, from Lemma 2.4, we get
$$\begin{aligned} \|(I - P)x\| = \|K_P(I - P)x\| &\leq \max\{\|q^{-1}\|_1, \|q^{-1}\|_\infty\} \|L(I - P)x\|_1 \\ &= \max\{\|q^{-1}\|_1, \|q^{-1}\|_\infty\} \|Lx\|_1 \\ &\leq \max\{\|q^{-1}\|_1, \|q^{-1}\|_\infty\} \|Nx\|_1. \end{aligned} \tag{3.7}$$
Hence, we have from (3.1) that
$$\begin{aligned} \|x\| &\leq \|Px\| + \|(I - P)x\| \\ &\leq M + \|q^{-1}\|_1 \|Nx\|_1 + \max\{\|q^{-1}\|_1, \|q^{-1}\|_\infty\} \|Nx\|_1 \\ &\leq M + \max\{2\|q^{-1}\|_1, \|q^{-1}\|_1 + \|q^{-1}\|_\infty\} \|Nx\|_1. \end{aligned} \tag{3.8}$$
Let $\Lambda = \max\{2\|q^{-1}\|_1, \|q^{-1}\|_1 + \|q^{-1}\|_\infty\}$. If (3.2) holds, then from (3.8), we get
$$\|x\| \leq M + \Lambda \|Nx\|_1 \leq M + \Lambda(\|\beta\|_1 \|x\|_\infty + \|\gamma\|_1 \|x'\|_\infty + \|\delta\|_1 \|x'\|_\infty^\theta + \|\rho\|_1). \tag{3.9}$$

Thus, from $\|x\|_\infty \le \|x\|$ and (3.9), we have

$$\|x\|_\infty \le \frac{M + \Lambda(\|\beta\|_1\|x\|_\infty + \|\gamma\|_1\|x'\|_\infty + \|\delta\|_1\|x'\|_\infty^\theta + \|\rho\|_1)}{1-\Lambda\|\beta\|_1}. \tag{3.10}$$

It follows from $\|x'\|_\infty \le \|x\|$, (3.9) and (3.10) that

$$\begin{aligned}\|x'\|_\infty &\le \Lambda\|\beta\|_1\|x\|_\infty + \Lambda\Big(\|\gamma\|_1\|x'\|_\infty + \|\delta\|_1\|x'\|_\infty^\theta + \|\rho\|_1 + \frac{M}{\Lambda}\Big)\\ &\le \frac{\Lambda\|\gamma\|_1}{1-\Lambda\|\beta\|_1}\|x'\|_\infty + \frac{\Lambda\|\delta\|_1}{1-\Lambda\|\beta\|_1}\|x'\|_\infty^\theta + \frac{\Lambda\|\rho\|_1 + M}{1-\Lambda\|\beta\|_1}.\end{aligned}$$

So

$$\|x'\|_\infty \le \frac{\Lambda\|\delta\|_1}{1-\Lambda(\|\beta\|_1+\|\gamma\|_1)}\|x'\|_\infty^\theta + \frac{\Lambda\|\rho\|_1+M}{1-\Lambda(\|\beta\|_1+\|\gamma\|_1)}. \tag{3.11}$$

Since $\theta \in [0,1)$, by (3.11), there exists $M_1 > 0$, such that

$$\|x'\|_\infty \le M_1. \tag{3.12}$$

Similar, by (3.10) and (3.12), there exists $M_2 > 0$, such that

$$\|x\|_\infty \le M_2. \tag{3.13}$$

Hence,

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\} \le \max\{M_1, M_2\}.$$

Then Ω_1 is bounded.

If (3.3) holds, similar to the above argument, we can prove that Ω_1 is bounded too. Let

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, then we have $x = c \in \mathbb{R}$, thus,

$$\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{q(s)}\int_s^\infty f(\tau,c,0)d\tau ds = 0. \tag{3.14}$$

Then, we have by (H3) and (3.14) that

$$\|x\| = |c| \le N^*,$$

which implies that Ω_2 is bounded. We define the isomorphism $J : \operatorname{Im} Q \to \ker L$ by

$$J(ch(t)) = c, \quad c \in \mathbb{R},\ t \in [0,\infty).$$

If (3.4) holds, set

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda)JQNx = 0,\ \lambda \in [0,1]\}.$$

For every $c_0 \in \Omega_3$, we obtain

$$\lambda c_0 = (1-\lambda)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{q(s)}\int_s^\infty f(\tau,c_0,0)d\tau ds.$$

If $\lambda = 1$, then $c_0 = 0$ and if $|c_0| > N^*$, in view of (3.4), one has

$$\lambda c_0^2 = (1-\lambda)c_0\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{q(s)}\int_s^\infty f(\tau,c_0,0)d\tau ds < 0,$$

which contradicts $\lambda c_0^2 \ge 0$. Thus, Ω_3 is bounded.

If (3.5) holds, then let

$$\Omega_3 = \{x \in \ker L : \lambda x + (1-\lambda)JQNx = 0,\ \lambda \in [0,1]\},$$

similar to the above argument, we can show that Ω_3 is bounded.

In the following, we shall prove that all conditions of Lemma 2.4 are satisfied. Let Ω to be a bounded open subset of X such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. Then by the above argument, we have

(1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$.

Lastly, we will prove that (3) of Lemma 2.4 is satisfied. Define

$$H(x, \lambda) = \pm\lambda x + (1 - \lambda)QNx.$$

It is obvious that $H(x, \lambda) \neq 0$ for every $x \in \partial\Omega \cap \ker L$. Thus,

$$\begin{aligned}\deg(JQN|_{\ker L\cap\partial\Omega}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0)\\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0)\\ &= \deg(\pm I, \Omega \cap \ker L, 0) \neq 0.\end{aligned}$$

Then by Lemma 2.4, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. In other words, (1.1)-(1.2) has at least one solution in $C^1[0, \infty)$. □

Acknowledgements. The author would like to express his sincere appreciation to the anonymous referee for his/her helpful comments in improving the presentation and quality of this article.

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