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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A SEMILINEAR ELLIPTIC SYSTEM

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Abstract. In this article, we show the existence and uniqueness of smooth solutions for boundary-value problems of semilinear elliptic systems.

1. Introduction and main results

We study the solvability for the semilinear elliptic system with homogeneous Dirichlet boundary value condition

$$\begin{gathered} L_1 u = f(x,u,v,Du,Dv), \quad x \in \Omega \\ L_2 v = g(x,u,v,Du,Dv), \quad x \in \Omega \\ u = v = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ denotes a bounded domain with smooth boundary, and $f, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, L_1 and L_2 are the uniformly elliptic operators of second order:

$$L_k u = \sum_{i,j=1}^{N} \partial_{x_j}(a_{i,j}^k(x)u), \ k = 1, 2,$$

with its first eigenvalue $\lambda_k > 0$ for $k = 1, 2$, and in the context, $\lambda =: \min\{\lambda_1, \lambda_2\}$. We suppose the following conditions:

(H1) $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ are Caratheodory functions which satisfy

$$\begin{gathered} |f(x,s,t,\xi,\eta)| \leq h_1(x,s,t) + k_1|\xi|^{\alpha_1} + k_2|\eta|^{\alpha_2}, \\ |g(x,s,t,\xi,\eta)| \leq h_2(x,s,t) + k_3|\xi|^{\alpha_3} + k_4|\eta|^{\alpha_4}, \end{gathered}$$

where constant $\alpha_i, k_i \in \mathbb{R}_0^+$, $i = 1, 2, 3, 4$; $h_1(x,s,t)$ and $h_2(x,s,t)$ are Caratheodory functions that satisfy the following conditions:

(H2) for every $r > 0$, $\sup_{|s|\leq r,\, |t|\leq r} h_i(\cdot, s, t) \in L^p(\Omega)$, $\frac{2N}{N+1} < p < N$;

(H3) $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} =: \alpha \leq 1$;

(H4) $\alpha_i \geq \frac{1}{p}$ or $\alpha_i = 0$, for $i = 1, 2, 3, 4$.

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Theorem 1.1. *Assume* (H1)–(H4). *If* (1.1) *has two pairs of subsolutions and supersolutions* $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$, *then* (1.1) *has at least one solution* $(u, v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$.

For the next theorem we need the assumption

(H5) $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ are Lipschitz continuous, with Lipschitz coefficients l_1 and l_2, and $L := \max\{l_1, l_2\} < \frac{\lambda}{4C+1}$, where $C = C(n, p, \Omega)$ is the coefficient for the Poincaré inequality.

Theorem 1.2. *Under Condition* (H5), *Problem* (1.1) *has at most one weak solution* $(u, v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$, $\frac{2N}{N+1} < p < N$.

2. The proof of Theorem 1.1

Proof. From (H2) and (H3), we know that $[\alpha p, p^*)$ is not empty, where $p^* = \frac{Np}{N-p}$. Fix $q_0 \in [\alpha p, p^*)$, let $T : W^{1,q_0}(\Omega) \mapsto W^{1,q_0}(\Omega) \cap L^\infty(\Omega)$ be the cut-off function about $\underline{u}, \overline{u}, \underline{v}, \overline{v}$; i.e.,

$$\begin{gathered} Tu(x) = \overline{u}(x), \quad \overline{u} \le u, \\ Tu(x) = u(x), \quad \underline{u} \le u \le \overline{u}, \\ Tu(x) = \underline{u}(x), \quad u \le \underline{u}, \\ Tv(x) = \overline{v}(x), \quad \overline{v} \le v, \\ Tv(x) = v(x), \quad \underline{v} \le v \le \overline{v}, \\ Tv(x) = \underline{v}(x), \quad v \le \underline{v}. \end{gathered}$$

