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SOME INEQUALITIES FOR SOBOLEV INTEGRALS

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Dedicated to Professor Nedyu Popivanov on his 60-th birthday

Abstract. We present some inequalities for Sobolev integrals for functions of one variable which are generalization of Dirichlet principle for harmonic functions.

1. Introduction

In this note we present some inequalities for Sobolev integrals which are generalizations of Dirichlet principle for functions of one variable. The Dirichlet principle for harmonic functions, also known as Thomson's principle, states that there exists a function u that minimizes the functional

$$E(u) = \int_\Omega |\nabla u(x)|^2 dx,$$

called the Dirichlet integral for $\Omega \subset \mathbb{R}^n$, $n \geq 2$, among all the functions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ which take given on values φ on the boundary $\partial\Omega$ of Ω. The minimizer u satisfies the Dirichlet problem for Laplace equation

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega, \\ u(x) &= \varphi(x), \quad x \in \partial\Omega, \end{aligned}$$

which is the Euler-Lagrange equation associated to the Dirichlet integral.

The term "Dirichlet principle" is identified by Bernhard Riemann in "Theory of Abelian Functions", published 1857. It is one of main steps in the history of potential theory and calculus of variations (see [3]). The direct method of the calculus of variations was developed near the middle of nineteenth century. The existence of a minimum of $E(u)$ was considered, in heuristic way, as a trivial consequence of its positivity. Weierstrass, gave in 1870, a counterexample that such an evidence is not valid for the one dimensional case, by showing that for C^2 functions $u : [-1, 1] \to \mathbb{R}$ such that $u(-1) = a$, $u(1) = b$, $a \neq b$, the integral

$$\int_{-1}^{1} |xu'(x)|^2 dx,$$

has no minimum.

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Let f be n times continuously differentiable functions of one variable x. We assume that the independent variable $x \in I := [0,1]$ and all derivatives of function f, except one, are zero at the end points 0 and 1 of the interval I. Denote by N the set $\{0,1,\dots,n-1\}$. Our main result is as follows.

Theorem 1.1. *(a) Let $f \in C^n(I)$ be a real-valued differentiable function such that*

$$f(0) = f(1) = \dots = f^{(n-2)}(0) = f^{(n-2)}(1) = 0$$

and $f^{(n-1)}(0) = A$, $f^{(n-1)}(1) = B$. Then

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq n^2A^2 + (-1)^n 2nAB + n^2B^2. \tag{1.1}$$

(b) Let $f \in C^n(I)$ be a real-valued differentiable function such that $f(0) = a$, $f(1) = b$ and

$$f'(0) = f'(1) = \dots = f^{(n-1)}(0) = f^{(n-1)}(1) = 0.$$

Then

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq (2n-1)! \binom{2n-2}{n-1} (a-b)^2. \tag{1.2}$$

(c) Let $f \in C^n(I)$ be a real-valued differentiable function such that

$$f^{(k)}(0) = A, \quad f^{(k)}(1) = B, \quad f^{(j)}(0) = f^{(j)}(1) = 0, \quad k \in N, \ j \in N \backslash \{k\}.$$

Then

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq \alpha_{n,k}A^2 - \gamma_{n,k}AB + \alpha_{n,k}B^2, \tag{1.3}$$

where

$$\begin{aligned}
\alpha_{n,k} &= \frac{(2n-k-1)!}{k!} \binom{2n-2k-2}{n-k-1} \binom{2n-k-1}{k}, \\
\gamma_{n,k} &= (-1)^k \frac{(2n-k-1)!}{k!} \Big\{ \binom{2n-2k-2}{n-k-1} \binom{2n-k-1}{k} \\
&\quad + \sum_{t=0}^{k} \binom{n-k-1+t}{t} \binom{2n-2k-2+t}{n-2k-1+t} \Big\}.
\end{aligned}$$

Theorem 1.1 is proved in Section 2. Direct calculations are used in the proof of (a) and (b). It is mentioned for which functions the equality holds. We prove Corollary 2.1 as a consequence of (1.1) and (1.2), which is a generalization of an inequality proved in [1, Lemma 3]. The method of divided differences (cf. [2]) in interpolation theory is used in the proof of general statement (c). We simplify some coefficients in (1.3) by reflection method. As a result, we have Corollary 2.2, which is a combinatorial identity, proved by "variational" tools. The present paper is a continuation of a problem, formulated by second author in [4].

