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MULTIPLE SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

WENJING CHEN, JIANFU YANG

Abstract. Let $(\mathcal{M}, g)$ be a compact, connected, orientable, Riemannian n-manifold of class C^∞ with Riemannian metric g $(n \geq 3)$. We study the existence of solutions to the equation

$$-\varepsilon^2 \Delta_g u + V(x)u = K(x)|u|^{p-2}u$$

on this Riemannian manifold. Here $2 < p < 2^* = 2n/(n-2)$, $V(x)$ and $K(x)$ are continuous functions. We show that the shape of $V(x)$ and $K(x)$ affects the number of solutions, and then prove the existence of multiple solutions.

1. Introduction

In this article, we consider the existence of solutions of the problem

$$-\varepsilon^2 \Delta_g u + V(x)u = K(x)|u|^{p-2}u \quad \text{in } \mathcal{M}, \tag{1.1}$$

where $(\mathcal{M}, g)$ is a compact, connected, orientable, Riemannian manifold of class C^∞ with Riemannian metric g, $\dim \mathcal{M} = n \geq 3$, $2 < p < 2^* = \frac{2n}{n-2}$ and Δ_g is the Laplace-Beltrami operator.

In the whole space $\mathbb{R}^n$, problem (1.1) is the so-called Schrödinger equation. The existence of solutions of Schrödinger problem (1.1) has been extensively investigated, mainly in the semiclassical limit $\varepsilon \to 0$, see for instance [1], [2], [7], [8], [10], [15], [17], [18]. In particular, it was found in [15] a mountain pass solution of problem (1.1) in the case $K(x) = 1$. Later on, it was shown in [17] that the maximum point of the mountain pass solution concentrates at the minimum point of V as $\varepsilon \to 0$. In the case $K(x) \neq$const., Wang and Zeng found in [18] a ground state solution for ε small. Furthermore, they studied the concentration behavior of such a solution as $\varepsilon \to 0$. In [8], it was shown that the number of solutions of problem (1.1) is affected by the shape of functions V and K. In fact, in [8] the number of solutions of problem (1.1) was related to the topology of the set of global minimum points of certain function. On the other hand, for a bounded domain Ω in $\mathbb{R}^N$ with rich topology, Benci and Cerami proved that problem (1.1) with $V = K = 1$ has at least *cat* Ω positive solutions. Such a result was recently generalized to compact manifolds. In [3], the authors showed that problem (1.1) with $V = K = 1$ and positive mass possesses at least $\text{cat}(\mathcal{M}) + 1$ solutions, while for the zero mass case,

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similar results were obtained in [16]. Inspired by [3], [8] and [16], we consider in this paper the effect of coefficients V, K on the existence of number of solutions.

Problem (1.1) is related to the problem

$$-\Delta u + V(\eta)u = K(\eta)|u|^{p-2}u \quad \text{in } \mathbb{R}^n \tag{1.2}$$

for fixed $\eta \in \mathcal{M}$. It is well known that the problem

$$-\Delta u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^n \quad u > 0, \tag{1.3}$$

has a positive radial solution U; see for instance [5]. The function U and its radial derivatives satisfy the following decaying law

$$U(r) \sim e^{-|r|}|r|^{-\frac{n-1}{2}}, \quad \lim_{r\to\infty} \frac{U'(r)}{U(r)} = 1, \quad r = |x|.$$

By a result in [13], U is the unique positive solution of problem (1.3). We may verify that $w(z) := \big(\frac{V(\eta)}{K(\eta)}\big)^{1/(p-2)} U\Big(\big(V(\eta)\big)^{1/2} z\Big)$ with $K(\eta) > 0$ is a ground state solution of problem (1.2); that is, it is the minimizer of the variational problem

$$c_\eta := \inf_{u\in N_\eta} E_\eta(u),$$

where

$$E_\eta(u) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + V(\eta)u^2)\,dz - \frac{1}{p}\int_{\mathbb{R}^n} K(\eta)|u|^p\,dz$$

is the associated energy functional of problem (1.2) and

$$N_\eta := \big\{u \in H^1(\mathbb{R}^n)\backslash\{0\} : \int_{\mathbb{R}^n} (|\nabla u|^2 + V(\eta)u^2)\,dz = \int_{\mathbb{R}^n} K(\eta)|u|^p\,dz\big\}$$

is the related Nehari manifold. In fact,

$$c_\eta = E_\eta(w) = \big(\frac{1}{2} - \frac{1}{p}\big)\frac{V^{\frac{p}{p-2}-\frac{n}{2}}(\eta)}{K^{\frac{2}{p-2}}(\eta)} \int_{\mathbb{R}^n} |U(z)|^p\,dz.$$

Let

$$c_0 = \inf_{\eta\in\mathcal{M}} c_\eta \quad \text{and} \quad \Omega := \{\eta \in \mathcal{M} : c_\eta = c_0\}.$$

For $\delta > 0$ let

$$\Omega_\delta := \{\xi \in \mathcal{M} : \inf_{\eta\in\Omega} \|\xi - \eta\|_g \le \delta\}.$$

We assume in this paper that $V, K \in C(\mathcal{M}, \mathbb{R})$ and there is a positive number $\nu > 0$ such that $V, K \ge \nu > 0$. Denote by $\operatorname{cat}_X(A)$ the Ljusternik-Schirelmann category of A in X. Let

$$K_{\max} = \max_{x\in\mathcal{M}} K(x), \quad K_{\min} = \min_{x\in\mathcal{M}} K(x).$$

Our main result is the following.

Theorem 1.1. *Problem* (1.1) *has at least* $\operatorname{cat}_{\Omega_\delta}(\Omega)$ *positive solutions for* $\varepsilon > 0$ *small.*

Solutions of problem (1.1) will be found as critical points of the associated functional

$$I_\varepsilon(u) = \frac{1}{\varepsilon^n}\Big(\frac{1}{2}\int_{\mathcal{M}} \big(\varepsilon^2|\nabla_g u(x)|^2 + V(x)u^2\big)\,d\mu_g - \frac{1}{p}\int_{\mathcal{M}} K(x)|u^+|^p\,d\mu_g\Big),$$

in the Hilbert space

$$H_g^1(\mathcal{M}) := \{u : \mathcal{M} \to \mathbb{R} : \int_{\mathcal{M}} (|\nabla_g u|^2 + u^2)\, d\mu_g < \infty\}$$

with the norm

$$\|u\|_g = \Big(\int_{\mathcal{M}} (|\nabla_g u|^2 + u^2)\, d\mu_g \Big)^{1/2},$$

where $d\mu_g = \sqrt{\det g} dz$ denotes the volume form on $\mathcal{M}$ associated with the metric g. For $\sigma > 0$, let

$$\Sigma_{\varepsilon,\sigma} := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) < c_0 + \sigma\}$$

be a subset of the Nehari manifold

$$\mathcal{N}_\varepsilon := \{u \in H_g^1(\mathcal{M})\backslash\{0\} : \int_{\mathcal{M}} (\varepsilon^2 |\nabla_g u(x)|^2 + V(x) u^2)\, d\mu_g = \int_{\mathcal{M}} K(x)|u^+|^p\, d\mu_g\}$$

related to the functional I_ε. To prove Theorem 1.1, we first show that problem (1.1) has at least $\operatorname{cat}_{\Sigma_{\varepsilon,\sigma}} \Sigma_{\varepsilon,\sigma}$ solutions, then we need to relate $\operatorname{cat}_{\Sigma_{\varepsilon,\sigma}} \Sigma_{\varepsilon,\sigma}$ with $\operatorname{cat}_{\Omega_\delta} \Omega$. By a result in [11], we know that $\mathcal{M}$ can be isometrically embedded in a Euclidean space $\mathbb{R}^N$ as a regular sub-manifold with $N > 2n$. For any set $\omega \subset \mathcal{M}$ and $r > 0$, we define

$$[\omega]_r := \{z \in \mathbb{R}^N : \operatorname{dist}(z, \omega) \le r\}$$

a subset of $\mathbb{R}^N$, where $\operatorname{dist}(z,\omega)$ denotes the distance between z and ω with respect to the Euclidian metric in $\mathbb{R}^N$. Let $r = r(\Omega_\delta)$ be the radius of topological invariance of Ω_δ, which is defined by

$$r(\Omega_\delta) := \sup\{l > 0 : \operatorname{cat}([\Omega_\delta]_l) = \operatorname{cat}(\Omega_\delta)\}.$$

We choose $r > 0$ so small that the metric projection

$$\Pi : [\Omega_\delta]_r \subset \mathbb{R}^N \to \Omega_\delta$$

is well defined. We will construct a function $\phi_\varepsilon : \Omega \to \Sigma_{\varepsilon,\sigma}$ and a function $\beta : \Sigma_{\varepsilon,\sigma} \to [\Omega_\delta]_r$ such that

$$\Omega \xrightarrow{\phi_\varepsilon} \Sigma_{\varepsilon,\sigma} \xrightarrow{\beta} [\Omega_\delta]_r \xrightarrow{\Pi} \Omega_\delta,$$

and $\Pi \circ \beta \circ \phi_\varepsilon$ is homotopic to the identity on Ω_δ. It implies that $\operatorname{cat}_{\Sigma_{\varepsilon,\sigma}} \Sigma_{\varepsilon,\sigma} \ge \operatorname{cat}_{\Omega_\delta} \Omega$.

