Electronic Journal of Differential Equations, Vol. 2009(2009), No. 141, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS ON THE HALF LINE

XINGQIU ZHANG

Abstract. In this article, we study the existence of positive solutions for Sturm-Liouville boundary-value problems of a second-order nonlinear differential equation on the half line. Our approach is based on the fixed point theorem and the monotone iterative technique. Without assuming the existence of lower and upper solution, we obtain the existence of positive solutions, and establish iterative schemes for approximating the solutions.

1. Introduction

In this article, we prove the existence positive solutions, and establish a an iterative scheme for their approximation, for the following Sturm-Liouville boundary value problem of second-order differential equation on the half line

$$\begin{gathered} x''(t) + q(t)f(t, x(t), x'(t)) = 0, \quad t \in J_+, \\ \alpha x(0) - \beta x'(0) = 0, \quad x'(\infty) = x_\infty \geq 0, \end{gathered} \tag{1.1}$$

where $J = [0, +\infty)$, $J_+ = (0, +\infty)$, $\alpha > 0$, $\beta \geq 0$, $x'(\infty) = \lim_{t\to+\infty} x'(t)$ and $f(t, u, v) : J \times J \times J \to J$ is continuous. Throughout this paper, we assume the following conditions.

- (H1) $f(t, u, v) \in C(J \times J \times J, J)$, $f(t, 0, 0) \not\equiv 0$ on any subinterval of J and, when u, v are bounded, $f(t,(1 + t)u, v)$ is bounded on J;
- (H2) $q(t)$ is a nonnegative measurable function defined in J_+ and $q(t)$ does not identically vanish on any subinterval of J_+ and

$$0 < \int_0^{+\infty} q(t)\mathrm{d}t < +\infty, \quad 0 < \int_0^{+\infty} tq(t)\mathrm{d}t < +\infty.$$

Boundary value problems on half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium; see for example [1, 6, 7, 11, 15]. In the past few years, the existence and multiplicity of bounded or unbounded positive solutions to

2000 *Mathematics Subject Classification.* 34B10, 34B15, 34B18, 34B40.
Key words and phrases. Successive iteration; singular differential equation; half line; positive solution.

Submitted October 22, 2008. Published October 30, 2009.
Supported by grant 10671167 from the National Natural Science Foundation of China, and 31805 from the the Natural Science Foundation of Liaocheng University.

nonlinear differential equations on the half line have been studied by different type of techniques, we refer the reader to [1, 3, 5, 6, 7, 8, 9, 11, 14, 15] and references therein. Most of papers only considered the existence of positive solutions of various boundary value problems. Seeing such a fact, we cannot but ask "How can we find the solutions when they are known to exist?" More recently, Ma, Du and Ge [10], Sun and Ge [12, 13] proved the existence of positive solutions for some second-order p-Laplacian boundary value problems which are defined on finite intervals by virtue of the iterative technique.

To the best of our knowledge, up to now, it seems that no results in the literature are available for the computation of positive solutions for boundary value problems on the half line. Motivated by above papers, the purpose of this paper is to fill this gap. As we know, it is very important to check the compactness of the corresponding operator when we use the monotone iterative technique and Ascoli-Arzela theorem plays a very important role. However, Ascoli-Arzela theorem is not suitable for operators on the half line. So, it is needed to list some new conditions to meet the requirement of compactness.

2. Preliminaries

First, we give some definitions.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$;
(2) $u, -u \in P$ implies $u = 0$.

A map $\alpha : P \to [0, +\infty)$ is said to be concave on P, if

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in P$ and $t \in [0, 1]$.