2. Proof of the main result

Proof of Theorem 1.1. (a) We divide the proof into the following steps.
Claim 1. The polynomial of order $2n-1$

$$P(x) = \frac{(x-x^2)^{n-1}}{(n-1)!} \big(A + x((-1)^{n-1}B - A)\big)$$

satisfies the boundary conditions of the problem.

Proof: We have
$$(x^n)^{(k)} = n(n-1)\dots(n-k+1)x^{n-k}$$
and
$$((1-x)^n)^{(k)} = (-1)^k n(n-1)\dots(n-k+1)(1-x)^{n-k}.$$
Then, by Leibnitz formula, $P^{(k)}(0) = P^{(k)}(1) = 0$ for $0 \le k \le n-2$. Further
$$P^{(n-1)}(0) = (1-x)^{n-1}(A + x((-1)^{n-1}B - A))|_{x=0} = A,$$
$$P^{(n-1)}(1) = (-1)^{n-1}x^n(A + x((-1)^{n-1}B - A))|_{x=1} = B.$$

Claim 2. We have
$$\int_0^1 |P^{(n)}(x)|^2 dx = n^2A^2 + (-1)^n 2nAB + n^2B^2.$$

Proof: By the boundary conditions for P and integration by parts we obtain
$$\begin{aligned}
I_n &= \int_0^1 |P^{(n)}(x)|^2 dx = \int_0^1 P^{(n)}(x) dP^{(n-1)}(x) \\
&= P^{(n)}(1)B - P^{(n)}(0)A - \int_0^1 P^{(n+1)}(x)P^{(n-1)}(x)dx \\
&= P^{(n)}(1)B - P^{(n)}(0)A - \int_0^1 P^{(n+2)}(x)P^{(n-2)}(x)dx \\
&= \dots \\
&= P^{(n)}(1)B - P^{(n)}(0)A.
\end{aligned}$$
Let us calculate $P^{(n)}(1)$ and $P^{(n)}(0)$. By the Leibnitz formula we have
$$\begin{aligned}
&P^{(n)}(0) \\
&= \frac{1}{(n-1)!}\sum_{j=0}^{n}\binom{n}{j}((x-x^2)^{n-1})^{(j)}(A + x((-1)^{n-1}B - A))^{(n-j)}|_{x=0} \\
&= \frac{1}{(n-1)!}\big[((x-x^2)^{n-1})^{(n)}|_{x=0}A + n((x-x^2)^{n-1})^{(n-1)}|_{x=0}((-1)^{n-1}B - A)\big] \\
&= -(n^2-n)A + n((-1)^{n-1}B - A) \\
&= -n^2A + (-1)^{n-1}nB,
\end{aligned}$$
and by the same argument,
$$P^{(n)}(1) = n^2B + (-1)^n nA.$$
Then
$$\begin{aligned}
I_n &= (n^2B + (-1)^n nA)B - (-n^2A + (-1)^{n-1}nB)A \\
&= n^2A^2 + (-1)^n 2nAB + n^2B^2.
\end{aligned}$$

Claim 3 (Dirichlet principle). Suppose that f satisfies the boundary conditions of the problem. Then
$$\int_0^1 |f^{(n)}(x)|^2 dx \ge \int_0^1 |P^{(n)}(x)|^2 dx.$$

Proof. Denote $h(x) = f(x) - P(x)$. The function h satisfies homogeneous boundary conditions
$$h^{(j)}(0) = h^{(j)}(1) = 0, \quad j \in N.$$

Then

$$\begin{aligned}\int_0^1 |f^{(n)}(x)|^2dx &= \int_0^1 |P^{(n)}(x)|^2dx + 2\int_0^1 P^{(n)}(x)h^{(n)}(x)dx + \int_0^1 |h^{(n)}(x)|^2dx\\ &\ge \int_0^1 |P^{(n)}(x)|^2dx + \int_0^1 |h^{(n)}(x)|^2dx\\ &\ge \int_0^1 |P^{(n)}(x)|^2dx,\end{aligned}$$

because $\int_0^1 P^{(n)}(x)h^{(n)}(x)dx = 0$. It follows by boundary conditions for h and integration by parts as follows:

$$\begin{aligned}\int_0^1 P^{(n)}(x)h^{(n)}(x)dx &= \int_0^1 P^{(n)}(x)dh^{(n-1)}(x)\\ &= -\int_0^1 P^{(n+1)}(x)h^{(n-1)}(x)dx\\ &= +\int_0^1 P^{(n+2)}(x)h^{(n-2)}(x)dx\\ &= \dots\\ &= (-1)^n\int_0^1 P^{(2n)}(x)h(x)dx = 0,\end{aligned}$$

because P is a polynomial of order $2n-1$. This completes the proof of (a).