In section 2, we outline our frame of work. The mappings ϕ_ε and β are constructed in section 3 and section 4 respectively.

2. The framework and preliminary results

Let $\mathcal{M}$ be a compact Riemannian manifolds of class C^∞. On the tangent bundle of $\mathcal{M}$ we define the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ which has the following properties: (i) exp is of class C^∞; (ii) there exists a constant $R > 0$ such that $\exp_x \big|_{B(0,R)} : B(0,R) \to B_g(x,R)$ is a diffeomorphism for all $x \in \mathcal{M}$. Fix such an R in this paper and denote by $B(0,R)$ the ball in $\mathbb{R}^n$ centered at 0 with radius R and $B_g(x,R)$ the ball in $\mathcal{M}$ centered at x with radius R with respect to the distance induced by the metric g. Let $\mathcal{C}$ be the atlas on $\mathcal{M}$ whose charts are given by the exponential map and $\mathcal{P} = \{\psi_C\}_{C \in \mathcal{C}}$ be a partition of unity subordinate to the atlas $\mathcal{C}$. For $u \in H_g^1(\mathcal{M})$, we have

$$\int_{\mathcal{M}} |\nabla_g u|^2\, d\mu_g = \sum_{C \in \mathcal{C}} \int_C \psi_C(x) |\nabla_g u|^2\, d\mu_g.$$

Moreover, if u has support inside one chart $C = B_g(\eta, R)$, then

$$\begin{aligned}&\int_{\mathcal{M}} |\nabla_g u|^2 \, d\mu_g \\ &= \int_{B(0,R)} \psi_C(\exp_{x_0}(z)) g^{ij}_{x_0}(z) \frac{\partial u(\exp_{x_0}(z))}{\partial z_i} \frac{\partial u(\exp_{x_0}(z))}{\partial z_j} |g_{x_0}(z)|^{1/2} \, dz,\end{aligned}$$

where g_{x_0} denotes the Riemannian metric reading in $B(0, R)$ through the normal coordinates defined by the exponential map $\exp_{x_0}$. In particular, $g_{x_0}(0) = Id$. We let $|g_{x_0}(z)| := det(g_{x_0}(z))$ and $(g^{ij}_{x_0})(z)$ is the inverse matrix of $g_{x_0}(z)$. Since $\mathcal{M}$ is compact, there are two strictly positive constants h and H such that

$$\forall x \in \mathcal{M}, \quad \forall \upsilon \in T_x\mathcal{M}, \quad h\|\upsilon\|^2 \le g_x(\upsilon, \upsilon) \le H\|\upsilon\|^2.$$

Hence, we have

$$\forall x \in \mathcal{M}, \quad h^n \le |g_x| \le H^n.$$

Theorem 1.1 will follow from the following result in [14].

Proposition 2.1. *Let $\mathcal{N}$ be a $C^{1,1}$ complete Riemannian manifold modeled on a Hilbert space and J be a C^1 functional on $\mathcal{N}$ bounded from below. If there exists $b > \inf_{\mathcal{N}} J$ such that J satisfies the Palais-Smale condition on the sublevel $J^{-1}(-\infty, b)$, then for any noncritical level a, with $a < b$, there exist at least $\operatorname{cat}_{J^a}(J^a)$ critical points of J in J^a, where $J^a := \{u \in \mathcal{N} | J(u) \le a\}$.*

We need also the following Lemma.

Lemma 2.2. *Let X and Y be topological spaces, $Z \subset Y$ be a closed set and $h_1 \in C(Z, X)$, $h_2 \in C(X, Y)$ with h_2 being a closed mapping. Suppose that $h_2 \circ h_1 : Z \to Y$ is homotopic to the identity mapping Id in Y, then $\operatorname{cat}_X(X) \ge \operatorname{cat}_Y(Z)$.*

Proof. Let $k = \operatorname{cat}_X(X)$, there exist closed sets $V_1, V_2, \cdots, V_k$ such that $X = \bigcup_{1 \le i \le k} V_i$ and each V_i is contractible in X. Since $h_2 \in C(X, Y)$ and h_2 being a closed mapping, each $h_2(V_i)$ is closed and contractible in Y, then

$$\operatorname{cat}_X(X) \ge \operatorname{cat}_Y(h_2(X)). \tag{2.1}$$

Since $h_2 \circ h_1(Z) \subset h_2(X)$, we have

$$\operatorname{cat}_Y(h_2(X)) \ge \operatorname{cat}_Y(h_2 \circ h_1(Z)). \tag{2.2}$$

On the other hand, $h_2 \circ h_1 : Z \to Y$ is homotopic to the identity mapping Id in Y, thus

$$\operatorname{cat}_Y(h_2 \circ h_1(Z)) \ge \operatorname{cat}_Y(Z). \tag{2.3}$$

By (2.1)-(2.3), $\operatorname{cat}_X(X) \ge \operatorname{cat}_Y(Z)$. □

3. The function ϕ_ε

We know that $\mathcal{N}_\varepsilon$ is a $C^{1,1}$ manifold. If $u \in \mathcal{N}_\varepsilon$, we have $\|u\|_g \ge C > 0$, C is independent of u. For $u \in H^1_g(\mathcal{M})$, there exists a unique $t_\varepsilon(u) > 0$, $t_\varepsilon : H^1_g(\mathcal{M})\backslash\{0\} \to \mathbb{R}^+$, such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$ and

$$I_\varepsilon(t_\varepsilon(u)u) = \max_{t \ge 0} I_\varepsilon(tu).$$

More precisely,

$$t^{p-2}_\varepsilon(u) = \frac{\int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u(x)|^2 + V(x)u^2\right) \, d\mu_g}{\int_{\mathcal{M}} K(x)|u^+|^p \, d\mu_g}. \tag{3.1}$$

The function $t_\varepsilon(u)$ is C^1. Let us define a smooth real function χ_R on $\mathbb{R}^+$ such that

$$\chi_R(t) := \begin{cases} 1 & \text{if } 0 \le t \le \frac{R}{2}; \\ 0 & \text{if } t \ge R\,. \end{cases} \tag{3.2}$$

and $|\chi_R'(t)| \le \frac{2}{R}$. Fixing $\eta \in \Omega$ and $\varepsilon > 0$, we define

$$W_{\eta,\varepsilon}(x) := \begin{cases} w_\varepsilon(\exp_\eta^{-1}(x))\chi_R(|\exp_\eta^{-1}(x)|) & \text{if } x \in B_g(\eta, R); \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $w(z)$ is the ground state solution of problem (1.2) and $w_\varepsilon(z) = w(\frac{z}{\varepsilon})$. We define $\phi_\varepsilon : \Omega \to \mathcal{N}_\varepsilon$ by

$$\phi_\varepsilon(\eta) = t_\varepsilon(W_{\eta,\varepsilon}(x))W_{\eta,\varepsilon}(x). \tag{3.4}$$

Lemma 3.1. *With the above notation, we have*

$$\frac{1}{\varepsilon^n}\int_{\mathcal{M}} \varepsilon^2|\nabla_g W_{\eta,\varepsilon}(x)|^2\,d\mu_g \to \int_{\mathbb{R}^n}|\nabla w|^2 dz \quad \text{as } \varepsilon \to 0. \tag{3.5}$$

$$\frac{1}{\varepsilon^n}\int_{\mathcal{M}} V(x)|W_{\eta,\varepsilon}(x)|^2\,d\mu_g \to \int_{\mathbb{R}^n} V(\eta)w^2(z)dz \quad \text{as } \varepsilon \to 0, \tag{3.6}$$

$$\frac{1}{\varepsilon^n}\int_{\mathcal{M}} K(x)|W_{\eta,\varepsilon}(x)|^p\ \mu_g \to \int_{\mathbb{R}^n} K(\eta)w^p(z)dz \quad \text{as } \varepsilon \to 0. \tag{3.7}$$