Next, we prove that $Tu, Tv \in W^{1,q_0}(\Omega) \cap L^\infty(\Omega)$. Firstly, we notice that

$$\begin{gathered} |Tu(x)| \le \max\{|\underline{u}|, |\overline{u}|\} =: M, \quad \text{a.e. } x \in \Omega, \\ |Tv(x)| \le \max\{|\underline{v}|, |\overline{v}|\} =: m, \quad \text{a.e. } x \in \Omega \end{gathered}$$

for every $u, v \in W^{1,q_0}(\Omega)$, then $Tu, Tv \in L^\infty(\Omega)$. Since the embedding of $W^{2,p}(\Omega)$ into $W^{1,q_0}(\Omega)$ is compact and $\overline{u}, \underline{u}, \overline{v}, \underline{v} \in W^{1,q_0}(\Omega)$, then by [8, A.6], we know $|u - v| \in W^{1,q_0}(\Omega)$. Also, from

$$\begin{aligned} Tu(x) &= \frac{u + \overline{u} + 2\underline{u} - |u - \overline{u}|}{4} + \frac{|u + \overline{u} - 2\underline{u} - |u - \overline{u}||}{4}, \\ Tv(x) &= \frac{v + \overline{v} + 2\underline{v} - |v - \overline{v}|}{4} + \frac{|v + \overline{v} - 2\underline{v} - |v - \overline{v}||}{4} \end{aligned}$$

we know that $Tu, Tv \in W^{1,q_0}(\Omega)$, hence $Tu, Tv \in W^{1,q_0}(\Omega) \cap L^\infty(\Omega)$.

Let $S : [0,1] \times [W^{1,q_0}(\Omega)]^2 \mapsto [W^{1,q_0}(\Omega)]^2$ be defined as $S(t, u, v) = (w_1, w_2)$, where (w_1, w_2) is the solution of the following boundary-value problem

$$\begin{gathered} L_1 w_1 = t f(x, Tu, Tv, D(Tu), D(Tv)), x \in \Omega, \\ w_1 = 0, x \in \partial\Omega, \end{gathered} \tag{2.1}$$

$$\begin{gathered} L_2 w_2 = t g(x, Tu, Tv, D(Tu), D(Tv)), x \in \Omega, \\ w_2 = 0, x \in \partial\Omega. \end{gathered} \tag{2.2}$$

According to (H1)-(H4), for every $u, v \in W^{1,q_0}(\Omega)$, we have $f, g \in L^p(\Omega)$. Then, based on [3, Theorem 6.4], (2.1) and (2.2) have a unique solution $(w_1, w_2) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ which means that S is a well-defined operator. Obviously

$S(0,u,v)=(0,0)$, then by the Sobolev embedding theorem, $W^{2,p}(\Omega)\hookrightarrow W^{1,q_0}(\Omega)$, we know that S is continuous.

Next we prove that $(u,v)\in[W^{1,q_0}(\Omega)]^2$, and for a certain $t\in[0,1]$, $S(t,u,v)=(u,v)$ and this (u,v) satisfies

$$\|u\|_{1,q_0}+\|v\|_{1,q_0}\le C.$$

According to the Sobolev embedding theorem and (H1), we have

$$\begin{aligned}\|u\|_{1,q_0}&\le C\|u\|_{2,p}\\&\le C(\|h_1(x,Tu,Tv)\|_p+k_1\||D(Tu)|^{\alpha_1}\|_p+k_2\||D(Tv)|^{\alpha_2}\|_p)\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\|v\|_{1,q_0}&\le C\|v\|_{2,p}\\&\le C(\|h_2(x,Tu,Tv)\|_p+k_3\||D(Tu)|^{\alpha_3}\|_p+k_4\||D(Tv)|^{\alpha_4}\|_p).\end{aligned}\tag{2.4}$$