(b) Suppose that f satisfies the boundary conditions and $Q(x) = cx^{2n-1} + a_1x^{2n-2} + \dots + a_{2n-1}$ be the unique polynomial of order $2n-1$ satisfying boundary conditions $Q(0) = a$, $Q(1) = b$ and $Q'(0) = Q'(1) = \dots = Q^{(n-1)}(0) = Q^{(n-1)}(1) = 0$. Then as in Claim 3 we can prove that

$$\int_0^1 |f^{(n)}(x)|^2dx \ge \int_0^1 |Q^{(n)}(x)|^2dx. \tag{2.1}$$

To compute the right hand side of this equation, we use Hermite interpolation formula and divided differences. Recall some notions and facts on divided differences (cf. [2, pp. 96–104]).

Let g be a sufficiently smooth function defined in $m+1$ points $x_0 \le x_1 \le \dots \le x_m$ points and $x_0 = \dots = x_{\nu_i-1} = t_1 < x_{\nu_1} = \dots = x_{\nu_1+\nu_2-1} = t_2 < \dots$. The Hermite interpolation polynomial $H(x)$ of order m of function g satisfies the assumptions:

$$H^{(k)}(t_j) = g^{(k)}(t_j), \quad j = 1,\dots,l,\ k = 0,\dots,\nu_j-1,\ m+1 = \sum_{j=1}^{l}\nu_j.$$

It is determined by the Hermite interpolation formula

$$H(x) = \sum_{k=0}^{m-1} g[x_0,\dots,x_k](x-x_0)\dots(x-x_k),$$

where the divided differences $g[x_0,\dots,x_k]$ are determined recursively by the formula:

$$g[x_0,\dots,x_k] = \begin{cases}\dfrac{g[x_1,\dots,x_k]-g[x_0,\dots,x_{k-1}]}{x_k-x_0}, & x_k < x_0,\\ \frac{g^{(k)}(x_0)}{k!}, & x_k = x_0.\end{cases} \tag{2.2}$$

Let $Q(x) = cx^{2n-1} + a_1x^{2n-2} + \dots + a_{2n-1}$ be the unique Hermite interpolation polynomial satisfying boundary conditions $Q(0) = a$, $Q(1) = b$ and $Q'(0) = Q'(1) = \dots = Q^{(n-1)}(0) = Q^{(n-1)}(1) = 0$. We choose $x_0 = \dots = x_{n-1} = 0$ and $x_n = \dots = x_{2n-1} = 1$. We have $a_0 = Q[x_0, \dots, x_{2n-1}]$ and it can be determined by formula (2.2) as follows

$$Q[x_0] = \dots = Q[x_{n-1}] = a, \quad Q[x_n] = \dots = Q[x_{2n-1}] = b.$$

Next $Q[x_{n-1}, x_n] = b-a$ and all other 2-divided differences are equal to 0. Further $Q[x_{n-1}, x_n, x_{n+1}] = a - b$ and $Q[x_{n-2}, x_{n-1}, x_n] = b - a$ and all other 3-divided differences are equal to 0. Next $Q[x_{n-1}, x_n, x_{n+1}, x_{n+2}] = a - b$, $Q[x_{n-2}, x_{n-1}, x_n, x_{n+1}] = 2(a - b)$ and $Q[x_{n-3}, x_{n-2}, x_{n-1}, x_n] = b - a$. Coefficients to $a - b$ and $b - a$ grow like binomial coefficients in Pascal triangle. Finally we have

$$a_0 = Q[x_0, \dots, x_{2n-1}] = (-1)^n \binom{2n-2}{n-1}(a - b).$$

Observe that

$$\begin{aligned}
\int_0^1 |Q^{(n)}(x)|^2 dx &= \int_0^1 Q^{(n)}(x)dQ^{(n-1)}(x) \\
&= -\int_0^1 Q^{(n+1)}(x)Q^{(n-1)}(x)dx \\
&= (-1)^{n-1}(Q^{(2n-1)}(1)b - Q^{(2n-1)}(0)a) \\
&= (-1)^n(2n-1)!(a-b)a_0 \\
&= (-1)^n(2n-1)!(a-b)(-1)^n\binom{2n-2}{n-1}(a-b) \\
&= (2n-1)!\binom{2n-2}{n-1}(a-b)^2,
\end{aligned}$$

and the inequality in (b) is proved.

