Electronic Journal of Differential Equations, Vol. 2009(2009), No. 144, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

WEAK SOLUTIONS FOR ANISOTROPIC NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. We study the anisotropic boundary-value problem

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial}{\partial x_i}u)=f \quad \text{in } \Omega,$$
$$u=0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 3)$. We obtain the existence and uniqueness of a weak energy solution for $f\in L^\infty(\Omega)$, and the existence of weak energy solution for general data f dependent on u.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N\geq 3)$ with smooth boundary $\partial\Omega$. Our aim is to prove existence and uniqueness of a weak energy solution to the anisotropic nonlinear elliptic problem

$$\begin{gathered}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial}{\partial x_i}u)=f \quad \text{in } \Omega\\ u=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where the right hand side f is in $L^\infty(\Omega)$. We assume that for $i=1,\dots,N$ the function $a_i:\Omega\times\mathbb{R}\to\mathbb{R}$ is Carathéodory; i.e., $a(x,.)$ is continuous for a.e. $x\in\Omega$ and $a(.,t)$ is measurable for every $t\in\mathbb{R}$ and satisfy the following conditions: $a_i(x,\xi)$ is the continuous derivative with respect to ξ of the mapping $A_i:\Omega\times\mathbb{R}\to\mathbb{R}$, $A_i=A_i(x,\xi)$; i.e., $a_i(x,\xi)=\frac{\partial}{\partial\xi}A_i(x,\xi)$ such that:

The following equatility holds

$$A_i(x,0)=0, \tag{1.2}$$

for almost every $x\in\Omega$.

There exists a positive constant C_1 such that

$$|a_i(x,\xi)|\leq C_1(j_i(x)+|\xi|^{p_i(x)-1}) \tag{1.3}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}$, where j_i is a nonnegative function in $L^{p_i'(.)}(\Omega)$, with $1/p_i(x)+1/p_i'(x)=1$.

2000 *Mathematics Subject Classification.* 35J20, 35J25, 35D30, 35B38, 35J60.

Key words and phrases. Anisotropic Sobolev spaces; weak energy solution; variable exponents; electrorheological fluids.

Submitted February 10, 2008. Published November 12, 2009.

The following inequality holds

$$\big(a_i(x,\xi)-a_i(x,\eta)\big).\big(\xi-\eta\big)>0 \tag{1.4}$$

for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}$, with $\xi\neq\eta$.

The following inequalities hold

$$|\xi|^{p_i(x)}\le a_i(x,\xi).\xi\le p_i(x)A_i(x,\xi) \tag{1.5}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}$.

For the exponent $p_1(.),\dots,p_N(.)$, we assume that $p_i(.):\overline{\Omega}\to\mathbb{R}$ are continuous functions such that:

$$2\le p_i(x)<N,\quad \sum_{i=1}^{N}\frac{1}{p_i^-}>1, \tag{1.6}$$

where

$$p_i^-:=\operatorname*{ess\,inf}_{x\in\Omega}p_i(x),p_i^+:=\operatorname*{ess\,sup}_{x\in\Omega}p_i(x).$$

A prototype example that is covered by our assumptions is the following anisotropic $(p_1(.),\dots,p_N(.))$-harmonic problem: Set

$$A_i(x,\xi)=\big(1/p_i(x)\big)|\xi|^{p_i(x)},\quad a_i(x,\xi)=|\xi|^{p_i(x)-2}\xi$$

where $p_i(x)\ge 2$. Then we obtain the problem:

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(|\frac{\partial}{\partial x_i}u|^{p_i(x)-2}\frac{\partial}{\partial x_i}u\Big)=f$$

which, in the particular case when $p_i=p$ for any $i\in\{1,\dots,N\}$ is the $p(.)$-Laplace equation.

The study of nonlinear elliptic equations involving the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. For the nonhomogeneous $p(.)$-Laplace operators, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$. The spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$ were thoroughly studied by Musielak [18], Edmunds et al [7, 8, 9], Kovacik and Rakosnik [13], Diening [5, 6] and the references therein.

Variable Sobolev spaces have been used in the last decades to model various phenomena. Chen, Levine and Rao [4] proposed a framework for image restoration based on a variable exponent Laplacian. An other application which uses nonhomogeneous Laplace operators is related to the modelling of electrorheological fluids. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For some technical applications, consult Pfeiffer et al [19]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA Laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer to Diening [5], Rajagopal and Ruzicka [20], and Ruzicka [21].