We will use the following space to study (1.1)

$$E = \big\{x \in C^1[0, +\infty) : \sup_{t\in J} \frac{|x(t)|}{t+1} < \infty,\ \lim_{t\to+\infty} x'(t) \text{ exists}\big\}.$$

Then E is a Banach space equipped with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty\}$, where $\|x\|_1 = \sup_{t\in J} \frac{|x(t)|}{t+1}$, $\|x'\|_\infty = \sup_{t\in J} |x'(t)|$. Let $E_+ = \{x \in E : x(t) \geq 0\}$. Define the cone $P \subset E$ by

$$\begin{aligned} P = \big\{&x \in E : x(t) \geq 0, t \in [0, +\infty), x \text{ is concave on } [0, +\infty), \\ &\alpha x(0) - \beta x'(0) = 0, \text{ and } \lim_{t\to+\infty} x'(t) \text{ exists}\big\}. \end{aligned}$$

Remark 2.2. If x satisfies (1.1), then $x''(t) = -q(t)f(t, x(t), x'(t)) \leq 0$ on $[0, +\infty)$, which implies that x is concave on $[0, +\infty)$. Moreover if $x'(\infty) = x_\infty \geq 0$, then $x'(t) \geq 0, t \in [0, +\infty)$ and so x is monotone increasing on $[0, +\infty)$.

Lemma 2.3. *Assume that* (H1)-(H2) *hold. Then $x \in E_+ \cap C^2[J_+, J]$ is a solution of* (1.1) *if and only if $x \in C[J, E]$ is a solution of the integral equation*

$$\begin{aligned} x(t) = &\frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)f(s, x(s), x'(s))\mathrm{d}s + x_\infty\Big) \\ &+ \int_0^t \int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty. \end{aligned} \tag{2.1}$$

Proof. Suppose that $x \in E_+ \cap C^2[J_+, J]$ is a solution of (1.1). For $t \in J$, integrating (1.1) from t to $+\infty$, we have

$$x'(t) = x_\infty + \int_t^{+\infty} f(s, x(s), x'(s))\mathrm{d}s. \tag{2.2}$$

Integrating from 0 to t,

$$x(t) = x(0) + tx_\infty + \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s. \tag{2.3}$$

Thus, by (2.2) we obtain

$$x'(0) = x_\infty + \int_0^{+\infty} f(s, x(s), x'(s))\mathrm{d}s,$$

which together with the boundary value condition implies

$$x(0) = \frac{\beta}{\alpha}\Big(x_\infty + \int_0^{+\infty} f(s, x(s), x'(s))\mathrm{d}s\Big). \tag{2.4}$$

Substituting the above expression into (2.3), we have

$$\begin{aligned} x(t) = &\frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s) f(s, x(s), x'(s))\mathrm{d}s + x_\infty\Big) \\ &+ \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s + tx_\infty. \end{aligned}$$

Next, we show that the integral $\int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s$ are convergent. Since $x \in E_+$, then there exists r_0 such that $\|x\| < r_0$. Set $B_{r_0} = \sup\{f(t,(1+t)u, v) | (t, u, v) \in J \times [0, r_0] \times [0, r_0]\}$, and we have by exchanging the integral order

$$\begin{aligned} \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s &\le \int_0^{+\infty} \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s \\ &\le \int_0^{+\infty} sq(s)\mathrm{d}s \cdot B_{r_0}. \end{aligned} \tag{2.5}$$

By (H2), we know that $\int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s$ is convergent. Thus, we have proved that the right term in (2.1) is well defined. Conversely, if x is a solution of integral equation, then direct differentiation gives the proof. $\square$

Now, we define an operator $A : P \to C^1[0, +\infty)$ by

$$\begin{aligned} (Ax)(t) = &\frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s) f(s, x(s), x'(s))\mathrm{d}s + x_\infty\Big) \\ &+ \int_0^t \int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s + tx_\infty. \end{aligned} \tag{2.6}$$

To obtain the complete continuity of A, the following lemma is needed.

Lemma 2.4 ([2, 8]). *Let W be a bounded subset of P. Then W is relatively compact in E if $\{W(t)/(1+t)\}$ and $\{W'(t)\}$ are both equicontinuous on any finite subinterval of $[0, +\infty)$ and for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$\Big|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\Big| < \varepsilon, \quad |x'(t_1) - x'(t_2)| < \varepsilon, \quad \forall\ t_1, t_2 \ge N,$$

uniformly with respect to $x \in W$ *as* $t_1, t_2 \geq N$, *where* $W(t) = \{x(t) | x \in W\}$, $W'(t) = \{x'(t) | x \in W\}$, $t \in [0, +\infty)$.