Note that the coefficient a_0 can be computed directly as follows. By the boundary conditions $Q(0) = a$ and $Q'(0) = \dots = Q^{(n-1)}(0) = 0$ we have

$$Q(x) = a_0x^{2n-1} + \dots + a_{n-1}x^n + a.$$

To satisfy the boundary conditions $Q(1) = b$ and $Q'(1) = \dots = Q^{(n-1)}(1) = 0$ one get the linear system for coefficients $a_0, \dots, a_{n-1}$:

$$\begin{gathered}
a_0 + a_1 + \dots + a_{n-1} = b - a, \\
(2n-1)a_0 + (2n-2)a_1 + \dots + na_{n-1} = 0, \\
\dots \\
(2n-1)\cdots(n-1)a_0 + (2n-2)\cdots(n-2)a_1 + \dots + n\dots 2a_{n-1} = 0.
\end{gathered}$$

Using Kramer formula one obtain that

$$a_0 = (b-a)(2n-2)\dots n\frac{D_{n-1}}{D_n},$$

where

$$D_n = \det\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2n-1 & 2n-2 & \dots & n \\ \dots & \dots & \dots & \dots \\ (2n-1)\cdots(n-1) & (2n-2)\cdots(n-2) & \dots & n\dots 2 \end{bmatrix}.$$

A direct computation shows that

$$D_n = (-1)^{n+1}(n-1)!D_{n-1} \tag{2.3}$$

which implies

$$a_0 = (b-a)(2n-2)\dots n\frac{(-1)^{n+1}}{(n-1)!} = (-1)^n\binom{2n-2}{n-1}(a-b).$$

Note that by (2.3), it follows that

$$D_n = \frac{(-1)^{n(n+3)/2}}{(n-1)!(n-2)!\dots 1!}.$$

which completes the proof of (b).

(c) We want to find $\min\{ \int_0^1 (f^{(n)}(x))^2 dx \}$ where

$$f \in C^n(I), \quad f^{(k)}(0) = A, \quad f^{(k)}(1) = B, \quad f^{(j)}(0) = f^{(j)}(1) = 0, \tag{2.4}$$

for $k \in N$ and $j \in N\backslash\{k\}$. As before the minimizer is a $2n-1$ order Hermite polynomial h which satisfies boundary conditions (2.4). We can show as in Claim 3 that

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq \int_0^1 |h^{(n)}(x)|^2 dx.$$

Integration by parts and the boundary conditions above imply

$$\int_0^1 |h^{(n)}(x)|^2 dx = (-1)^{n-k-1}(h^{(2n-k-1)}(1)B - h^{(2n-k-1)}(0)A) \tag{2.5}$$

We will find the coefficients of h with order greater or equal to $2n-k-1$ using divided differences. By boundary conditions (2.4) all differences in $k-$points are equal to 0. By (2.2) for $(k+1)-$points divided differences we have:

$$\begin{gathered}
h[x_0,\dots,x_k] = \dots = h[x_{n-k-1},\dots,x_{n-1}] = A/k!,\\
h[x_{n-k},\dots,x_n] = \dots = h[x_{n-1},\dots,x_{n+k-1}] = 0,\\
h[x_{n-k},\dots,x_n] = \dots = h[x_{n-1},\dots,x_{n+k-1}] = 0,\\
h[x_n,\dots,x_{n+k}] = \dots = h[x_{2n-k-1},\dots,x_{n+k-1}] = B/k!.
\end{gathered}$$