Proof. We have

$$\begin{aligned}
&\Big|\frac{1}{\varepsilon^n}\int_{\mathcal{M}} \varepsilon^2|\nabla_g W_{\eta,\varepsilon}(x)|^2\,d\mu_g - \int_{\mathbb{R}^n}|\nabla w|^2 dz\Big| \\
&= \Big|\frac{1}{\varepsilon^n}\int_{B_g(\eta,R)} \varepsilon^2\big|\nabla_g\left(w_\varepsilon(\exp_\eta^{-1}(x))\chi_R(|\exp_\eta^{-1}(x)|)\right)\big|^2\,d\mu_g - \int_{\mathbb{R}^n}|\nabla w|^2 dz\Big| \\
&= \Big|\frac{1}{\varepsilon^n}\int_{B(0,R)} \varepsilon^2\big|\nabla\left(w_\varepsilon(z)\chi_R(|z|)\right)\big|_g^2 |g_\eta(z)|^{1/2}\,dz - \int_{\mathbb{R}^n}|\nabla w|^2 dz\Big| \\
&= \Big|\int_{B(0,\frac{R}{\varepsilon})} \Big|\nabla\Big(w(z)\chi_{\frac{R}{\varepsilon}}(|z|)\Big)\Big|_g^2 |g_\eta(\varepsilon z)|^{1/2}\ dz - \int_{\mathbb{R}^n}|\nabla w|^2 dz\Big| \\
&\le \int_{\mathbb{R}^n}\Big|\sum_{i,j=1}^n \frac{\partial w(z)}{\partial z_i}\frac{\partial w(z)}{\partial z_j}\Big|\chi_{\frac{R}{\varepsilon}}^2(|z|)g_\eta^{ij}(\varepsilon z)|g_\eta(\varepsilon z)|^{1/2} - \delta_{ij}\Big|\Big|dz \\
&\quad + \int_{\mathbb{R}^n}\Big|\sum_{i,j=1}^n g_\eta^{ij}(\varepsilon z)\chi_{\frac{R}{\varepsilon}}(|z|)w(z)\left(\frac{\partial w}{\partial z_i}\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_j} + \frac{\partial w}{\partial z_j}\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_i}\right)\Big||g_\eta(\varepsilon z)|^{1/2}\,dz \\
&\quad + \int_{\mathbb{R}^n}\Big|\sum_{i,j=1}^n g_\eta^{ij}(\varepsilon z)w^2(z)\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_i}\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_j}\Big||g_\eta(\varepsilon z)|^{1/2}\,dz := I_1 + I_2 + I_3.
\end{aligned}$$

By the compactness of the manifold $\mathcal{M}$ and regularity of the exponential map of the Riemannian metric g, we have

$$\lim_{\varepsilon\to 0}\big|\chi_{\frac{R}{\varepsilon}}^2(|z|)g_\eta^{ij}(\varepsilon z)|g_\eta(\varepsilon z)|^{1/2} - \delta_{ij}\big| = 0$$

uniformly with respect to $\eta \in \Omega$, so $I_1 \to 0$ as $\varepsilon \to 0$. By the definition of $\chi_R(t)$,

$$I_2 \le \frac{H^{n/2}}{h}\int_{\mathbb{R}^n}\Big|\sum_{i,j=1}^n w(z)\Big(\frac{\partial w}{\partial z_i}\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_j} + \frac{\partial w}{\partial z_j}\frac{\partial \chi_{\frac{R}{\varepsilon}}(|z|)}{\partial z_i}\Big)\Big|\,dz$$

$$
\begin{aligned}
&\le \frac{4H^{n/2}\varepsilon}{Rh}\int_{\mathbb{R}^n}|w(z)|\,|\nabla w(z)|\,dz\\
&= \frac{4H^{n/2}\varepsilon}{Rh}\Big(\frac{V(\eta)}{K(\eta)}\Big)^{2/(p-2)} V(\eta)^{-n/2}\int_{\mathbb{R}^n}|U(z)|\,|\nabla U(z)|\,dz\\
&\le \frac{2H^{n/2}\varepsilon}{Rh}\frac{V^{\frac{2}{p-2}-\frac{n}{2}}(\eta)}{K^{\frac{2}{p-2}}(\eta)}\int_{\mathbb{R}^n}(|\nabla U(z)|^2+|U(z)|^2)\,dz.
\end{aligned}
$$

Similarly,

$$
I_3 \le \frac{H^{n/2}}{h}\frac{4\varepsilon^2}{R^2}\frac{V^{\frac{2}{p-2}-\frac{n}{2}}(\eta)}{K^{\frac{2}{p-2}}(\eta)}\int_{\mathbb{R}^n}U(z)^2\,dz.
$$

Hence, $I_2+I_3\to 0$ uniformly with respect to $\eta\in\Omega$ as $\varepsilon\to 0$ and (3.5) follows.

Next, we prove (3.6). We have

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon^n}\int_{\mathcal{M}}V(x)|W_{\eta,\varepsilon}(x)|^2\,d\mu_g-\int_{\mathbb{R}^n}V(\eta)w^2(z)dz\Big|\\
&=\Big|\frac{1}{\varepsilon^n}\int_{B_g(\eta,R)}V(x)|w_\varepsilon(\exp_\eta^{-1}(x))\chi_R(|\exp_\eta^{-1}(x)|)|^2\,d\mu_g-\int_{\mathbb{R}^n}V(\eta)w^2(z)dz\Big|\\
&=\Big|\frac{1}{\varepsilon^n}\int_{B(0,R)}V(\exp_\eta(z))|w_\varepsilon(z)\chi_R(|z|)|^2|g_\eta(z)|^{1/2}\,dz-\int_{\mathbb{R}^n}V(\eta)w^2(z)dz\Big|\\
&=\Big|\int_{B(0,\frac{R}{\varepsilon})}V(\exp_\eta(\varepsilon z))|w(z)\chi_R(|\varepsilon z|)|^2|g_\eta(\varepsilon z)|^{1/2}\,dz-\int_{\mathbb{R}^n}V(\eta)w^2(z)\,dz\Big|\\
&\le\Big|\int_{\mathbb{R}^n}\Big[V(\exp_\eta(\varepsilon z))|\chi_R(|\varepsilon z|)|^2|g_\eta(\varepsilon z)|^{1/2}-V(\eta)\Big]\,w^2(z)dz\Big|\\
&\quad+\Big|\int_{\mathbb{R}^n\setminus B(0,\frac{R}{\varepsilon})}\Big[V(\exp_\eta(\varepsilon z))|\chi_R(|\varepsilon z|)|^2|g_\eta(\varepsilon z)|^{1/2}-V(\eta)\Big]\,w^2(z)dz\Big|\\
&:=I_4+I_5.
\end{aligned}
$$

We note that $\exp_\eta(\varepsilon z)\to\eta$ and $g_\eta(\varepsilon z)\to\delta_{ij}$ as $\varepsilon\to 0$, by the continuity of V, $I_4\to 0$. Obviously, $I_5\to 0$. So (3.6) holds. (3.7) can be proved in the same way. □

Proposition 3.2. *For $\varepsilon>0$, the map $\phi_\varepsilon:\Omega\to\mathcal{N}_\varepsilon$ is continuous; and for any $\sigma>0$, there exists $\varepsilon_0>0$ such that if $\varepsilon<\varepsilon_0$ $\phi_\varepsilon(\eta)\in\Sigma_{\varepsilon,\sigma}$ for all $\eta\in\Omega$.*

Proof. The continuity of ϕ_ε can be proved as [3, Proposition 4.2], so we omit the details. Now, we show $\phi_\varepsilon(\eta)\in\Sigma_{\varepsilon,\sigma}$ for $\forall\eta\in\Omega$. By Lemma 3.1,

$$
\begin{aligned}
t_\varepsilon^{p-2}(W_{\eta,\varepsilon}(x))&=\frac{\frac{1}{\varepsilon^n}\int_{\mathcal{M}}\varepsilon^2|\nabla_g W_{\eta,\varepsilon}(x)(x)|^2d\mu_g+\frac{1}{\varepsilon^n}\int_{\mathcal{M}}V(x)\left(W_{\eta,\varepsilon}(x)\right)^2\,d\mu_g}{\frac{1}{\varepsilon^n}\int_{\mathcal{M}}K(x)|W^+_{\eta,\varepsilon}(x)|^p\,d\mu_g}\\
&\to\frac{\int_{\mathbb{R}^n}|\nabla w(z)|^2\,dz+\int_{\mathbb{R}^n}V(\eta)w^2(z)\,dz}{\int_{\mathbb{R}^n}K(\eta)w^p(z)\,dz}=1.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_\varepsilon(\phi_\varepsilon(\eta))&=I_\varepsilon(t_\varepsilon(W_{\eta,\varepsilon}(x))W_{\eta,\varepsilon}(x))\\
&=\frac12\int_{\mathbb{R}^n}(|\nabla w(z)|^2+V(\eta)w^2(z))\,dz-\frac1p\int_{\mathbb{R}^n}K(\eta)w^p(z)\,dz+o(1)\\
&=c_\eta+o(1)=c_0+o(1)
\end{aligned}
$$

uniformly with respect to $\eta \in \Omega$ and the proof is completed. $\square$

4. The function β

Let us define the center of mass $\beta(u) \in \mathbb{R}^N$ for $u \in \mathcal{N}_\varepsilon$ by

$$\beta(u) := \frac{\int_{\mathcal{M}} x|u^+(x)|^p \, d\mu_g}{\int_{\mathcal{M}} |u^+(x)|^p \, d\mu_g}.$$

The function β is well defined on $u \in \mathcal{N}_\varepsilon$ since $u^+ \not\equiv 0$ if $u \in \mathcal{N}_\varepsilon$. Let

$$m_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u), \tag{4.1}$$

which is achieved as $\mathcal{M}$ is compact. Since $K(x), V(x)$ are bounded, we may show the following result as in [3, Lemma 5.1].