Lemma 2.5. *Assume that* (H1)-(H2) *hold. Then* $A : P \to P$ *is completely continuous.*

Proof. It is clear that $(Ax)(t) \geq 0$ for any $x \in P$, $t \in J$. By (2.6), we have

$$(Ax)'(t) = \int_t^{+\infty} q(s)f(s, x(s), x'(s))\mathrm{d}s + x_\infty \geq 0, \tag{2.7}$$

$$(Ax)''(t) = -q(t)f(t, x(t)) \leq 0. \tag{2.8}$$

These two inequalities imply that $(TP) \subset P$. Now, we prove that A is continuous and compact respectively. Let $x_n \to x$ as $n \to \infty$ in P, then there exists r_0 such that $\sup_{n \in N \setminus \{0\}} \|x_n\| < r_0$. Let $B_{r_0} = \sup\{f(t, (1+t)u, v) | (t, u, v) \in J \times [0, r_0] \times [0, r_0]\}$. By (H2), we have

$$\int_0^{+\infty} q(\tau)|f(\tau, x_n(\tau), x_n'(\tau)) - f(\tau, x(\tau), x'(\tau))| d\tau \leq 2B_{r_0} \cdot \int_0^{+\infty} q(s)\mathrm{d}s < +\infty, \tag{2.9}$$

and

$$\begin{aligned} &\int_0^t \int_s^{+\infty} q(\tau)|f(\tau, x_n(\tau), x_n'(\tau)) - f(\tau, x(\tau), x'(\tau))|\mathrm{d}\tau\mathrm{d}s \\ &\leq \int_0^{+\infty} \int_s^{+\infty} q(\tau)|f(\tau, x_n(\tau)) - f(\tau, x(\tau), x'(\tau))|\mathrm{d}\tau\mathrm{d}s \\ &\leq 2B_{r_0} \int_0^{+\infty} sq(s)\mathrm{d}s < +\infty. \end{aligned} \tag{2.10}$$

It follows from (2.6), (2.9), (2.10), and the Lebesgue dominated convergence theorem that

$$\begin{aligned} \|Ax_n - Ax\|_1 &= \sup_{t \in J} \Big\{ \frac{1}{1+t} \Big| \frac{\beta}{\alpha} \Big(\int_0^{+\infty} q(s)(f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)))\mathrm{d}s \Big) \\ &\quad + \int_0^t \int_s^{+\infty} q(\tau)(f(\tau, x_n(\tau), x_n'(\tau)) - f(\tau, x(\tau), x'(\tau)))\mathrm{d}\tau\mathrm{d}s \Big| \Big\} \\ &\leq \sup_{t \in J} \Big\{ \frac{1}{1+t} \frac{\beta}{\alpha} \Big(\int_0^{+\infty} q(s)|f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s))|\mathrm{d}s \Big) \Big\} \\ &\quad + \sup_{t \in J} \Big\{ \int_0^t \int_s^{+\infty} q(\tau)|f(\tau, x_n(\tau)) - f(\tau, x(\tau), x'(\tau))|\mathrm{d}\tau\mathrm{d}s \Big\} \to 0, \end{aligned}$$

as $n \to \infty$. Also

$$\begin{aligned} \|(Ax_n)' - (Ax)'\|_\infty &= \sup_{t \in J} \Big\{ \int_t^{+\infty} q(s)|f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s))|\mathrm{d}s \Big\} \\ &\leq 2B_{r_0} \int_0^{+\infty} q(s)\mathrm{d}s < +\infty. \end{aligned}$$

Therefore, A is continuous.