For $(k+2)$-point divided differences, we have

$$h[x_{n-k-1},\dots,x_n] = -A/k!, \quad h[x_{n-1},\dots,x_{n+k}] = B/k!$$

and all others are equal to 0 (by Newton's interpolation formula). Next, for $(k+3)$-point divided differences

$$\begin{gathered}
h[x_{n-k-2},\dots,x_n] = -A/k!, \quad h[x_{n-2},\dots,x_{n+k}] = B/k!,\\
h[x_{n-k-1},\dots,x_{n+1}] = A/k!, \quad h[x_{n-1},\dots,x_{n+k+1}] = -B/k!,
\end{gathered}$$

and all others are equal to 0. By (2.2) we calculate other divided differences the same way. The coefficient of x^n is $h[x_0,\dots,x_n] = -A/k!$ and the coefficient of $x^n(x-1)^j$ for $n-k-1 \leq j \leq n-1$ is

$$h[x_0,\dots,x_{n+j}] = \frac{1}{k!}[(-1)^{j+1}\binom{n-k-1+j}{j}A + (-1)^{j+k}\binom{n-k-1+j}{j-k}B].$$

Let

$$\begin{aligned} g(x) &= x^n \sum_{j=n-k-1}^{n-1} h[x_0,\dots,x_{n+j}](x-1)^j, \\ g_A(x) &= \frac{x^n}{k!}\sum_{j=n-k-1}^{n-1}(-1)^{j+1}\binom{n-k-1+j}{j}(x-1)^j, \\ g_B(x) &= \frac{x^n}{k!}\sum_{j=n-k-1}^{n-1}(-1)^{j+k}\binom{n-k-1+j}{j-k}(x-1)^j. \end{aligned}$$

We have $g(x) = g_A(x)A + g_B(x)B$. It is clear that $h^{(2n-k-1)}(a) = g^{(2n-k-1)}(a)$ where a is 0 or 1. Now we calculate $g_A^{(2n-k-1)}(x)$ and $g_B^{(2n-k-1)}(x)$. Observe that

$$\begin{aligned} g_A(x) &= \frac{(-1)^{n+k}x^n}{k!}\sum_{t=0}^{k}(-1)^t\binom{2n-2k-2+t}{n-k-1+t}(x-1)^{n-k-1+t}, \\ g_B(x) &= \frac{(-1)^{n+k}x^n}{k!}\sum_{t=0}^{k}(-1)^t\binom{2n-2k-2+t}{n-2k-1+t}(x-1)^{n-k-1+t}. \end{aligned}$$

Then, by the Leibnitz formula we obtain

$$\begin{aligned} & g_A^{(2n-k-1)}(1) \\ &= \frac{(-1)^{n+k}n!}{k!}\sum_{t=0}^{k}(-1)^t\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-k-1+t}, \\ & g_B^{(2n-k-1)}(1) \\ &= \frac{(-1)^{n-1}n!}{k!}\sum_{t=0}^{k}(-1)^t\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-2k-1+t}, \end{aligned}$$

and

$$\begin{aligned} h^{(2n-k-1)}(1) &= g^{(2n-k-1)}(1) \\ &= (-1)^n\frac{n!}{k!}\sum_{t=0}^{k}(-1)^t\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t} \\ &\quad \times\Big[(-1)^k\binom{2n-2k-2+t}{n-k-1+t}A-\binom{2n-2k-2+t}{n-2k-1+t}B\Big], \end{aligned}$$

$$\begin{aligned} h^{(2n-k-1)}(0) &= g^{(2n-k-1)}(0) \\ &= (-1)^n\frac{(2n-k-1)!}{k!}\sum_{t=0}^{k}\binom{n-k-1+t}{t} \\ &\quad \times\Big[(-1)^k\binom{2n-2k-2+t}{n-k-1+t}A-\binom{2n-2k-2+t}{n-2k-1+t}B\Big]. \end{aligned}$$

Finally, by (2.5), we obtain

$$\begin{aligned}
&\int_0^1 |h^{(n)}(x)|^2 dx \\
&= \frac{(2n-k-1)!}{k!}\sum_{t=0}^{k}\binom{n-k-1+t}{t}\binom{2n-2k-2+t}{n-k-1+t}A^2 \\
&\quad + \frac{n!}{k!}\sum_{t=0}^{k}(-1)^{k+t}\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-2k-1+t}B^2 \qquad (2.6)\\
&\quad - \Big[\frac{n!}{k!}\sum_{t=0}^{k}(-1)^{t}\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-k-1+t} \\
&\quad + (-1)^k\frac{(2n-k-1)!}{k!}\sum_{t=0}^{k}\binom{n-k-1+t}{t}\binom{2n-2k-2+t}{n-2k-1+t}\Big]AB
\end{aligned}$$