Lemma 4.1. *There exists a number $\alpha > 0$ such that for any $\varepsilon > 0$, $m_\varepsilon \geq \alpha$.*

For a given $\varepsilon > 0$, let $\mathcal{P}_\varepsilon = \{P_j^\varepsilon\}_{j \in \Lambda_\varepsilon}$ be a finite good partition of the manifold $\mathcal{M}$ introduced in [3]: if for any $j \in \Lambda_\varepsilon$ the set partition P_j^ε is closed; $P_j^\varepsilon \cap P_i^\varepsilon \subseteq \partial P_j^\varepsilon \cap \partial P_i^\varepsilon$ for any $i \neq j$; there exist $r_1(\varepsilon) \geq r_2(\varepsilon) > 0$ such that there are points $q_j^\varepsilon \in P_j^\varepsilon$ for any j, satisfying $B_g(q_j^\varepsilon, \varepsilon) \subset P_j^\varepsilon \subset B_g(q_j^\varepsilon, r_2(\varepsilon)) \subset B_g(q_j^\varepsilon, r_1(\varepsilon))$ and any point $x \in \mathcal{M}$ is contained in at most $N_{\mathcal{M}}$ balls $B_g(q_j^\varepsilon, r_1(\varepsilon))$, where $N_{\mathcal{M}}$ does not depend on ε. This last condition can be satisfied for ε small enough by the compactness of $\mathcal{M}$, and $r_1(\varepsilon)$, $r_2(\varepsilon)$ can be chosen so that $r_1(\varepsilon) \geq r_2(\varepsilon) \geq (1 + \frac{1}{\Theta})\varepsilon$ with a constant Θ independent on ε. We may assume that the value ε_0 of Proposition 3.2 is small enough for the manifold $\mathcal{M}$ to have good partitions.

Lemma 4.2. *There exists a constant $\gamma > 0$ such that for any fixed $\sigma > 0$, $\varepsilon \in (0, \varepsilon_0)$ and function $u \in \Sigma_{\varepsilon,\sigma}$, there exists a set $\tilde{P}_\sigma^\varepsilon \in \mathcal{P}_\varepsilon$ such that*

$$\frac{1}{\varepsilon^n} \int_{\tilde{P}_\sigma^\varepsilon} K(x)|u^+|^p \, d\mu_g \geq \gamma.$$

Proof. Fixed $\sigma > 0$ and $0 < \varepsilon < \varepsilon_0$. Then for any $u \in \mathcal{N}_\varepsilon$ and any good partition $\mathcal{P}_\varepsilon = \{P_j^\varepsilon\}_{j \in \Lambda_\varepsilon}$, let $u_j^+ = u^+$ on the set P_j^ε. Then

$$\begin{aligned} &\frac{1}{\varepsilon^n} \int_{\mathcal{M}} (\varepsilon^2 |\nabla_g u(x)|^2 + V(x) u^2) \, d\mu_g \\ &= \frac{1}{\varepsilon^n} \int_{\mathcal{M}} K(x)|u^+|^p \, d\mu_g \\ &= \frac{1}{\varepsilon^n} \sum_{j \in \Lambda_\varepsilon} \int_{P_j^\varepsilon} K(x)|u^+|^p \, d\mu_g \\ &\leq \max_j \Big(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g \Big)^{\frac{p-2}{p}} \sum_{j \in \Lambda_\varepsilon} \Big(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g \Big)^{2/p}. \end{aligned} \tag{4.2}$$

Let

$$\chi_\varepsilon(t) := \begin{cases} 1 & \text{if } t \leq r_2(\varepsilon); \\ 0 & \text{if } t > r_1(\varepsilon) \end{cases}$$

be a smooth cutoff function, where $r_1(\varepsilon), r_2(\varepsilon)$ are defined above for good partitions, and assume that $|\chi_\varepsilon'| \leq \frac{\Theta}{\varepsilon}$ uniformly. Let

$$\tilde{u}_j(x) = u^+(x) \chi_\varepsilon(|x - q_j^\varepsilon|).$$

We know that $\tilde{u}_j(x) \in H_g^1(\mathcal{M})$, and $supt(\tilde{u}_j(x)) = B_g(q_j^\varepsilon, r_1(\varepsilon))$. By the definition of u_j^+, we have $u_j^+ = u^+$ on the set $P_j^\varepsilon \subset B_g(q_j^\varepsilon, r_2(\varepsilon)) \subset B_g(q_j^\varepsilon, r_1(\varepsilon))$. By the Sobolev inequality there exists a positive constant C such that for any j,

$$\begin{aligned}
&\Big(\frac{1}{\varepsilon^n}\int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g\Big)^{2/p} \\
&= \Big(\frac{1}{\varepsilon^n}\int_{P_j^\varepsilon} K(x)|u^+|^p \, d\mu_g\Big)^{2/p} \\
&\le \Big(\frac{1}{\varepsilon^n}\int_{B_g(q_j^\varepsilon, r_2(\varepsilon))} K(x)|u^+\chi_\varepsilon(|x-q_j^\varepsilon|)|^p \, d\mu_g\Big)^{2/p} \\
&\le \Big(\frac{1}{\varepsilon^n}\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))} K(x)|u^+\chi_\varepsilon(|x-q_j^\varepsilon|)|^p \, d\mu_g\Big)^{2/p} \\
&= \Big(\frac{1}{\varepsilon^n}\int_{\mathcal{M}} K(x)|\tilde{u}_j|^p \, d\mu_g\Big)^{2/p} \\
&\le K_{\max}^{2/p}\Big(\frac{1}{\varepsilon^n}\int_{\mathcal{M}} |\tilde{u}_j|^p \, d\mu_g\Big)^{2/p} \\
&\le K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{\mathcal{M}} \left(\varepsilon^2|\nabla_g \tilde{u}_j|^2 + |\tilde{u}_j|^2\right) d\mu_g \\
&= K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{P_j^\varepsilon} \left(\varepsilon^2|\nabla_g \tilde{u}_j|^2 + |\tilde{u}_j|^2\right) d\mu_g \\
&\quad + K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \left(\varepsilon^2|\nabla_g \tilde{u}_j|^2 + |\tilde{u}_j|^2\right) d\mu_g \\
&\le K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{\mathcal{M}} \left(\varepsilon^2|\nabla_g u_j^+|^2 + |u_j^+|^2\right) d\mu_g \\
&\quad + K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \left(\varepsilon^2|\nabla_g \tilde{u}_j|^2 + |\tilde{u}_j|^2\right) d\mu_g.
\end{aligned} \tag{4.3}$$

Moveover

$$\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} |\tilde{u}_j|^2 d\mu_g \le \int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} |u^+|^2 d\mu_g, \tag{4.4}$$

and

$$\begin{aligned}
&\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \varepsilon^2 |\nabla_g \tilde{u}_j|^2 d\mu_g \\
&= \int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \varepsilon^2 \left|\nabla_g \left(u^+(x)\chi_\varepsilon(|x-q_j^\varepsilon|)\right)\right|^2 d\mu_g \\
&\le 2\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \varepsilon^2 \left(|\nabla_g u^+|^2 \chi_\varepsilon^2(|x-q_j^\varepsilon|) + \left(\chi_\varepsilon'(|x-q_j^\varepsilon|)\right)^2 |u^+|^2\right) d\mu_g \\
&\le 2\int_{B_g(q_j^\varepsilon, r_1(\varepsilon))\setminus P_j^\varepsilon} \left(\varepsilon^2 |\nabla_g u^+|^2 + \Theta^2 |u^+|^2\right) d\mu_g.
\end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we get

$$\Big((\frac{1}{\varepsilon^n}\int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g\Big)^{2/p} \le K_{\max}^{2/p} C \frac{1}{\varepsilon^n}\int_{\mathcal{M}} \left(\varepsilon^2|\nabla_g u_j^+|^2 + |u_j^+|^2\right) d\mu_g$$

$$+ K_{\max}^{2/p} C C' \frac{1}{\varepsilon^n} \int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u^+|^2 + |u^+|^2\right) d\mu_g,$$

where $C' = \max\{2, 2\Theta^2 + 1\}$. Hence,

$$\begin{aligned}
&\sum_{j\in\Lambda_\varepsilon} \Big(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g\Big)^{2/p} \\
&\leq K_{\max}^{2/p} C \sum_{j\in\Lambda_\varepsilon} \frac{1}{\varepsilon^n} \int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u_j^+|^2 + |u_j^+|^2\right) d\mu_g \\
&\quad + K_{\max}^{2/p} C C' N_{\mathcal{M}} \frac{1}{\varepsilon^n} \int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u^+|^2 + |u^+|^2\right) d\mu_g \\
&\leq K_{\max}^{2/p} C (C'+1) N_{\mathcal{M}} \frac{1}{\varepsilon^n} \int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u^+|^2 + |u^+|^2\right) d\mu_g \\
&\leq K_{\max}^{2/p} C (C'+1) N_{\mathcal{M}} \max\Big\{1, \frac{1}{\nu}\Big\} \frac{1}{\varepsilon^n} \int_{\mathcal{M}} \left(\varepsilon^2 |\nabla_g u|^2 + V(x)|u|^2\right) d\mu_g
\end{aligned} \tag{4.6}$$