In this paper, the operator involved in (1.1) is more general than the $p(.)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is not adequate to study nonlinear problems of this type. This lead us to seek weak solutions for problems (1.1) in a more general variable exponent Sobolev space which was introduced for the first time by Mihaïlescu et al [16]. Note that, Antontsev and Shmarev [2] studied the following problem which is quite close to (1.1):

$$\begin{gathered} -\sum_i D_i(a_i(x,u))|D_iu|^{p_i(x)-2}D_iu + c(x,u)|u|^{\sigma(x)-2}u = F(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

in a bounded domain $\Omega \in \mathbb{R}^N$, and elliptic systems of the same structure,

$$\begin{gathered} -\sum_j D_j(a_{ij}(x,\nabla u)) = f^i(x,u) \quad \text{in } \Omega,\ i = 1,\dots,n. \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.8}$$

In [2], the authors proved among others result, existence of (bounded) weak solutions and establish sufficient conditions of uniqueness of a weak solution, where the variational set considered is

$$\mathbf{V}(\Omega) = \{u \in L^{\sigma(x)}(\Omega) \cap W_0^{1,1}(\Omega), D_i(u) \in L^{p_i(x)}(\Omega), i = 1,\dots,n\}$$

equipped with the norm $\|u\|_V = \|u\|_{\sigma(.)} + \sum_{i=1}^n \|D_iu\|_{p_i(.)}$.

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries including, among other things, a brief discussion of variable exponent Lebesgue, Sobolev and anisotropic Sobolev variables exponent spaces. The main existence and uniqueness result is stated and proved in section 3. Finally, in section 4, we discuss some extensions.

2. Preliminaries

In this section, we define the Lebesgue and Sobolev spaces with variable exponent and give some of their properties. Roughly speaking, anistropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Given a measurable function $p(.) : \Omega \to [1,\infty)$. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}dx$$

is finite. If the exponent is bounded; i.e., if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \le 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)|u|_{p(.)}|v|_{p'(.)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. Now, let

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result (cf. [11]).

Lemma 2.1. *If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < +\infty$ then the following relations hold*

(i) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_+}$;
(ii) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_-}$;
(iii) $|u_n - u|_{p(.)} \to 0 \Rightarrow \rho_{p(.)}(u_n - u) \to 0$;
(iv) $|u|_{L^{p(.)}(\Omega)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*)*;
(v) $|u_n|_{L^{p(.)}(\Omega)} \to 0$ *(respectively* $\to +\infty$*)* $\Leftrightarrow \rho_{p(.)}(u_n) \to 0$ *(respectively* $\to +\infty$*)*;
(vi) $\rho_{p(.)}\big(u/|u|_{L^{p(.)}(\Omega)}\big) = 1$.

Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ under the norm $\|u\|_{1,p(.)}$. Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1\}.$$

Furthermore, if $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(.)}(\Omega)$, that is $H_0^{1,p(.)}(\Omega) = W_0^{1,p(.)}(\Omega)$ (cf. [12]). Since Ω is an open bounded set and $p \in C_+(\overline{\Omega})$ is logarithmic Hölder, the $p(.)$-Poincaré inequality

$$|u|_p \leq C|\nabla u|_{p(.)}$$

holds for all $u \in W_0^{1,p(.)}(\Omega)$, where C depends on p, $|\Omega|$, $\operatorname{diam}(\Omega)$ and N (see [12]), and so

$$\|u\| := |\nabla u|_{p(.)},$$

is an equivalent norm in $W_0^{1,p(.)}(\Omega)$. Of course also the norm

$$\|u\|_{p(.)} := \sum_{i=1}^N \big|\frac{\partial}{\partial x_i} u\big|_{p(.)}$$

is an equivalent norm in $W_0^{1,p(.)}(\Omega)$. Hence the space $W_0^{1,p(.)}(\Omega)$ is a separable and reflexive Banach space.

Finally, let us present a natural generalization of the variable exponent Sobolev space $W_0^{1,p(.)}(\Omega)$ (cf. [16]) that will enable us to study with sufficient accuracy problem (1.1). First of all, we denote by $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overrightarrow{p} = (p_1, \dots, p_N)$. The *anisotropic variable exponent Sobolev space* $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(.)} := \sum_{i=1}^N \big|\frac{\partial}{\partial x_i} u\big|_{p_i(.)}.$$

The space $(W_0^{1,\overrightarrow{p}(.)}(\Omega), \|u\|_{\overrightarrow{p}(.)})$ is a reflexive Banach space (cf. [16]).