Note, that by

$$\binom{n-k-1+t}{t}\binom{2n-2k-2+t}{n-k-1+t} = \binom{2n-2k-2}{n-k-1}\binom{2n-2k-2+t}{t}$$

and

$$\sum_{t=0}^{k}\binom{2n-2k-2+t}{t} = \binom{2n-2k-2+k+1}{k} = \binom{2n-k-1}{k}$$

the coefficient to A^2 is

$$\frac{(2n-k-1)!}{k!}\binom{2n-2k-2}{n-k-1}\binom{2n-k-1}{k}.$$

We summarize above considerations as follows.

Claim 4. Let $f \in C^n(I)$ and

$$f^{(k)}(0) = A, \quad f^{(k)}(1) = B, \quad f^{(j)}(0) = f^{(j)}(1) = 0, \quad j \in N\backslash\{k\}. \qquad (2.7)$$

Then

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq \alpha_{n,k}A^2 - \gamma_{n,k}AB + \beta_{n,k}B^2, \qquad (2.8)$$

where

$$\begin{aligned}
\alpha_{n,k} &= \frac{(2n-k-1)!}{k!}\binom{2n-2k-2}{n-k-1}\binom{2n-k-1}{k}, \\
\beta_{n,k} &= \frac{n!}{k!}\sum_{t=0}^{k}(-1)^{k+t}\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-2k-1+t}, \\
\gamma_{n,k} &= \frac{n!}{k!}\sum_{t=0}^{k}(-1)^{t}\frac{(n-k-1+t)!}{t!}\binom{2n-k-1}{n-k-1+t}\binom{2n-2k-2+t}{n-k-1+t} \\
&\quad + (-1)^k\frac{(2n-k-1)!}{k!}\sum_{t=0}^{k}\binom{n-k-1+t}{t}\binom{2n-2k-2+t}{n-2k-1+t}.
\end{aligned}$$

The equality in (2.8) holds for the Hermit polynomial of degree $2n-1$ which satisfies the boundary conditions (2.7).

Let $p(x) = f(1-x)$. It is clear that $p^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ and

$$p^{(k)}(0) = (-1)^k B, \quad p^{(k)}(1) = (-1)^k A, \quad p^{(j)}(0) = p^{(j)}(1) = 0, \quad j \in N\backslash\{k\} \tag{2.9}$$

We obtain

$$\int_0^1 |p^{(n)}(x)|^2 dx = \int_0^1 [(-1)^n f^{(n)}(1-x)]^2 dx = -\int_1^0 |f^{(n)}(t)|^2 dt = \int_0^1 |f^{(n)}(t)|^2 dt.$$

Then

$$\begin{aligned} &\min\Big\{\int_0^1 |p^{(n)}(x)|^2 dx : p \text{ satisfies (2.9)}\Big\} \\ &= \min\Big\{\int_0^1 |f^{(n)}(x)|^2 dx : f \text{ satisfies (2.7)}\Big\}, \end{aligned} \tag{2.10}$$

and if we calculate $\min\int_0^1 |p^{(n)}(x)|^2 dx$ we will have $(-1)^k B$ instead of A and $(-1)^k A$ instead of B in (2.6):

$$\int_0^1 |p^{(n)}(x)|^2 dx \geq \beta_{n,k} A^2 - \gamma_{n,k} AB + \alpha_{n,k} B^2.$$

Finally, by (2.10) we obtain

$$\alpha_{n,k} A^2 - \gamma_{n,k} AB + \beta_{n,k} B^2 = \beta_{n,k} A^2 - \gamma_{n,k} AB + \alpha_{n,k} B^2,$$

and we have $\alpha_{n,k} = \beta_{n,k}$. The above equation shows that

$$\begin{aligned} \gamma_{n,k} &= (-1)^k \frac{(2n-k-1)!}{k!} \binom{2n-2k-2}{n-k-1} \binom{2n-k-1}{k} \\ &\quad + (-1)^k \frac{(2n-k-1)!}{k!} \sum_{t=0}^{k} \binom{n-k-1+t}{t} \binom{2n-2k-2+t}{n-2k-1+t}. \end{aligned}$$

and

$$\int_0^1 |f^{(n)}(x)|^2 dx \geq \alpha_{n,k} A^2 - \gamma_{n,k} AB + \alpha_{n,k} B^2,$$

which completes the proof.