From (4.2) and (4.6) we have

$$\begin{aligned}
\max_j \Big\{ \Big(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} K(x)|u^+|^p \, d\mu_g\Big)^{\frac{p-2}{p}} \Big\} &\geq \frac{\frac{1}{\varepsilon^n} \int_{\mathcal{M}} (\varepsilon^2 |\nabla_g u(x)|^2 + V(x) u^2) \, d\mu_g}{\sum_{j\in\Lambda_\varepsilon} \Big(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} K(x)|u_j^+|^p \, d\mu_g\Big)^{2/p}} \\
&\geq \frac{1}{K_{\max}^{2/p} C(C'+1) N_{\mathcal{M}} \max\{1, \frac{1}{\nu}\}}.
\end{aligned}$$

Thus, the proof is completed. □

Lemma 4.3. *Let σ and ε be fixed, and $I_\varepsilon^{m_\varepsilon + 2\sigma} := \{u \in \mathcal{N}_\varepsilon | I_\varepsilon(u) < m_\varepsilon + 2\sigma\}$, where m_ε is defined in (4.1). For any $u \in \Sigma_{\varepsilon,\sigma} \cap I_\varepsilon^{m_\varepsilon+2\sigma}$ there exists $u_\sigma \in \mathcal{N}_\varepsilon$ such that*

$$I_\varepsilon(u_\sigma) < I_\varepsilon(u), \quad |||u_\sigma - u|||_\varepsilon < 4\sqrt{\sigma}, \tag{4.7}$$

where $|||u|||_\varepsilon^2 = \frac{1}{\varepsilon^n} \int_{\mathcal{M}} (\varepsilon^2 |\nabla_g u|^2 + u^2) \, d\mu_g$, and

$$\big|\nabla|_{\mathcal{N}_\varepsilon} I_\varepsilon(u_\sigma)\big| < \sqrt{\sigma} |||\xi|||_\varepsilon. \tag{4.8}$$

The above result follows by the Ekeland principle, also by the proof in [3, Lemma 5.4].

Let $u_k \in \Sigma_{\varepsilon_k,\sigma_k} \cap I_{\varepsilon_k}^{m_{\varepsilon_k}+2\sigma_k}$, where $\varepsilon_k, \sigma_k \to 0$ as $k \to \infty$. For all k, the map $\exp_{\eta_k} : T_{\eta_k}\mathcal{M} \to \mathcal{M}$ is a diffeomorphism on the ball $B_g(\eta_k, R)$. Let $\{\psi_c\}$ be a partition of unity induced on $\mathcal{M}$ by the cover of balls of radius R. By the compactness of $\mathcal{M}$, we can assume that there exists $\rho > 0$ such that for all k

$$\min\big\{\psi_{B_g(\eta_k,R)}(x) | x \in B_g(\eta_k, \frac{R}{\rho})\big\} \geq \psi_0 > 0. \tag{4.9}$$

Let

$$\varphi_k : B_g\big(\eta_k, \frac{R}{\rho}\big) \to B\big(0, \frac{R}{\varepsilon_k \rho}\big) \subset \mathbb{R}^n, \quad \varphi_k := \frac{\exp_{\eta_k}^{-1}}{\varepsilon_k}$$

and define $w_k : \mathbb{R}^n \to \mathbb{R}$ by

$$w_k(z) := \chi_k(z) u_k(\varphi_k^{-1}(z)) = \chi_R(\varepsilon_k |z| \rho)\, u_k(\exp_{\eta_k}(\varepsilon_k z)) = \chi_{\frac{R}{\rho}}(|\exp_{\eta_k}^{-1}(x)|) u_k(x),$$

where $x = \exp_{\eta_k}(\varepsilon_k z) \in \Omega$ and $\chi_k(z) := \chi_{\frac{R}{\varepsilon_k \rho}}(|z|)$. Then, $w_k \in H_0^1\left(B\left(0, \frac{R}{\varepsilon_k \rho}\right)\right) \subset H^1(\mathbb{R}^n)$.

Lemma 4.4. *There exists $\tilde{w} \in H^1(\mathbb{R}^n)$ such that, up to a subsequence, w_k tends to $\tilde{w}$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^p_{loc}(\mathbb{R}^n)$. The limit function $\tilde{w}$ is a ground state solution of the problem*

$$-\Delta u + V(\eta)u = K(\eta)|u|^{p-2}u, \quad \text{on } \mathbb{R}^n. \tag{4.10}$$

Proof. We first show that w_k is bounded in $H^1(\mathbb{R}^n)$. There holds

$$I_{\varepsilon_k}(u_k) = \big(\frac{1}{2} - \frac{1}{p}\big)\frac{1}{\varepsilon_k^n}\int_{\mathcal{M}} \left(\varepsilon^2|u_k|^2 + V(x)u_k^2\right) d\mu_g < c_0 + \sigma_k,$$

which, together with the boundedness of $V(x)$, yield

$$\begin{aligned}\frac{1}{\varepsilon_k^n}\int_{\mathcal{M}} |u_k|^2\, d\mu_g &\le \frac{C}{\varepsilon_k^n}\int_{\mathcal{M}} V(x)|u_k|^2\, d\mu_g \\ &\le \frac{C}{\varepsilon_k^n}\int_{\mathcal{M}} \left(\varepsilon^2|\nabla_g u_k|^2 + V(x)u_k^2\right) d\mu_g \\ &\le C\left(c_0 + \sigma\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon_k^n}\int_{\mathcal{M}} |u_k(x)|^2\, d\mu_g \ge \frac{1}{\varepsilon_k^n}\int_{B_g(\eta_k, \frac{R}{\rho})} \chi_k^2(\varphi_k(x))|u_k(x)|^2\, d\mu_g \\ &= \frac{1}{\varepsilon_k^n}\int_{B(0,\frac{R}{\rho})} \chi_k^2(\varphi_k(\exp_{\eta_k}(z)))|u_k(\exp_{\eta_k}(z))|^2\ |g_{\eta_k}(z)|^{1/2}\, dz \\ &= \int_{B(0,\frac{R}{\varepsilon_k\rho})} \chi_k^2(z)|u_k(\varphi_k^{-1}(z))|^2\ |g_{\eta_k}(\varepsilon_k z)|^{1/2}\, dz \ge h^{n/2}\int_{\mathbb{R}^n} |w_k|^2\, dz.\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla w_k|^2\, dz \\ &= \int_{B(0,\frac{R}{\varepsilon_k\rho})} \sum_{i,j} \frac{\partial\left(\chi_k(z)u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\frac{\partial\left(\chi_k(z)u_k(\varphi_k^{-1}(z))\right)}{\partial z_j}\, dz \\ &= \int_{B(0,\frac{R}{\varepsilon_k\rho})} \sum_{i,j} \chi_k^2(z)\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_j}\, dz \\ &\quad + \int_{B(0,\frac{R}{\varepsilon_k\rho})} \sum_{i,j} u_k(\varphi_k^{-1}(z))\chi_k(z)\Big(\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\frac{\partial\big(\chi_k(z)\big)}{\partial z_j} \\ &\quad + \frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_j}\frac{\partial\big(\chi_k(z)\big)}{\partial z_i}\Big)\, dz \\ &\quad + \int_{B(0,\frac{R}{\varepsilon_k\rho})} \sum_{i,j} u_k^2(\varphi_k^{-1}(z))\frac{\partial\big(\chi_k(z)\big)}{\partial z_i}\frac{\partial\big(\chi_k(z)\big)}{\partial z_j}\, dz := I_6 + I_7 + I_8.\end{aligned}$$