Let us introduce the following notation.

$$\begin{gathered}
\overrightarrow{P}_+ = (p_1^+, \dots, p_N^+), \quad \overrightarrow{P}_- = (p_1^-, \dots, p_N^-), \\
P_+^+ = \max\{p_1^+, \dots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, \dots, p_N^-\}, \\
P_-^- = \min\{p_1^-, \dots, p_N^-\}, \quad P_{-,\infty} = \max\{P_-^+, P_-^*\}, \\
P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}.
\end{gathered}$$

We have the following result (cf. [16]).

Theorem 2.2. *Assume* $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ *is a bounded domain with smooth boundary. Assume relation* (1.6) *is fulfilled. For any* $q \in C(\overline{\Omega})$ *verifying*

$$1 < q(x) < P_{-,\infty} \quad \text{for all } x \in \overline{\Omega},$$

then the embedding

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$$

is continuous and compact.

We remark that Assumption (1.4) and relation $a_i(x,\xi) = \nabla_\xi A_i(x,\xi)$ imply in particular that for $i = 1, \dots, N$, $A_i(x,\xi)$ is convex with respect to the second variable.

3. Existence and uniqueness of weak energy solution

In this section, we study the weak energy solution of (1.1).

Definition 3.1. A weak energy solution of (1.1) is a function $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that

$$\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u). \frac{\partial}{\partial x_i}\varphi dx = \int_\Omega f(x)\varphi dx, \text{ for all } \varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega). \tag{3.1}$$

The main result of this section is the following.

Theorem 3.2. *Assume* (1.2)-(1.6) *and* $f \in L^\infty(\Omega)$. *Then there exists a unique weak energy solution of* (1.1).

Proof of Existence. Let E denote the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. Define the energy functional $J : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u) dx - \int_\Omega f u \, dx.$$

We first establish some basic properties of J.

Proposition 3.3. *The functional* J *is well-defined on* E *and* $J \in C^1(E, \mathbb{R})$ *with the derivative given by*

$$\langle J'(u), \varphi\rangle = \int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u). \frac{\partial}{\partial x_i}\varphi dx - \int_\Omega f\varphi dx,$$

for all $u, \varphi \in E$.

To prove the above proposition, we define for $i = 1, \dots, N$ the functionals $\Lambda_i : E \to \mathbb{R}$ by

$$\Lambda_i(u) = \int_\Omega A_i(x, \frac{\partial}{\partial x_i} u) dx, \quad \text{for all } u \in E.$$

Lemma 3.4. *For $i = 1, \dots, N$,*

(i) *the functional Λ_i is well-defined on E;*
(ii) *the functional Λ_i is of class $C^1(E, \mathbb{R})$ and*

$$\langle \Lambda_i'(u), \varphi \rangle = \int_\Omega a_i(x, \frac{\partial}{\partial x_i} u) . \frac{\partial}{\partial x_i} \varphi dx,$$

for all $u, \varphi \in E$.

Proof. (i) For any $x \in \Omega$ and $\xi \in \mathbb{R}$, we have

$$A_i(x, \xi) = \int_0^1 \frac{d}{dt} A_i(x, t\xi) dt = \int_0^1 a_i(x, t\xi).\xi dt.$$

Then by (1.3),

$$A_i(x, \xi) \le C_1 \int_0^1 (j_i(x) + |\xi|^{p_i(x)-1} t^{p_i(x)-1}) |\xi| dt \le C_1 j_i(x) |\xi| + \frac{C_1}{p_i(x)} |\xi|^{p_i(x)}.$$

The above inequality and (1.6) imply

$$0 \le \int_\Omega A_i(x, \frac{\partial}{\partial x_i} u) dx \le C_1 \int_\Omega j_i(x) |\frac{\partial}{\partial x_i} u| dx + \frac{C_1}{p_i^-} \int_\Omega |\frac{\partial}{\partial x_i} u|^{p_i(x)} dx,$$

for all $u \in E$. Using (2.1) and Lemma 2.1, we deduce that Λ_i is well-defined on E, for $i = 1, \dots, N$.