Corollary 2.1. *Let $f \in C^n(I)$ be a differentiable function such that $f(0) = a$, $f(1) = b$, $f^{(n-1)}(0) = A$, $f^{(n-1)}(1) = B$ and*

$$f'(0) = f'(1) = \cdots = f^{(n-2)}(0) = f^{(n-2)}(1) = 0.$$

Then

$$\begin{aligned} \int_0^1 |f^{(n)}(x)|^2 dx &\geq n^2 A^2 + (-1)^n 2nAB + n^2 B^2 \\ &\quad + 2(-1)^n \frac{(2n-1)!}{(n-1)!} \big(B + (-1)^n A\big)(a-b) \\ &\quad + (2n-1)! \binom{2n-2}{n-1} (a-b)^2. \end{aligned}$$

Proof. As in steps (a) and (b) of Theorem 1.1, the polynomial $P(x) + Q(x)$ satisfies the boundary conditions $f(0) = a$, $f(1) = b$, $f^{(n-1)}(0) = A$, $f^{(n-1)}(1) = B$ and

$f'(0) = f'(1) = \dots = f^{(n-2)}(0) = f^{(n-2)}(1) = 0$. It is the minimizer of the functional $\int_0^1 |f^{(n)}(x)|^2 dx$ and

$$\begin{aligned}
&\int_0^1 |P^{(n)}(x) + Q^{(n)}(x)|^2 dx \\
&= \int_0^1 (|P^{(n)}(x)|^2 + 2P^{(n)}(x)Q^{(n)}(x) + |Q^{(n)}(x)|^2) dx \\
&= \int_0^1 |P^{(n)}(x)|^2 dx + 2(-1)^{n-1} P^{(2n-1)}(x)Q(x)|_{x=0}^{x=1} + \int_0^1 |Q^{(n)}(x)|^2 dx \\
&= n^2A^2 + (-1)^n 2nAB + n^2B^2 \\
&\quad + 2(-1)^n \frac{(2n-1)!}{(n-1)!}(B + (-1)^n A)(a-b) + (2n-1)!\binom{2n-2}{n-1}(a-b)^2,
\end{aligned}$$

which completes the proof. ☐

Corollary 2.2. *We have the equality*

$$\begin{aligned}
&\sum_{t=0}^{k} (-1)^{k+t} \frac{(n-k-1+t)!}{t!} \binom{2n-k-1}{n-k-1+t} \binom{2n-2k-2+t}{n-2k-1+t} \\
&= \frac{(2n-k-1)!}{n!} \binom{2n-2k-2}{n-k-1} \binom{2n-k-1}{k}.
\end{aligned}$$

The proof of the above corollary follows from the proof of (c) in Theorem 1.1, which is a variational proof of a combinatorial identity.

Remark. Numerical computations show that if $n = 4$ and $k = 2$ both sides of last identity are equal to 100. Straightforward computations show that

$$\begin{gathered}
\alpha_{n,n-1} = n^2, \quad \gamma_{n,n-1} = (-1)^{n-1} 2n, \\
\alpha_{n,0} = (2n-1)!\binom{2n-2}{n-1}, \\
\gamma_{n,0} = 2(2n-1)!\binom{2n-2}{n-1}.
\end{gathered}$$

Numerical computations for coefficients $\alpha_{n,k}$ and $\gamma_{n,k}$ for $n = 3$ and $n = 4$ are presented on following tables:

TABLE 1. $\alpha_{3,k}$ and $\gamma_{3,k}$ for $k = 0, 1, 2$.

k	2	1	0
$\alpha_{3,k}$	9	192	720
$\gamma_{3,k}$	6	-336	1440

TABLE 2. $\alpha_{4,k}$ and $\gamma_{4,k}$ for $k = 0, 1, 2, 3$.

k	3	2	1	0
$\alpha_{4,k}$	16	1200	25920	100800
$\gamma_{4,k}$	-8	1680	-48960	201600

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