Let Ω be any bounded subset of P. Then, there exists $r > 0$ such that $\|x\| \leq r$ for any $x \in \Omega$. Therefore, we have

$\|Ax\|_1$

$$
\begin{aligned}
&= \sup_{t\in J}\Big\{\frac{1}{1+t}\Big|\frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s + x_\infty\Big) \\
&\quad + \int_0^t\int_s^{+\infty} q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s + t x_\infty\Big|\Big\} \\
&\le \frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)\mathrm{d}s\cdot B_r + x_\infty\Big) + \sup_{t\in J}\Big\{\frac{t}{1+t}\int_0^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s\Big\} + x_\infty \\
&\le \Big(\frac{\beta}{\alpha}+1\Big)\Big(\int_0^{+\infty} q(s)\mathrm{d}s\cdot B_r + x_\infty\Big),
\end{aligned}
$$

and

$$
\|(Ax)'\|_\infty = \sup_{t\in J}\Big\{\Big|\int_t^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s + x_\infty\Big|\Big\} \le \int_0^{+\infty} q(s)\mathrm{d}s\cdot B_r + x_\infty.
$$

So, $T\Omega$ is bounded. Remembering that the integral $\int_0^{+\infty}\int_s^{+\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s$ is convergent, so for any $T\in J_+$ and $t_1,t_2\in[0,T]$, by the absolute continuity of the integral, we have

$$
\begin{aligned}
&\Big|\frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2}\Big| \\
&\le \frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s + x_\infty\Big)\cdot\Big|\frac{1}{1+t_1} - \frac{1}{1+t_2}\Big| \\
&\quad + \Big|\frac{1}{1+t_1}\int_0^{t_1}\int_s^{+\infty} q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s \\
&\quad - \frac{1}{1+t_2}\int_0^{t_2}\int_s^{+\infty} q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s\Big| \\
&\quad + \Big|\frac{t_1}{1+t_1} - \frac{t_1}{1+t_1}\Big|x_\infty \\
&\le \frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)\mathrm{d}s\cdot B_r + x_\infty\Big)\Big|\frac{1}{1+t_1} - \frac{1}{1+t_2}\Big| + \frac{1}{1+t_1}\Big|\int_{t_1}^{t_2}\int_s^{+\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s\cdot B_r\Big| \\
&\quad + \int_0^{t_2}\int_s^{+\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s\cdot B_r\Big|\frac{1}{1+t_1} - \frac{1}{1+t_2}\Big| + \Big|\frac{t_1}{1+t_1} - \frac{t_1}{1+t_1}\Big|x_\infty
\end{aligned}
$$

which approaches zero, uniformly as $t_1\to t_2$. Also

$$
|(Ax)'(t_1) - (Ax)'(t_2)| = \Big|\int_{t_1}^{t_2} q(s)f(s,x(s),x'(s))\mathrm{d}s\Big| \le B_r\cdot\Big|\int_{t_1}^{t_2} q(s)\mathrm{d}s\Big|
$$

which approaches zero, uniformly as $t_1\to t_2$. Thus, we have proved that $T\Omega$ is equicontinuous on any finite subinterval of $[0,+\infty)$.

Next, we prove that for any $\varepsilon>0$, there exits sufficiently large $N>0$ such that

$$
\Big|\frac{(Ax)(t_1)}{1+t_1} - \frac{(Ax)(t_2)}{1+t_2}\Big| < \varepsilon, \quad |(Ax)'(t_1) - (Ax)'(t_2)| < \varepsilon, \tag{2.11}
$$

for all $t_1,t_2\ge N$ and all $x\in\Omega$. For any $x\in\Omega$, we have

$$
\lim_{t\to\infty}\Big|\frac{(Ax)(t)}{1+t}\Big| = \lim_{t\to\infty}\frac{1}{1+t}\int_0^t\int_s^{+\infty} q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s + x_\infty. \tag{2.12}
$$

Similar to (2.5), we get

$$\int_0^t \int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s \le B_r \int_0^{+\infty} \tau q(\tau)\mathrm{d}\tau < +\infty,$$

which shows that

$$\lim_{t\to\infty} \frac{1}{1+t} \int_0^t \int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))\mathrm{d}\tau \mathrm{d}s = 0. \tag{2.13}$$

On the other hand, we arrive at

$$\begin{aligned}\lim_{t\to\infty} |(Ax)'(t)| &= \lim_{t\to\infty} \int_t^{+\infty} q(s)f(s, x(s), x'(s))\mathrm{d}s + x_\infty \\ &\le B_r \cdot \lim_{t\to\infty} \int_t^{+\infty} q(s)\mathrm{d}s + x_\infty = x_\infty.\end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13), and (2.14) that $\left|\frac{(Ax)(t)}{1+t}\right|$ and $|(Ax)'(t)|$ tend to x_∞ uniformly as $t \to \infty$. So, for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\Big|\frac{(Ax)(t)}{1+t} - x_\infty\Big| < \frac{\varepsilon}{2}, \quad \forall\, t \ge N_1.$$