(ii) Existence of the Gâteaux derivative. Let $u, \varphi \in E$. Fix $x \in \Omega$ and $0 < |r| < 1$. Then by the mean value theorem, there exists $\nu \in [0, 1]$ such that

$$\begin{aligned}
&\big|a_i(x, \frac{\partial}{\partial x_i} u(x) + \nu r \frac{\partial}{\partial x_i} \varphi(x))\big| \big|\frac{\partial}{\partial x_i} \varphi(x)\big| \\
&= \big|A_i(x, \frac{\partial}{\partial x_i} u(x) + r \frac{\partial}{\partial x_i} \varphi(x)) - A_i(x, \frac{\partial}{\partial x_i} u(x))\big| / |r| \\
&\le \Big[C_1 j_i(x) + C_1 2^{P_+^+} \Big(\big|\frac{\partial}{\partial x_i} u(x)\big|^{p_i(x)-1} + \big|\frac{\partial}{\partial x_i} \varphi(x)\big|^{p_i(x)-1} \Big) \Big] \big|\frac{\partial}{\partial x_i} \varphi(x)\big|.
\end{aligned}$$

Next, by (2.1), we have

$$\int_\Omega C_1 j_i(x) |\frac{\partial}{\partial x_i} \varphi(x)| dx \le \beta |C_1 j_i|_{p_i'(x)} . |\frac{\partial}{\partial x_i} \varphi|_{p_i(x)}$$

and

$$\int_\Omega |\frac{\partial}{\partial x_i} u|^{p_i(x)-1} |\frac{\partial}{\partial x_i} \varphi| dx \le \alpha \big| |\frac{\partial}{\partial x_i} u|^{p_i(x)-1} \big|_{p_i'(x)} . |\frac{\partial}{\partial x_i} \varphi|_{p_i(x)}.$$

The above inequalities imply

$$C_1 \Big[j_i(x) + 2^{P_+^+} \Big(\big|\frac{\partial}{\partial x_i} u(x)\big|^{p_i(x)-1} + \big|\frac{\partial}{\partial x_i} \varphi(x)\big|^{p_i(x)-1} \Big) \Big] \big|\frac{\partial}{\partial x_i} \varphi(x)\big| \in L^1(\Omega).$$

It follow from the Lebesgue theorem that

$$\langle \Lambda_i'(u), \varphi \rangle = \int_\Omega a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} \varphi dx, \quad \text{for } i = 1, \dots, N.$$

Assume now $u_n \to u$ in E. Let us define $\psi_i(x,u) = a_i(x, \frac{\partial}{\partial x_i}u)$. Using assumption (1.3), [13, theorems 4.1 and 4.2], we deduce that $\psi_i(x,u_n) \to \psi_i(x,u)$ in $L^{p_i'(x)}(\Omega)$. By (2.1), we obtain

$$|\langle \Lambda_i'(u_n) - \Lambda_i'(u), \varphi\rangle| \le C|\psi_i(x,u_n) - \psi_i(x,u)|_{p_i'(x)}|\frac{\partial}{\partial x_i}\varphi|_{p_i(x)},$$

and so

$$\|\Lambda_i'(u_n) - \Lambda_i'(u)\| \le C|\psi_i(x,u_n) - \psi_i(x,u)|_{p_i'(x)} \to 0,$$

as $n \to \infty$ for $i = 1, \dots, N$. The proof is complete. □

By Lemma 3.4, it is clear that Proposition 3.3 holds true and then, the proof of Proposition 3.3 is also complete.

Lemma 3.5. *For $i = 1, \dots, N$ the functional Λ_i is weakly lower semi-continuous.*

Proof. By [3, corollary III.8], it is sufficient to show that Λ_i is lower semi-continuous. For this, fix $u \in E$ and $\epsilon > 0$. Since Λ_i is convex (by Remark 2.3), we deduce that for any $v \in E$, the following inequality holds

$$\int_\Omega A_i(x, \frac{\partial}{\partial x_i}v)dx \ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx + \int_\Omega a_i(x, \frac{\partial}{\partial x_i}u).(\frac{\partial}{\partial x_i}v - \frac{\partial}{\partial x_i}u)dx.$$