By the hypotheses on u_k, $\psi(x)$ denotes the functions of the partition of unity associated to $B_g(\eta_k, R)$, using (4.9), we obtain

$$\begin{aligned}&\frac{\varepsilon_k^2}{\varepsilon_k^n}\int_{\mathcal{M}} |\nabla_g u_k(x)|^2\, d\mu_g \\ &\ge \frac{\varepsilon_k^2}{\varepsilon_k^n}\int_{B_g(\eta_k,\frac{R}{\rho})} \psi(x)|\nabla_g u_k(x)|^2\, d\mu_g\end{aligned}$$

$$\begin{aligned}
&\geq \psi_0 \int_{B(0,\frac{R}{\varepsilon_k\rho})} \Big(\sum_{i,j} g^{ij}_{\eta_k}(\varepsilon_k z)\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_j}\Big)|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz\\
&\geq C(\mathcal{M})\psi_0 I_6
\end{aligned}$$

for a positive constant $C(\mathcal{M})$ depending only on the manifold. By the Minkowski and Hölder inequalities,

$$\begin{aligned}
&|I_7|\\
&\leq \Big|2\int_{B(0,\frac{R}{\varepsilon_k\rho})}\sum_{i,j} u_k(\varphi_k^{-1}(z))\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\frac{\partial\big(\chi_k(z)\big)}{\partial z_j}\,dz\Big|\\
&\leq 2\sum_{i,j}\Big(\int_{B(0,\frac{R}{\varepsilon_k\rho})}|u_k(\varphi_k^{-1}(z))|^2\,dz\Big)^{1/2}\Big(\int_{B(0,\frac{R}{\varepsilon_k\rho})}\frac{2\varepsilon_k\rho}{R}\Big|\frac{\partial\left(u_k(\varphi_k^{-1}(z))\right)}{\partial z_i}\Big|^2\,dz\Big)^{1/2}
\end{aligned}$$

and

$$|I_8| \leq \frac{4n\varepsilon_k^2\rho^2}{R^2}\int_{B(0,\frac{R}{\varepsilon_k\rho})}\left|u_k(\varphi_k^{-1}(z))\right|^2\,dz.$$

Hence, w_k is uniformly bounded in $H^1(\mathbb{R}^n)$ since $I_{\varepsilon_k}(u_k) \leq 2c_0$ for all k.

Suppose now that $w_k \rightharpoonup \tilde{w}$ in $H^1(\mathbb{R}^n)$. We show $\tilde{w}$ is a solution of problem (4.10). Let $\omega_{\varepsilon_k} := \{y \in \mathbb{R}^N | \varepsilon_k y \in [\Omega]_r\}$ and denote by $\widetilde{\exp}$ the exponential map associated to ω_{ε_k}. We set $v(y) := u(\varepsilon_k y)$ for $u \in H^1_g(\mathcal{M})$, $y \in \omega_{\varepsilon_k}$ and let $J_{\varepsilon_k}(v(y)) := I_{\varepsilon_k}(u(\varepsilon_k y))$. For each $\eta_k \in \Omega$, we define

$$\varphi_{k,\varepsilon_k} : B_{g_{\varepsilon_k}}\left(\frac{\eta_k}{\varepsilon_k}, \frac{R}{\varepsilon_k\rho}\right) \to B\left(0, \frac{R}{\varepsilon_k\rho}\right), \quad \varphi_{k,\varepsilon_k} := \Big(\widetilde{\exp}_{\frac{\eta_k}{\varepsilon_k}}|_{B\left(0,\frac{R}{\varepsilon_k\rho}\right)}\Big)^{-1}. \tag{4.11}$$

For any $\xi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\xi \subset \{\chi_k(z) = 1\}$ for k large enough. Hence, $w_k(z) = u_k(\varphi_{k,\varepsilon_k}^{-1}(z))$ for $z \in \operatorname{supp}\xi \subset B(0, \frac{R}{\varepsilon_k\rho})$ and k large enough. So we have

$$\begin{aligned}
J'_{\varepsilon_k}\left(w_k(\varphi_{k,\varepsilon_k}(y))\right)[\xi(\varphi_{k,\varepsilon_k}(y))] &= J'_{\varepsilon_k}\left(u_k\left(\varphi_k^{-1}(\varphi_{k,\varepsilon_k}(y))\right)\right)[\xi(\varphi_{k,\varepsilon_k}(y))]\\
&= I'_{\varepsilon_k}\left(u_k(x)\right)\Big[\xi\Big(\varphi_{k,\varepsilon_k}(\frac{x}{\varepsilon_k})\Big)\Big]
\end{aligned}$$

where if $y \in \omega_{\varepsilon_k}$ then $y \in \frac{x}{\varepsilon_k}$ for a $x \in \Omega$. By the Ekeland principle,

$$\left|J'_{\varepsilon_k}\left(w_k(\varphi_{k,\varepsilon_k}(y))\right)[\xi(\varphi_{k,\varepsilon_k}(y))]\right| < \sqrt{\sigma_k}|||\xi\Big(\varphi_{k,\varepsilon_k}(\frac{x}{\varepsilon_k})\Big)|||_{\varepsilon_k},$$

while

$$\||||\xi\Big(\varphi_{k,\varepsilon_k}(\frac{x}{\varepsilon_k})\Big)|||_{\varepsilon_k} \to \Big[\int_{\mathbb{R}^n}(|\nabla\xi|^2 + |\xi|^2)\,dz\Big]^{1/2}$$

as $k \to \infty$. Therefore,

$$J'_{\varepsilon_k}\left(w_k(\varphi_{k,\varepsilon_k}(y))\right)[\xi(\varphi_{k,\varepsilon_k}(y))] \to 0 \tag{4.12}$$

for $\xi \in C_0^\infty(\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
&|J'_{\varepsilon_k}\left(w_k(\varphi_{k,\varepsilon_k}(y))\right)[\xi(\varphi_{k,\varepsilon_k}(y))] - J'(\tilde{w})[\xi]|\\
&\leq \Big|\int_{B(0,\frac{R}{\varepsilon_k})\cap\operatorname{supp}\xi}\sum_{i,j} g^{ij}_{\eta_k}(\varepsilon_k z)\frac{\partial w_k(z)}{\partial z_i}\frac{\partial\xi(z)}{\partial z_j}|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz\\
&\quad - \int_{\mathbb{R}^n}\nabla\tilde{w}(z)\nabla\xi(z)\,dz\Big|
\end{aligned}$$

$$
\begin{aligned}
&+ \Big| \int_{B(0,\frac{R}{\varepsilon_k})\cap \operatorname{supp}\xi} V(\exp_{\eta_k}(\varepsilon_k z)) w_k(z)\xi(z)|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz \\
&- \int_{\mathbb{R}^n} V(\eta)\tilde{w}(z)\xi(z)\,dz\Big| \\
&+ \Big| \int_{B(0,\frac{R}{\varepsilon_k})\cap \operatorname{supp}\xi} K(\exp_{\eta_k}(\varepsilon_k z)) |w_k(z)|^{p-1}\xi(z)|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz \\
&- \int_{\mathbb{R}^n} K(\eta)|\tilde{w}|^{p-1}(z)\xi(z)\,dz\Big| \\
&\le \int_{\mathbb{R}^n} \sum_{i,j} \Big| g^{ij}_{\eta_k}(\varepsilon_k z)\zeta_{B(0,\frac{R}{\varepsilon_k})}(z) \frac{\partial w_k(z)}{\partial z_i}\frac{\partial \xi(z)}{\partial z_j}|g_{\eta_k}(\varepsilon_k z)|^{1/2} - \delta_{ij}\frac{\partial \tilde{w}(z)}{\partial z_i}\frac{\partial \xi(z)}{\partial z_j}\Big|\,dz \\
&+ \int_{\mathbb{R}^n} \Big|\xi(z)\Big(V(\exp_{\eta_k}(\varepsilon_k z))\zeta_{B(0,\frac{R}{\varepsilon_k})}(z) w_k(z)|g_{\eta_k}(\varepsilon_k z)|^{1/2} - V(\eta)\tilde{w}(z)\Big)\Big|\,dz \\
&+ \int_{\mathbb{R}^n} \Big|\xi(z)\Big(\zeta_{B(0,\frac{R}{\varepsilon_k})}(z)|g_{\eta_k}(\varepsilon_k z)|^{1/2} K(\exp_{\eta_k}(\varepsilon_k z))|w_k(z)|^{p-1} \\
&- K(\eta)|\tilde{w}(z)|^{p-1}\Big)\Big|\,dz \\
&:= I_9 + I_{10} + I_{11}
\end{aligned}
$$

where $\zeta_{B(0,\frac{R}{\varepsilon_k})}(z)$ denotes the characteristic function of the set $B(0,\frac{R}{\varepsilon_k}) \subset \mathbb{R}^n$. We see that I_9, I_{10} and I_{11} tend to zero as $k \to \infty$. By the fact that

$$
\lim_{k\to\infty} |g^{ij}_{\eta_k}(\varepsilon_k z)\zeta_{B(0,\frac{R}{\varepsilon_k})}(z)|g_{\eta_k}(\varepsilon_k z)|^{1/2} - \delta_{ij}| = 0
$$

and $\exp_{\eta_k}(\varepsilon_k z) - \eta_k \to 0$ as $k \to \infty$, we obtain

$$
J'_{\varepsilon_k}\left(w_k(\varphi_{k,\varepsilon_k}(y))\right)[\xi(\varphi_{k,\varepsilon_k}(y))] \to J'(\tilde{w})[\xi] \quad \text{for } \forall \xi \in C_0^\infty(\mathbb{R}^n). \tag{4.13}
$$

Equations (4.12) and (4.13) imply $\tilde{w}$ is a solution of (4.10).