Consequently, for any $t_1, t_2 \ge N_1$, we have

$$\Big|\frac{(Ax)(t_1)}{1+t_1} - x_\infty\Big| < \frac{\varepsilon}{2}, \quad \Big|\frac{(Ax)(t_2)}{1+t_2} - x_\infty\Big| < \frac{\varepsilon}{2}. \tag{2.15}$$

Similarly, we can prove that there exists $N_2 > 0$ such that

$$\Big|(Ax)'(t_1) - x_\infty\Big| < \frac{\varepsilon}{2}, \quad \Big|(Ax)'(t_2) - x_\infty\Big| < \frac{\varepsilon}{2}, \quad \forall\, t_1, t_2 \ge N_2. \tag{2.16}$$

Choose $N = \max\{N_1, N_2\}$, then (2.11) can be easily seen by (2.15) and (2.16). By Lemma 2.4, we know that $A : P \to P$ is completely continuous. $\square$

3. Main results

For notational convenience, we denote

$$m = \Big(\frac{\beta}{\alpha} + 1\Big)x_\infty, \quad n = \frac{\beta}{\alpha}\int_0^{+\infty} q(\tau)\mathrm{d}\tau + \max\Big\{\int_0^{+\infty} q(\tau)\mathrm{d}\tau, \int_0^{+\infty} \tau q(\tau)\mathrm{d}\tau\Big\}.$$

We will prove the following existence results.

Theorem 3.1. *Assume that* (H1)-(H2) *hold, and there exists* $a > 2m$ *such that*

- (S1) $f(t, x_1, y_1) \le f(t, x_2, y_2)$ *for any* $0 \le t < +\infty$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$;
- (S2) $f(t, (1+t)u, v) \le \frac{a}{2n}$, $(t, u, v) \in [0, +\infty) \times [0, a] \times [0, a]$.

Then the boundary value problem (1.1) *has two positive nondecreasing on* $[0, +\infty)$ *and concave solutions* w^* *and* v^*, *such that* $0 < \|w^*\| \le a$, *and* $\lim_{n\to\infty} w_n = \lim_{n\to\infty} A^n w_0 = w^*$, *where*

$$w_0(t) = \frac{a}{2}(t+1) + \frac{\beta}{\alpha}x_\infty + tx_\infty, \quad t \in J,$$

and $0 < \|v^*\| \le a$, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^*$, *where* $v_0(t) = 0$, $t \in J$.

Proof. By Lemma 2.5, we know that $A : P \to P$ is completely continuous. For any $x_1, x_2 \in P$ with $x_1 \le x_2$, $x_1' \le x_2'$, from the definition of A and (S1), we can easily get that $Ax_1 \le Ax_2$. We denote

$$\overline{P}_a = \{x \in P : \|x\| \le a\}.$$

First, we prove that $A : \overline{P}_a \to \overline{P}_a$. If $x \in \overline{P}_a$, then $\|x\| \le a$. By (2.1), (S1) and (S_2), we get

$$\begin{aligned}
\|Ax\|_1 &= \sup_{t\in J}\Big\{\frac{1}{1+t}\Big|\frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s + x_\infty\Big) \\
&\quad + \int_0^t\int_s^{+\infty} q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty\Big|\Big\} \\
&\le \Big(\frac{\beta}{\alpha}+1\Big)x_\infty + \Big(\frac{\beta}{\alpha}+1\Big)\int_0^{+\infty} q(\tau)\mathrm{d}\tau\cdot\frac{a}{2n} \\
&\le m + n\cdot\frac{a}{2n} \le a,
\end{aligned}$$

and

$$\begin{aligned}
\|(Ax)'\|_\infty &= \sup_{t\in J}\Big\{\Big|\int_t^{+\infty} q(s)f(s,x(s),x'(s))\mathrm{d}s + x_\infty\Big|\Big\} \\
&\le \int_0^{+\infty} q(s)\mathrm{d}s\cdot\frac{a}{2n} + x_\infty \le a.
\end{aligned}$$