Using (1.3) and (2.1), we have

$$\begin{aligned}
\int_\Omega A_i(x, \frac{\partial}{\partial x_i}v)dx &\ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx - \int_\Omega |a_i(x, \frac{\partial}{\partial x_i}u)||\frac{\partial}{\partial x_i}v - \frac{\partial}{\partial x_i}u|dx \\
&\ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx - C_1\int_\Omega j_i(x)|\frac{\partial}{\partial x_i}(v-u)|dx \\
&\quad - C_1\int_\Omega |\frac{\partial}{\partial x_i}u|^{p_i(x)-1}|\frac{\partial}{\partial x_i}(v-u)|dx \\
&\ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx - C_2|j_i|_{p_i'(x)}|\frac{\partial}{\partial x_i}(v-u)|_{p_i(x)} \\
&\quad - C_3||\frac{\partial}{\partial x_i}u|^{p_i(x)-1}|_{p_i'(x)}|\frac{\partial}{\partial x_i}(v-u)|_{p_i(x)} \\
&\ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx - C_4\|v-u\|_{\overrightarrow{p}(.)} \\
&\ge \int_\Omega A_i(x, \frac{\partial}{\partial x_i}u)dx - \epsilon,
\end{aligned}$$

for all $v \in E$ with $\|v-u\|_{\overrightarrow{p}(.)} < \delta = \epsilon/C_4$, where C_2, C_3 and C_4 are positive constants. We conclude that Λ_i is weakly lower semi-continuous for $i = 1, \dots, N$. The proof is complete. □

Proposition 3.6. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

.

Proof. Using (1.5), we have

$$\begin{aligned}
J(u) &= \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i}u)dx - \int_\Omega fu\,dx \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\frac{\partial}{\partial x_i}u|^{p_i(x)}dx - \int_\Omega fu\,dx \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\frac{\partial}{\partial x_i}u|^{p_i(x)}dx - \|f\|_{q'}\|u\|_q,
\end{aligned}$$

where $\|u\|_q = \big(\int_\Omega |u|^q dx\big)^{1/q}$ and $1 < q < P_-^+$. For each $i \in 1,\dots,N$, we define

$$\alpha_i = \begin{cases} P_+^+ & \text{if } |\frac{\partial}{\partial x_i}u| < 1, \\ P_-^- & \text{if } |\frac{\partial}{\partial x_i}u| > 1. \end{cases}$$

For the coerciveness of J, we focus our attention on the case when $u \in E$ and $\|u\|_{\overrightarrow{p}(.)} > 1$. Then, by Lemma 2.1 we obtain

$$\begin{aligned}
J(u) &\geq \frac{1}{P_+^+}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{\alpha_i} - \|f\|_{q'}\|u\|_q \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-} - \frac{1}{P_+^+}\sum_{\{i:\alpha_i = P_+^+\}}\Big(|\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-} - |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_+^+}\Big) - \|f\|_{q'}\|u\|_q \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-} - \frac{1}{P_+^+}\sum_{\{i:\alpha_i = P_+^+\}}\Big(|\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-}\Big) - \|f\|_{q'}\|u\|_q \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-} - \frac{N}{P_+^+} - \|f\|_{q'}\|u\|_q \\
&\geq \frac{1}{P_+^+}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}^{P_-^-} - \frac{N}{P_+^+} - C'\|u\|_q \\
&\geq \frac{1}{P_+^+}\Big(\frac{1}{N}\sum_{i=1}^N |\frac{\partial}{\partial x_i}u|_{p_i(.)}\Big)^{P_-^-} - \frac{N}{P_+^+} - C'\|u\|_q \\
&\geq \frac{1}{P_+^+ N^{P_-^-}}\|u\|_{\overrightarrow{p}(.)}^{P_-^-} - \frac{N}{P_+^+} - C'\|u\|_{\overrightarrow{p}(.)},
\end{aligned}$$

since E is continuously embedded in $L^q(\Omega)$. As $P_-^- > 1$, then J is coercive. It is obvious that J is bounded from below. By Lemma 3.5, Λ_i is weakly lower semi-continuous for $i = 1,\dots,N$. We show that J is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to u in E. Since for $i = 1,\dots,N$ Λ_i is weakly lower semi-continuous, we have

$$\Lambda_i(u) \leq \liminf_{n\to+\infty} \Lambda_i(u_n). \tag{3.2}$$

On the other hand, E is embedded in $L^q(\Omega)$ for $1 < q < P_{-,\infty}$. This fact together with relation (3.2) imply

$$J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

Therefore, J is weakly lower semi-continuous. The proof is complete. $\square$

Since J is proper, weakly lower semi-continuous and coercive, then J has a minimizer which is a weak energy solution of (1.1). The proof of existence is then complete.