Then from the definition of Tu and Tv, and the condition (H2) we know that

$$\|h_i(x,Tu,Tv)\|_p\le C,\quad i=1,2.\tag{2.5}$$

Where C depends only on $\overline{u},\underline{u},\overline{v},\underline{v}$ and p. When $i=1,3$, we have

$$\|D(Tu)^{\alpha_i}\|_p=[\|D(Tu)\|_{\alpha_ip}]^{\alpha_i}=\begin{cases}\|D\overline{u}\|_{\alpha_ip}]^{\alpha_i}, & u\ge\overline{u}\\ [\|Du\|_{\alpha_ip}]^{\alpha_i}, & \underline{u}\le u\le\overline{u}\\ [\|D\underline{u}\|_{\alpha_ip}]^{\alpha_i}, & u\le\underline{u}.\end{cases}\tag{2.6}$$

When $i=2,4$, we have

$$||D(Tv)^{\alpha_i}||_p=[\|D(Tv)\|_{\alpha_ip}]^{\alpha_i}=\begin{cases}\|D\overline{v}\|_{\alpha_ip}]^{\alpha_i}, & v\ge\overline{v}\\ [\|Dv\|_{\alpha_ip}]^{\alpha_i}, & \underline{v}\le v\le\overline{v}\\ [\|D\underline{v}\|_{\alpha_ip}]^{\alpha_i}, & v\le\underline{v}.\end{cases}\tag{2.7}$$

Then by [1, Theorem 4.14] (Ehrling-Nirenberg-Gagliardo), we obtain

$$\begin{aligned}\|Du\|_{\alpha_ip}\le k_1\epsilon\|u\|_{2,\alpha_ip}+k_2(\epsilon)\|u\|_{\alpha_ip},\ \underline{u}\le u\le\overline{u},\\ \|Dv\|_{\alpha_ip}\le k_3\epsilon\|v\|_{2,\alpha_ip}+k_4(\epsilon)\|v\|_{\alpha_ip},\ \underline{v}\le v\le\overline{v}.\end{aligned}\tag{2.8}$$

Since $\underline{u}\le u\le\overline{u},\underline{v}\le v\le\overline{v}$, $\alpha_i\le 1$, $i=1,2,3,4$, by $\alpha_ip\le q_0$ and $\overline{u},\underline{u},\overline{v},\underline{v}\in W^{1,q_0}(\Omega)$, we obtain

$$\begin{gathered}\|u\|_{\alpha_ip}\le C,\quad \|v\|_{\alpha_ip}\le C,\quad \|D\overline{u}\|_{\alpha_ip}\le C,\\ \|D\underline{u}\|_{\alpha_ip}\le C,\quad \|D\overline{v}\|_{\alpha_ip}\le C,\quad \|D\underline{v}\|_{\alpha_ip}\le C,\\ \|u\|_{2,\alpha_ip}\le C\|u\|_{2,p},\quad \|v\|_{2,\alpha_ip}\le C\|v\|_{2,p},\quad i=1,2,3,4.\end{gathered}\tag{2.9}$$

Without loss of generality, we assume that $\|u\|_{2,p}\ge 1,\|v\|_{2,p}\ge 1$. By (2.5), (2.6), (2.7), (2.8), (2.9), we know from (2.3) and (2.4) that

$$\|u\|_{2,p}+\|v\|_{2,p}\le C_1\varepsilon(\|u\|_{2,p}+\|v\|_{2,p})+\frac{C}{2}.$$

Select $\varepsilon=\frac{1}{2C_1}$, we can write

$$\|u\|_{2,p}+\|v\|_{2,p}\le C.$$

Then according to (2.3) and (2.4),

$$\|u\|_{1,q_0}+\|v\|_{1,q_0}\le C.$$

From the Leray-Schauder fixed point theorem [7, Theorem 11.3], there exists a solution $(u,v) \in [W^{1,q_0}(\Omega)]^2$ satisfying $S(1,u,v) = (u,v)$; i. e.,

$$
\begin{gathered}
L_1 u = f(x, Tu, Tv, D(Tu), D(Tv)), \quad x \in \Omega, \\
L_2 v = g(x, Tu, Tv, D(Tu), D(Tv)), \quad x \in \Omega, \\
u = v = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{2.10}
$$

Then $(u,v) \in [W^{1,q_0}(\Omega)]^2$ implies $f, g \in L^p(\Omega)$ and $(u,v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$.