Hence, we have proved that $A : \overline{P}_a \to \overline{P}_a$. Let $w_0(t) = \frac{a}{2}(t+1) + \frac{\beta}{\alpha}x_\infty + tx_\infty, 0 \le t < +\infty$, then $w_0(t) \in \overline{P}_a$. Let $w_1 = Aw_0, w_2 = A^2w_0$, then by Lemma 2.5, we have that $w_1 \in \overline{P}_a$ and $w_2 \in \overline{P}_a$. We denote $w_{n+1} = Aw_n = A^n w_0, n = 0, 1, 2, \dots$. Since $A : \overline{P}_a \to \overline{P}_a$, we have $w_n \in A(\overline{P}_a) \subset \overline{P}_a, n = 1, 2, 3, \dots$. It follows from the complete continuity of A that $\{w_n\}_{n=1}^\infty$ is a sequentially compact set.

By (2.1) and (S2), we get

$$\begin{aligned}
w_1(t) &= \frac{\beta}{\alpha}\Big(\int_0^{+\infty} q(s)f(s,\omega_0(s),\omega_0'(s))\mathrm{d}s + x_\infty\Big) \\
&\quad + \int_0^t\int_s^{+\infty} q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty \\
&\le \frac{\beta}{\alpha}\int_0^{+\infty} q(s)\mathrm{d}s\cdot\frac{a}{2n} + \frac{\beta}{\alpha}x_\infty + \int_0^{+\infty}\tau q(\tau)\mathrm{d}\tau\cdot\frac{a}{2n} + tx_\infty \\
&\le \frac{a}{2} + \frac{\beta}{\alpha}x_\infty + tx_\infty \le w_0(t),\ 0 \le t < +\infty,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\omega_1'(t) = (A\omega_0)'(t) &= \int_t^{+\infty} q(s)f(s,\omega_0(s),\omega_0'(s))\mathrm{d}s + x_\infty \\
&\le \int_0^{+\infty} q(s)\mathrm{d}s\cdot\frac{a}{2n} + x_\infty \\
&\le \frac{a}{2} + x_\infty = \omega_0'(t), \quad 0 \le t < +\infty.
\end{aligned} \tag{3.2}$$

So, by (3.1), (3.2) and (S1) we have

$$\begin{gathered}
w_2(t) = (Aw_1)(t) \le (Aw_0)(t) = w_1(t), \quad 0 \le t < +\infty, \\
w_2'(t) = (Aw_1)'(t) \le (Aw_0)'(t) = (w_1)'(t),\ 0 \le t < +\infty.
\end{gathered}$$

By induction, we get

$$w_{n+1}(t) \le w_n(t), \quad w'_{n+1}(t) \le w'_n(t), \quad 0 \le t < +\infty,\ n = 0, 1, 2, \dots.$$

Thus, there exists $w^* \in \overline{P}_a$ such that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of A and $w_{n+1} = Aw_n$, we get that $Aw^* = w^*$.

Let $v_0(t) = 0, 0 \le t < +\infty$, then $v_0(t) \in \overline{P}_a$. Let $v_1 = Av_0, v_2 = A^2 v_0$, then by Lemma 2.5, we have that $v_1 \in \overline{P}_a$ and $v_2 \in \overline{P}_a$. We denote $v_{n+1} = Av_n = A^n v_0, n = 0, 1, 2, \dots$. Since $A : \overline{P}_a \to \overline{P}_a$, we have $v_n \in A(\overline{P}_a) \subset \overline{P}_a, n = 1, 2, 3, \dots$. It follows from the complete continuity of A that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $v_1 = Av_0 \in \overline{P}_a$, we have

$$\begin{gathered} v_1(t) = (Av_0)(t) = (A0)(t) \ge 0, \quad 0 \le t < +\infty, \\ v'_1(t) = (Av_0)'(t) = (A0)'(t) = v'_0(t) \ge 0, \quad 0 \le t < +\infty. \end{gathered}$$