Finally, we show $\tilde{w}$ is a ground state solution of (4.10). For $u_k \in \Sigma_{\varepsilon_k,\sigma_k}$ we have

$$
\begin{aligned}
(c_0 + \sigma_k) \ge I_{\varepsilon_k}(u_k) &= \big(\frac{1}{2} - \frac{1}{p}\big)\frac{1}{\varepsilon_k^n}\int_{\mathcal{M}} K(x)|u_k^+|^p\,d\mu_g \\
&\ge \big(\frac{1}{2} - \frac{1}{p}\big)\frac{1}{\varepsilon_k^n}\int_{B_g(\eta_k,\frac{R}{\rho})} K(x)|u_k^+|^p\,d\mu_g \\
&= \big(\frac{1}{2} - \frac{1}{p}\big)\int_{B(0,\frac{R}{\varepsilon_k\rho})} K(\exp_{\eta_k}(\varepsilon_k z))|u_k^+(\varphi_k^{-1}(z))|^p|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz\,.
\end{aligned}
$$

The sequence of functions

$$
F_k(z) := \left(K(\exp_{\eta_k}(\varepsilon_k z))\right)^{1/p} u_k^+(\varphi_k^{-1}(z)) g_{\eta_k}^{1/(2p)}(\varepsilon_k z)\zeta_{B(0,\frac{R}{\varepsilon_k\rho})}(z) \in L^p(\mathbb{R}^n),
$$

is bounded in $L^p(\mathbb{R}^n)$, so there exists $F \in L^p(\mathbb{R}^n)$ which is the L^p- weak limit of the sequence F_k. However, for $\xi \in C_0^\infty(\mathbb{R}^n)$, as w_k tends to $\tilde{w}$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^p_{loc}(\mathbb{R}^n)$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} F_k(z)\xi(z)\,dz &= \int_{\mathbb{R}^n} \left(K(\exp_{\eta_k}(\varepsilon_k z))\right)^{1/p} w_k^+(z) g_{\eta_k}^{1/(2p)}(\varepsilon_k z)\xi(z)\,dz \\
&\to \int_{\mathbb{R}^n} K(\eta)^{1/p}\tilde{w}^+(z)\xi(z)\,dz \quad \text{as } k \to \infty\,.
\end{aligned}
$$

Hence, $F \equiv K^{\frac{1}{p}}(\eta)\tilde{w}^+ \equiv K^{\frac{1}{p}}(\eta)\tilde{w}$ and for any k,

$$\big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^n} K(\eta)|\tilde{w}|^p\,dz \le \liminf_{k\to\infty}\big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^n}|F_k(z)|^p\,dz \le c_0+\sigma_k,$$

namely,

$$\int_{\mathbb{R}^n} K(\eta)|\tilde{w}|^p\,dz \le \frac{2p}{p-2}(c_0+\sigma_k). \tag{4.14}$$

Hence, $\tilde{w}\in N_\eta\cup\{0\}$ and $J(\tilde{w})\le c_0$. If $\tilde{w}\not\equiv 0$, $\tilde{w}$ is a ground state solution.

Now we show that $\tilde{w}\not\equiv 0$. Given $T>0$, we can choose $\eta_k\in\mathcal{M}$ such that for k big enough $\eta_k\in\tilde{P}_\sigma^{\varepsilon_k}\subset B_g(\eta_k,\varepsilon_k T), \varepsilon_k<\frac{R}{\rho}$. By Lemma 4.2,

$$\begin{aligned}
\|w_k^+\|^p_{L^p(B(0,T))} &= \int_{B(0,T)} \chi_k^p(z)\left|u_k^+(\varphi_k^{-1}(z))\right|^p\,dz\\
&= \frac{1}{\varepsilon_k^n}\int_{B(0,\varepsilon_k T)}\Big|u_k^+\Big(\varphi_k^{-1}(\frac{z}{\varepsilon_k})\Big)\Big|^p\,dz\\
&\ge \frac{1}{H^{n/2}}\frac{1}{\varepsilon_k^n}\int_{B(0,\varepsilon_k T)}\Big|u_k^+\Big(\varphi_k^{-1}(\frac{z}{\varepsilon_k})\Big)\Big|^p|g_{\eta_k}(\varepsilon_k z)|^{1/2}\,dz\\
&\ge \frac{1}{K_{\max}H^{n/2}}\frac{1}{\varepsilon_k^n}\int_{B_g(\eta_k,\varepsilon_k T)} K(x)\left|u_k^+(x)\right|^p\,d\mu_g\\
&\ge \frac{1}{K_{\max}H^{n/2}}\frac{1}{\varepsilon_k^n}\int_{\tilde{P}_\sigma^{\varepsilon_k}} K(x)\left|u_k^+(x)\right|^p\,d\mu_g\\
&\ge \frac{\gamma}{K_{\max}H^{n/2}}
\end{aligned}$$

This implies $\tilde{w}\not\equiv 0$ because w_k converges strongly to $\tilde{w}$ in $L^p(B(0,T))$. The assertion then follows. □

Proposition 4.5. *For $\theta\in(0,1)$ there exists $\sigma_0<c_0$ such that for $\sigma\in(0,\sigma_0)$, $\varepsilon\in(0,\varepsilon_0)$ and $u=u_{\varepsilon,\sigma}\in\Sigma_{\varepsilon,\sigma}$ we can find $\eta=\eta(u)\in\Omega$ such that*

$$\frac{1}{\varepsilon^n}\int_{B_g(\eta,\frac{R}{2})} K(x)|u^+|^p\,d\mu_g > \frac{2p(1-\theta)}{p-2}c_0.$$

Proof. First, we show that the result holds for $u\in\Sigma_{\varepsilon,\sigma}\cap I_\varepsilon^{m_\varepsilon+2\sigma}$. Suppose by contradiction that there exists $\theta\in(0,1)$ such that we can find sequences ε_k and σ_k, which are positive and tending to zero as $k\to\infty$, and a sequence $\{u_k\}\subset\Sigma_{\varepsilon_k,\sigma_k}\cap I_{\varepsilon_k}^{m_{\varepsilon_k}+2\sigma_k}$ such that for any $\eta\in\Omega$ there holds

$$\frac{1}{\varepsilon^n}\int_{B_g(\eta,\frac{R}{2})} K(x)|u_k^+|^p\,d\mu_g \le \frac{2p(1-\theta)}{p-2}c_0. \tag{4.15}$$

By Lemma 4.3, we may assume that

$$\Big|\nabla|_{\mathcal{N}_{\varepsilon_k}} I_{\varepsilon_k}(u_k)\Big| < \sqrt{\sigma_k}\||\xi|\|_{\varepsilon_k}\quad \forall\xi\in H_g^1(\mathcal{M}). \tag{4.16}$$

Lemma 4.2 implies that there exists a set P_k of the partition $\mathcal{P}_\varepsilon$ such that

$$\frac{1}{\varepsilon_k^n}\int_{P_k} K(x)|u_k^+|^p\,d\mu_g > \gamma,$$

and we may choose $\eta_k\in P_k$. By the compactness of $\mathcal{M}$, we may assume that $\eta_k\to\eta\in\mathcal{M}$ as $k\to\infty$.

By the hypothesis on K, $K_{\min} > 0$. We claim that for any $T > 0$ and $\tau \in (0,1)$ it holds

$$|w_k^+|^p_{L^p(B(0,T))} \le \frac{1}{K_{\min}} \frac{1}{1-\tau}(1-\theta)\frac{2p}{p-2}c_0$$

for k large enough. Indeed, we note $|g_{\eta_k}(\varepsilon_k z)| \to |g_\eta(0)| = 1$ for all $z \in B(0,R)$ and fixed $\tau \in (0,1)$. For k large enough, $|g_{\eta_k}(z)| > (1-\tau)$ if $z \in B(0, \varepsilon_k T)$. By this fact and (4.15) we have

$$\begin{aligned}
|w_k^+|^p_{L^p(B(0,T))} &= \int_{B(0,T)} \chi_k^p(z) \left|u_k^+(\varphi_k^{-1}(z))\right|^p dz \\
&= \frac{1}{\varepsilon_k^n} \int_{B(0,\varepsilon_k T)} \chi^p_{\frac{R}{\rho}}(z) \left|u_k^+(\exp_{\eta_k}(z))\right|^p dz \\
&\le \frac{1}{\varepsilon_k^n} \int_{B(0,\varepsilon_k T)} \frac{|g_{\eta_k}(z)|^{1/2}}{1-\tau} \left|u_k^+(\exp_{\eta_k}(z))\right|^p dz \\
&= \frac{1}{1-\tau} \frac{1}{\varepsilon_k^n} \int_{B_g(\eta_k, \varepsilon_k T)} |u_k^+|^p \, d\mu_g \\
&\le \frac{1}{(1-\tau)\varepsilon_k^n K_{\min}} \int_{B_g(\eta_k, \frac{R}{2})} K(x) |u_k^+|^p \, d\mu_g \\
&\le \frac{1}{K_{\min}} \frac{1-\theta}{1-\tau} \frac{2p}{p-2} c_0.
\end{aligned} \tag{4.17}$$

We know from Lemma 4.4 that $\tilde{w}$ is a ground state solution of problem (4.10); that is,

$$E_\eta(\tilde{w}) = \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^n} K(\eta) |\tilde{w}^+|^p \, dz = c_0.$$

By Lemma 4.4, there exists $T > 0$ such that for k large enough

$$\frac{2p}{p-2}c_0 = \int_{\mathbb{R}^n} K(\eta)|\tilde{w}^+|^p \, dz \le \int_{B(0,T)} K(\eta)|w_k^+|^p \, dz \le K_{\max} \int_{B(0,T)} |w_k^+|^p \, dz.$$

Choosing $\mu > K_{\max}/K_{\min}$ and τ such that $\frac{1-\theta}{1-\tau} < \frac{1-\theta}{1-\tau}\mu < 1$, we obtain

$$\frac{1}{K_{\min}} \frac{1-\theta}{1-\tau} \frac{2p}{p-2} c_0 < \frac{\mu}{K_{\max}} \frac{1-\theta}{1-\tau} \frac{2p}{p-2} c_0 < \int_{B(0,T)} |w_k^+|^p \, dz \tag{4.18}$$

a contradiction to (4.17).