Next we prove that (u,v) satisfies

$$
\underline{u} \le u \le \overline{u}, \underline{v} \le v \le \overline{v}.
$$

Firstly we prove $u \le \overline{u}$. Let $w = u - \overline{u}$, then $w \in W^{2,p}(\Omega)$, define $w^+(x) = \max\{0, w(x)\}$, then we need only to prove $w^+ = 0$. Previously,

$$
\begin{gathered}
L_1 u = f(x, Tu, Tv, D(Tu), D(Tv)), \\
L_1 \overline{u} \ge f(x, \overline{u}, Tv, D\overline{u}, D(Tv)).
\end{gathered}
$$

We obtain the inequality

$$
L_1 w \le [f(x, Tu, Tv, D(Tu), D(Tv)) - f(x, \overline{u}, Tv, D\overline{u}, D(Tv))]. \tag{2.11}
$$

Multiply this inequality by w^+, and integrate on Ω. On the left-hand side, we have

$$
\begin{aligned}
\int_\Omega L_1 \omega \cdot \omega^+ &= \int_\Omega \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega \cdot D_j \omega^+ - \int_{\partial\Omega} \sum a_{ij}^1(x) D_i \omega \cdot \omega^+ \\
&= \int_\Omega \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega \cdot D_j \omega^+
\end{aligned}
$$

Then we can rewrite (2.11) as

$$
\begin{aligned}
\int_\Omega \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega \cdot D_j \omega^+ \le \int_\Omega &[f(x, Tu, Tv, D(Tu), D(Tv)) \\
&- f(x, \overline{u}, Tv, D\overline{u}, D(Tv))] w^+ dx.
\end{aligned} \tag{2.12}
$$

Let $A = \{x \in \Omega : u(x) \le \overline{u}(x)\}$ and $B = \{x \in \Omega : u(x) > \overline{u}(x)\}$. Then $\Omega = A \cup B$. Obviously on A, $w^+ = 0$. In B, $Tu = \overline{u}$. Then the righthand side of (2.12) is zero. That is,

$$
\int_\Omega \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega \cdot D_j \omega^+ = 0.
$$

On A, $w^+ = 0$; on B, $\omega = \omega^+$. We can write the previous equation as

$$
\int_{\Omega^+} \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega^+ \cdot D_j \omega^+ = 0.
$$

Then according to the definition of the uniform elliptic operator,

$$
\lambda |D\omega^+|^2 \le \int_{\Omega^+} \sum_{i,j=1}^N a_{ij}^1(x) D_i \omega^+ \cdot D_j \omega^+ = 0.
$$

Consequently, $w^+ = 0, x \in \Omega$. That is in Ω, $u \le \overline{u}$. Similarly, we can prove that $\underline{u} \le u$ and $\underline{v} \le v \le \overline{v}$. From the definition of T, we know $Tu = u$ and $Tv = v$. Then by (2.10), we obtain that $(u,v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ is the solution of (1.1). The proof is completed. □

An example. In this section, we illustrate Theorem 1.1.

$$
\begin{gathered}
L_1 u = \lambda_1\phi_1(x) + \frac{2\lambda_1}{9}u + v + \lambda_1\phi_1|Du|^{\frac{1}{2}}, \quad x \in \Omega, \\
L_2 v = \frac{3}{4}\lambda_2^2\phi_2(x) + \frac{\lambda_2^2}{12}u + \frac{\lambda_2}{4}v + \frac{\sqrt{3}}{4}\lambda_2^{\frac{3}{2}}\phi_2(x)|Dv|^{\frac{1}{2}}, \quad x \in \Omega, \\
u = v = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{2.13}
$$