So, that we have

$$\begin{gathered} v_2(t) = (Av_1)(t) \ge (A0)(t) = v_1(t), \quad 0 \le t < +\infty, \\ v'_2(t) = (Av_1)'(t) \ge (A0)'(t) = v'_1(t), \quad 0 \le t < +\infty. \end{gathered}$$

By induction, we get

$$v_{n+1}(t) \ge v_n(t), \quad v'_{n+1}(t) \ge v'_n(t), \quad 0 \le t < +\infty,\ n = 0, 1, 2, \dots.$$

Thus, there exists $v^* \in \overline{P}_a$ such that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of A and $v_{n+1} = Av_n$, we get that $Av^* = v^*$.

If $f(t, 0, 0) \not\equiv 0$, $0 \le t < \infty$, then the zero function is not the solution of (1.1). Thus, v^* is a positive solution of (1.1). It is well known that each fixed point of A in P is a solution of (1.1). Hence, we assert that w^* and v^* are two positive, nondecreasing on $[0, +\infty)$ and concave solutions of (1.1). □

Remark 3.2. The iterative schemes in Theorem 3.1 are $w_0(t) = \frac{a}{2}(t+1) + \frac{\beta}{\alpha} x_\infty + t x_\infty, w_{n+1} = Aw_n = A^n w_0, n = 0, 1, 2, \dots$ and $v_0(t) = 0, v_{n+1} = Av_n = A^n v_0, n = 0, 1, 2, \dots$. They start off with a known simple linear function and the zero function respectively. It is convenient in application. We can easily get that w^* and v^* are the maximal and minimal solutions of the boundary value problem (1.1). Of course w^* and v^* may coincide and then the boundary value problem (1.1) has only one solution in P.

The following theorem can be obtained directly from Theorem 3.1.

Theorem 3.3. *Assume that* (H1)-(H2) *hold and there exists* $2m < a_1 < a_2 < \dots < a_n$ *such that*

- (S1) $f(t, x_1, y_1) \le f(t, x_2, y_2)$ *for any* $0 \le t < +\infty$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$*;*
- (S2) $f(t, (1+t)u, v) \le \frac{a_k}{2n}$, $(t, u, v) \in [0, +\infty) \times [0, a_k] \times [0, a_k]$, $k = 1, 2, \dots, n$.

Then the boundary value problem (1.1) *has* $2n$ *positive nondecreasing on* $[0, +\infty)$ *and concave solutions* w_k^* *and* v_k^**, such that* $0 < \|w_k^*\| \le a_k$*, and* $\lim_{n\to\infty} w_{kn} = \lim_{n\to\infty} A^n w_{k0} = w_k^*$*, where*

$$w_{k0}(t) = \frac{a_k}{2}(t+1) + \frac{\beta}{\alpha} x_\infty + t x_\infty, \quad t \in J,$$

and $0 < \|v_k^*\| \le a_k$, $\lim_{n\to\infty} v_{kn} = \lim_{n\to\infty} A^n v_0 = v_k^*$*, where* $v_{k0}(t) = 0$, $t \in J$.

4. Example

Consider the boundary-value problem

$$\begin{gathered} x''(t) + \frac{1}{\sqrt{t}(1+t)^2} f(t, x(t), x'(t)) = 0, \quad t \in J_+, \\ 2x(0) - 3x'(0) = 0, \quad x'(\infty) = 0, \end{gathered} \tag{4.1}$$

where

$$f(t,u,v) = \begin{cases} 10^{-2}|\cos(2t+1)| + 10^{-2}\Big(\frac{u}{1+t}\Big)^4 + \frac{1}{10}\Big(\frac{v}{400}\Big), & u \le 3, \\ 10^{-2}|\cos(2t+1)| + 10^{-2}\Big(\frac{3}{1+t}\Big)^4 + \frac{1}{10}\Big(\frac{v}{400}\Big), & u \ge 3. \end{cases}$$