Next, we show that $\Sigma_{\varepsilon,\sigma} \cap I_\varepsilon^{m_\varepsilon + 2\sigma} = \Sigma_{\varepsilon,\sigma}$. In fact, for $u \in \Sigma_{\varepsilon,\sigma} \cap I_\varepsilon^{m_\varepsilon+2\sigma}$, we have $I_\varepsilon(u) < c_0 + \sigma$ and $I_\varepsilon(u) < m_\varepsilon + 2\sigma$, which yield $m_\varepsilon \ge (1-\theta)c_0$ for any $\theta \in (0,1)$. By Proposition 3.2, $\limsup_{\varepsilon \to 0} m_\varepsilon \le c_0$, and then $\lim_{\varepsilon \to 0} m_\varepsilon = c_0$, which implies $\Sigma_{\varepsilon,\sigma} \subset I_\varepsilon^{m_\varepsilon+2\sigma}$ for σ, ε small enough. The proof is completed. □

Proposition 4.6. *There exists $\sigma_0 \in (0, c_0)$ such that for $\sigma \in (0, \sigma_0)$, $\varepsilon \in (0, \varepsilon_0)$ and $u \in \Sigma_{\varepsilon,\sigma}$ there holds $\beta(u) \in [\Omega_\delta]_r$.*

Proof. By Proposition 4.5, for $\theta \in (0,1)$ and $u \in \Sigma_{\varepsilon,\sigma}$ with ε and σ suitably small, there exists $\eta \in \Omega$ such that

$$(1-\theta)\frac{2p}{p-2}c_0 < \frac{1}{\varepsilon^n} \int_{B_g(\eta, \frac{R}{2})} K(x)|u^+|^p \, d\mu_g. \tag{4.19}$$

On the other hand, for $u \in \Sigma_{\varepsilon,\sigma}$, we have

$$I_\varepsilon(u) = \frac{1}{\varepsilon^n}\frac{p-2}{2p}\int_{\mathcal{M}} K(x)|u^+|^p\,d\mu_g < c_0 + \sigma,$$

therefore,

$$\frac{1}{\varepsilon^n}\int_{\mathcal{M}} |u^+|^p\,d\mu_g \le \frac{1}{K_{\min}}\frac{1}{\varepsilon^n}\int_{\mathcal{M}} K(x)|u^+|^p\,d\mu_g < \frac{1}{K_{\min}}\frac{2p}{p-2}\left(c_0+\sigma\right). \tag{4.20}$$

Let

$$f\left(u(x)\right) := \frac{|u^+(x)|^p}{\int_{\mathcal{M}} |u^+(x)|^p\,d\mu_g}.$$

By (4.19) and (4.20),

$$\int_{B_g(\eta,\frac{R}{2})} f\left(u(x)\right)\,d\mu_g \ge \frac{\frac{1}{K_{\max}}\frac{1}{\varepsilon^n}\int_{B_g(\eta,\frac{R}{2})} K(x)|u^+(x)|^p\,d\mu_g}{\frac{1}{\varepsilon^n}\int_{\mathcal{M}}|u^+(x)|^p\,d\mu_g} > \frac{K_{\min}(1-\theta)c_0}{K_{\max}(c_0+\sigma)}.$$

Therefore,

$$\begin{aligned}
|\beta(u)-\eta| &\le \Big|\int_{B_g(\eta,\frac{R}{2})}(x-\eta)f\left(u(x)\right)\,d\mu_g\Big| + \Big|\int_{\mathcal{M}\setminus B_g(\eta,\frac{R}{2})}(x-\eta)f\left(u(x)\right)\,d\mu_g\Big| \\
&\le \frac{r(\Omega_\delta)}{2} + D\Big(1-\frac{K_{\min}(1-\theta)c_0}{K_{\max}(c_0+\sigma)}\Big),
\end{aligned}$$

where D is the diameter of Ω_δ as a subset of $\mathcal{M}$. The assertion follows by choosing θ and σ suitably small. □

Proof of Theorem 1.1. We know that $I_\varepsilon \in C^1$ and $\mathcal{N}_\varepsilon$ is a $C^{1,1}$ complete Riemannian manifold. Also I_ε is bounded from below on $\mathcal{N}_\varepsilon$ and satisfies the (PS) condition. By Proposition 2.1, I_ε has at least $\operatorname{cat}_{\Sigma_{\varepsilon,\sigma}}(\Sigma_{\varepsilon,\sigma})$ critical points.

By Propositions 3.2 and 4.5, $\beta\circ\phi_\varepsilon : \Omega \to [\Omega_\delta]_r$ is well defined and $\beta\circ\phi_\varepsilon(\eta) \in [\Omega_\delta]_r \subset \mathbb{R}^N$ for $\eta \in \Omega$. Now we show that $\Pi\circ\beta\circ\phi_\varepsilon$ is homotopic to the identity on Ω_δ. Indeed,

$$\begin{aligned}
\Pi\circ\beta\circ\phi_\varepsilon(\eta)-\eta &= \int_{\mathcal{M}}(x-\eta)f\left(\phi_\varepsilon(\eta)\right)\,d\mu_g \\
&= \int_{\mathcal{M}}(x-\eta)f\Big(t_\varepsilon(w_\varepsilon(\exp_\eta^{-1}(x))\chi_R(|\exp_\eta^{-1}(x)|)) \\
&\quad \times w_\varepsilon(\exp_\eta^{-1}(x))\chi_R(|\exp_\eta^{-1}(x)|)\Big)\,d\mu_g \\
&= \frac{\int_{\mathcal{M}}(x-\eta)w_\varepsilon^p(\exp_\eta^{-1}(x))\chi_R^p(|\exp_\eta^{-1}(x)|)\,d\mu_g}{\int_{\mathcal{M}} w_\varepsilon^p(\exp_\eta^{-1}(x))\chi_R^p(|\exp_\eta^{-1}(x)|)\,d\mu_g} \\
&= \frac{\int_{B_g(\eta,R)}(x-\eta)w_\varepsilon^p(\exp_\eta^{-1}(x))\chi_R^p(|\exp_\eta^{-1}(x)|)\,d\mu_g}{\int_{B_g(\eta,R)} w_\varepsilon^p(\exp_\eta^{-1}(x))\chi_R^p(|\exp_\eta^{-1}(x)|)\,d\mu_g} \\
&= \frac{\int_{B(0,R)} z w_\varepsilon^p(z)\chi_R^p(|z|)|g_\eta(z)|^{1/2}\,dz}{\int_{B(0,R)} w_\varepsilon^p(z)\chi_R^p(|z|)|g_\eta(z)|^{1/2}\,dz} \\
&= \frac{\varepsilon\int_{B(0,\frac{R}{\varepsilon})} z w^p(z)\chi_R^p(|\varepsilon z|)|g_\eta(\varepsilon z)|^{1/2}\,dz}{\int_{B(0,\frac{R}{\varepsilon})} w^p(z)\chi_R^p(|\varepsilon z|)|g_\eta(\varepsilon z)|^{1/2}\,dz}.
\end{aligned}$$

Hence, $|\Pi \circ \beta \circ \phi_\varepsilon(\eta) - \eta| \leq \varepsilon C \to 0$, where $C > 0$ does not depend on η. Applying Lemma 2.2 with $X = \Sigma_{\varepsilon,\sigma}$, $Y = \Omega_\delta$, $Z = \Omega$ and $h_1 = \phi_\varepsilon$, $h_2 = \Pi \circ \beta$, we obtain $\operatorname{cat}_{\Sigma_{\varepsilon,\sigma}}(\Sigma_{\varepsilon,\sigma}) \geq \operatorname{cat}_{\Omega_\delta}(\Omega)$. The proof is complete. □

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