Here Ω is a regular domain in $R^N (N > 2)$ with smooth boundary $\partial\Omega$, and

$$
\phi_i(x) = \frac{\varphi_i(x)}{\sup_\Omega |\varphi_i| + \sup_\Omega |D\varphi_i|} \le 1.
$$

In addition, $\lambda_i > 0, \varphi_i(x) > 0$ are the first eigenvalue and the corresponding eigenfunction of operator L_i in Ω with zero-Dirichlet boundary value condition. Therefore,

$$
L_i\phi_i(x) = \frac{L_i\varphi_i(x)}{\sup_\Omega |\varphi_i| + \sup_\Omega |D\varphi_i|} = \frac{\lambda_i\varphi_i(x)}{\sup_\Omega |\varphi_i| + \sup_\Omega |D\varphi_i|} = \lambda_i\phi_i(x).
$$

When $2 < p < N$, we can verify that problem (2.13) satisfies condition (H1)–(H4). Let

$$
\underline{u} = 0; \quad \underline{v} = 0; \quad \overline{u} = 9\phi; \quad \overline{v} = 3\lambda_1\phi; \quad \phi = \max(\phi_1, \phi_2).
$$

It is not difficult to verify that $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$, based on this definition, is a pair of super-solution and sub-solution for problem(2.13). Hence according to Theorem 1.1, problem (2.13) has at least one solution $(u,v) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$.

3. The proof of Theorem 1.2

Proof. Assume $(u_1, v_1), (u_2, v_2) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ are solutions for problem (1.1); therefore

$$
\begin{gathered}
L_1 u_1 = f(x, u_1, v_1, Du_1, Dv_1), \quad x \in \Omega, \\
L_2 v_1 = g(x, u_1, v_1, Du_1, Dv_1), \quad x \in \Omega, \\
u_1 = v_1 = 0, \quad x \in \partial\Omega
\end{gathered}
$$

and

$$
\begin{gathered}
L_1 u_2 = f(x, u_2, v_2, Du_2, Dv_2), \quad x \in \Omega, \\
L_2 v_2 = g(x, u_2, v_2, Du_2, Dv_2), \quad x \in \Omega, \\
u_2 = v_2 = 0, \quad x \in \partial\Omega.
\end{gathered}
$$

Then

$$
L_1(u_1 - u_2) = f(x, u_1, v_1, Du_1, Dv_1) - f(x, u_2, v_2, Du_2, Dv_2), \tag{3.1}
$$
$$
L_2(v_1 - v_2) = g(x, u_1, v_1, Du_1, Dv_1) - g(x, u_2, v_2, Du_2, Dv_2), \tag{3.2}
$$
$$
(u_1 - u_2)\mid_{\partial\Omega} = (v_1 - v_2)\mid_{\partial\Omega} = 0. \tag{3.3}
$$

Multiply (3.1) by $(u_1 - u_2)$ and (3.2) by $(v_1 - v_2)$, and then integrate them on Ω yield

$$\int_\Omega (u_1 - u_2) \cdot L_1(u_1 - u_2) = \int_\Omega \sum_{i,j=1}^N a^1_{ij}(x) D_i(u_1 - u_2) \cdot D_j(u_1 - u_2),$$

$$\int_\Omega (v_1 - v_2) \cdot L_2(v_1 - v_2) = \int_\Omega \sum_{i,j=1}^N a^2_{ij}(x) D_i(v_1 - v_2) \cdot D_j(v_1 - v_2).$$