Set $q(t) = \frac{1}{\sqrt{t}(1+t)^2}$. It is clear that (H1) and (S2) hold. Let $\alpha = 2$, $\beta = 3$, $x_\infty = 0$. By direct computation, we can obtain that

$$\int_0^{+\infty} q(t)\mathrm{d}t = \int_0^{+\infty} \frac{1}{\sqrt{t}(1+t)^2}\mathrm{d}t < \int_0^1 \frac{1}{\sqrt{t}}\mathrm{d}t + \int_1^{+\infty} \frac{1}{\sqrt{t}\cdot t^2}\mathrm{d}t = \frac{8}{3}, \tag{4.2}$$

$$\int_0^{+\infty} tq(t)\mathrm{d}t = \int_0^{+\infty} \frac{t}{\sqrt{t}(1+t)^2}\mathrm{d}t < \int_0^1 \sqrt{t}\mathrm{d}t + \int_1^{+\infty} \frac{\sqrt{t}}{t^2}\mathrm{d}t = \frac{8}{3}. \tag{4.3}$$

By (4.2) and (4.3), we have $m = 0$, $n < \frac{20}{3}$. Choose $a = 300$ and check check (S2). Since nonlinear term f satisfies

$$f(t,(1+t)u,v) \le \frac{1}{10^2} + \frac{81}{100} + \frac{3}{40} = \frac{179}{200} < \frac{300}{2\cdot\frac{20}{3}} < \frac{300}{2n},$$

for $t \in [0,+\infty)$, $u, v \in [0, 300]$; then all the conditions in Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3.1 holds.

References

[1] R. P. Agarwal, D. O'Regan; *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, (2001).
[2] C. Corduneanu; *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.
[3] H. Lian, H. Pang, W. Ge; *Triple positive solutions for boundary value problems on infinite intervals*, Nonlinear Anal., 67, 2199-2207 (2007).
[4] S. H. Liang, J. H. Zhang; *The existence of countably many positive solutions for some nonlinear three-point boundary problems on the half-line*, Nonlinear Analysis (2008) doi:10.1016/j.na.2008.04.016
[5] S. H. Liang, J. H. Zhang; *The existence of countably many positive solutions for one-dimensional p-Laplacian with infinitely many singularities on the half-line*, Appl. Math. Comput., 201, 210-220 (2008).
[6] X. Liu; *Solutions of impulsive boundary value problems on the half-line*, J. Math. Anal. Appl., 222, 411-430 (1998).
[7] Y. S. Liu; *Existence and unboundedness of positive solutions for singular boudary value problems on half-line*, Appl. Math. Comput., 144, 543-556 (2003).
[8] Y. S. Liu; *Boundary value probelm for second order differential equations on unbounded domain*, Acta Anal. Funct. Appl., 4 (3), 211-216 (2002) (in Chinese).
[9] L. S. Liu, Z. B. Liu, Y. H. Wu; *Infinite boundary value problems for nth-order nonlinear impulsive integro-differential equations in Banach spaces*, Nonlinear Anal., 67, 2670-2679 (2007).
[10] D. X. Ma, Z. J. Du, W. G. Ge; *Existence and iteration of monotone positive solutions for multipoint boundary value problems with p-Laplacian operator*, Comput. Math. Appl., 50, 729-739 (2005).
[11] D. O'Regan; *Theory of Singular Boundary Value Problems*, World Scientific, Singapore, 1994.

[12] B. Sun, W. G. Ge; *Existence and iteration of positive solutions for some p-Laplacian boundary value problems*, Nonlinear Anal., 67, 1820-1830 (2007).

[13] B. Sun, W. Ge; *Successive iteration and positive pseudo-symmetric solutions for a three-point second-order p-Laplacian boundary value problems*, Appl. Math. Comput., 188, 1772-1779 (2007).

[14] B. Q. Yan; *Boundary value problems on the half-line with impulses and infinite delay*, J. Math. Anal. Appl., 259, 94-114 (2001).

[15] M. Zima; *On positive solution of boundary value problems on the half-line*, J. Math. Anal. Appl., 259, 127-136 (2001).

Xingqiu Zhang
School of Mathematics, Liaocheng University, Liaocheng, Shandong, China
E-mail address: zhxq197508@163.com