Proof of uniqueness for Theorem 3.2. Let u_1, u_2 be two weak energy solutions of (1.1). Then

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,\frac{\partial}{\partial x_i}u_1)-a_i(x,\frac{\partial}{\partial x_i}u_2)\Big).\big(\frac{\partial}{\partial x_i}u_1-\frac{\partial}{\partial x_i}u_2\big)dx=0. \tag{3.3}$$

Using (1.4) in (3.3), we obtain

$$\|u_1-u_2\|_{\overrightarrow{p}(.)}=\sum_{i=1}^{N}|\frac{\partial}{\partial x_i}u_1-\frac{\partial}{\partial x_i}u_2|_{p_i(.)}=0. \tag{3.4}$$

From (3.4) it follows that $u_1=u_2$.

4. An extension

In this section, we show that the existence result obtained for (1.1) can be extended to more general anisotropic elliptic problem of the form

$$\begin{gathered}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial}{\partial x_i}u)=f(x,u)\quad\text{in }\Omega\\ u=0\quad\text{on }\partial\Omega.\end{gathered} \tag{4.1}$$

We assume that the nonlinearity $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Let

$$F(x,t)=\int_0^t f(x,s)ds.$$

We assume that there exists $C_1>0$, $C_2>0$ such that

$$|f(x,t)|\le C_1+C_2|t|^{\beta-1}, \tag{4.2}$$

where $1<\beta<P_-^-$. We have the following result.

Theorem 4.1. *Under assumptions* (1.2)-(1.6) *and* (4.2), *Problem* (4.1) *has at least one weak energy solution.*

Proof. Let $g(u)=\int_\Omega F(x,u)dx$, then $g':E\to E^*$ is completely continuous; i.e., $u_n\rightharpoonup u\Rightarrow g'(u_n)\to g'(u)$, and thus the functional g is weakly continuous. Consequently,

$$J(u)=\sum_{i=1}^{N}\int_\Omega A_i(x,\frac{\partial}{\partial x_i}u)dx-\int_\Omega F(x,u)dx,\quad u\in E$$

is such that $J\in C^1(E,\mathbb{R})$ and is weakly lower semi-continuous. We then have to prove that J is bounded from below and coercive in order to complete the proof. From (4.2), we have $|F(x,t)|\le C(1+|t|^\beta)$ and then

$$J(u)\ge\frac{1}{P_+^+N^{P_-^-}}\|u\|_{\overrightarrow{p}(.)}^{P_-^-}-\frac{N}{P_+^+}-C\int_\Omega|u|^\beta dx-C_3,$$

for all $u\in E$ such that $\|u\|_{\overrightarrow{p}(.)}>1$.

We know that E is continuously embedded in $L^{\beta}(\Omega)$. It follows from inequality above that

$$J(u) \geq C_5 \|u\|^{P_-^-}_{\overrightarrow{p}(.)} - \frac{N}{P_+^+} - C_4 \|u\|^{\beta}_{\overrightarrow{p}(.)} - C_3 \to +\infty$$

as $\|u\|_{\overrightarrow{p}(.)} \to +\infty$. Consequently, J is bounded from below and coercive. The proof is then complete. $\square$

Assume now that $F^+(x,t) = \int_0^t f^+(x,s)ds$ is such that there exists $C_1 > 0$, $C_2 > 0$ such that

$$|f^+(x,t)| \leq C_1 + C_2 |t|^{\beta - 1}, \tag{4.3}$$

where $1 < \beta < P_-^-$. Then we have the following result.

Theorem 4.2. *Under assumptions* (1.2)-(1.6) *and* (4.3)*, Problem* (4.1) *has at least one weak energy solution.*

Proof. As $f = f^+ - f^-$, let $F^-(x,t) = \int_0^t f^-(x,s)ds$. Then

$$\begin{aligned} I(u) &= \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u)dx + \int_\Omega F^-(x,u)dx - \int_\Omega F^+(x,u)dx \\ &\geq \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u)dx - \int_\Omega F^+(x,u)dx. \end{aligned}$$

Therefore, similarly as in the proof of Theorem 4.1, the conclusion follows immediately. $\square$

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