By the uniformly elliptic condition, we get

$$\int_\Omega \sum_{i,j=1}^N a^1_{ij}(x) D_i(u_1 - u_2) \cdot D_j(u_1 - u_2) \geq \lambda \|Du_1 - Du_2\|^2,$$

$$\int_\Omega \sum_{i,j=1}^N a^2_{ij}(x) D_i(v_1 - v_2) \cdot D_j(v_1 - v_2) \geq \lambda \|Dv_1 - Dv_2\|^2.$$

Using the Lipschitz condition on f, g, it yields

$$\begin{aligned}
&\int_\Omega (f(x, u_1, v_1, Du_1, Dv_1) - f(x, u_2, v_2, Du_2, Dv_2))(u_1 - u_2) dx \\
&\leq L \int_\Omega (|u_1 - u_2| + |v_1 - v_2| + |Du_1 - Du_2| + |Dv_1 - Dv_2|) \cdot |u_1 - u_2| dx \\
&\leq L \int_\Omega (3|u_1 - u_2|^2 + |v_1 - v_2|^2 + \frac{|Du_1 - Du_2|^2 + |Dv_1 - Dv_2|^2}{2}) dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega (g(x, u_1, v_1, Du_1, Dv_1) - g(x, u_2, v_2, Du_2, Dv_2))(v_1 - v_2) dx \\
&\leq L \int_\Omega (|u_1 - u_2| + |v_1 - v_2| + |Du_1 - Du_2| + |Dv_1 - Dv_2|) \cdot |v_1 - v_2| dx \\
&\leq L \int_\Omega (|u_1 - u_2|^2 + 3|v_1 - v_2|^2 + \frac{|Du_1 - Du_2|^2 + |Dv_1 - Dv_2|^2}{2}) dx.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\lambda \|Du_1 - Du_2\|^2 \\
&\leq \int_\Omega \sum_{i,j=1}^N a^1_{ij}(x) D_i(u_1 - u_2) \cdot D_j(u_1 - u_2) \\
&\leq L \int_\Omega (3|u_1 - u_2|^2 + |v_1 - v_2|^2 + \frac{|Du_1 - Du_2|^2 + |Dv_1 - Dv_2|^2}{2}) dx
\end{aligned}$$

and

$$\begin{aligned}
&\lambda \|Dv_1 - Dv_2\|^2 \\
&\leq \int_\Omega \sum_{i,j=1}^N a^2_{ij}(x) D_i(v_1 - v_2) \cdot D_j(v_1 - v_2) \\
&\leq L \int_\Omega (|u_1 - u_2|^2 + 3|v_1 - v_2|^2 + \frac{|Du_1 - Du_2|^2 + |Dv_1 - Dv_2|^2}{2}) dx.
\end{aligned}$$

Summing these two formulas yields

$$\begin{aligned}&\lambda\|Du_1-Du_2\|^2+\lambda\|Dv_1-Dv_2\|^2\\&\le L\int_\Omega(4|u_1-u_2|^2+4|v_1-v_2|^2+|Du_1-Du_2|^2+|Dv_1-Dv_2|^2)dx.\end{aligned}\tag{3.4}$$

Using the Poincaré inequality,

$$\|u\|^2_{L^2(\Omega)}\le C\|Du\|^2_{L^2(\Omega)},\quad \|v\|^2_{L^2(\Omega)}\le C\|Dv\|^2_{L^2(\Omega)}.$$

According to this formula and (3.4), we have

$$\int_\Omega[|D(u_1-u_2)|^2+|D(v_1-v_2)|^2]\,dx\le L\frac{4C+1}{\lambda}\int_\Omega[|D(u_1-u_2)|^2+|D(v_1-v_2)|^2]\,dx$$

By condition $(H5)$, $L\frac{4C+1}{\lambda}<1$, we get $D(u_1-u_2)=0, D(v_1-v_2)=0$, $x\in\Omega$. Since $u_i=v_i=0$ on $\partial\Omega$ for $i=1,2$, it follows that $u_1=u_2$ and $v_1=v_2$, a.e. $x\in\Omega$. This completes the proof. □

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