Electronic Journal of Differential Equations, Vol. 2009(2009), No. 145, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OSCILLATORY BEHAVIOR OF SECOND-ORDER NEUTRAL DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. Oscillation criteria are established for solutions of forced and unforced second-order neutral difference equations with positive and negative coefficients. These results generalize some existing results in the literature. Examples are provided to illustrate our results.

1. Introduction

Neutral difference and differential equations arise in many areas of applied mathematics, such as population dynamics [7], stability theory [13, 14], circuit theory [4], bifurcation analysis [3], dynamical behavior of delayed network systems [16], and so on. Therefore, these equations have attracted a great interest during the last few decades. In the present paper, we focus on the neutral type delay difference equation

$$\Delta(a_n\Delta(x_n + c_n x_{n-k})) + p_n f(x_{n-l}) - q_n f(x_{n-m}) = 0, \tag{1.1}$$

$$\Delta(a_n\Delta(x_n - c_n x_{n-k})) + p_n f(x_{n-l}) - q_n f(x_{n-m}) = 0, \tag{1.2}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$, n_0 is a nonnegative integer, k, l, m are positive integers, $\{a_n\}, \{c_n\}, \{p_n\}, \{q_n\}$ are real sequences, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, l, m\}$. By a solution of equation (1.1) ((1.2)) we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1) ((1.2)) for all $n \in \mathbb{N}(n_0)$. It is also known that equation (1.1) ((1.2)) has a unique solution $\{x_n\}$ if an initial sequence $\{x_0(n)\}$ is given to hold $x_n = x_0(n), n = n_0 - \theta, n_0 - \theta + 1, \dots, n_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) ((1.2)) is said to be oscillatory if it is neither eventually positive nor eventually negative and it is non-oscillatory otherwise.

Determining oscillation criteria for difference equations has received a great deal of attention in the last few years, see for example [1, 2] and the references quoted therein. Sufficient conditions for oscillation of solutions of first order neutral delay

2000 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Oscillation; neutral difference equations;
positive and negative coefficients.

Submitted August 7, 2009. Published November 12, 2009.

difference equations with positive and negative coefficients have been investigated by many authors [2, 9, 10, 11, 13]. On the other hand in the recent papers [5, 6, 8, 12] the authors obtain some sufficient conditions for the existence of nonoscillatory solutions and oscillation of all bounded solutions of second order linear neutral difference equations with positive and negative coefficients. To the best knowledge of the authors, there are no results in literature dealing with the oscillatory behavior of equations (1.1) and (1.2). The purpose of this paper is to derive sufficient conditions for every solution of equation (1.1) and (1.2) to be oscillatory. Our results improve and generalize the known results in the literature.

In Section 2, we present sufficient conditions for oscillation of all solutions of equations (1.1) and (1.2). In Section 3, we establish oscillation results for equations (1.1) and (1.2) with forcing terms. Examples are provided in Section 4 to illustrate the results.

2. Oscillation Results for Equations (1.1) and (1.2)

In this section, we obtain oscillation criteria for the solutions of (1.1) and (1.2). We shall use the following assumptions in this article:

- (H1) $\{a_n\}$ is a positive sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$;
- (H2) $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences;
- (H3) $l \geq m$;
- (H4) $p_n - q_{n-m+l} \geq b > 0$, where b is a constant;
- (H5) there exist positive constants M_1 and M_2 such that $M_1 \leq \frac{f(u)}{u} \leq M_2$ for $u \neq 0$.

We begin with the following theorem.

Theorem 2.1. *With respect to the difference equation* (1.1) *assume* (H1)-(H5). *If*

$$m + 1 \geq k, \quad 0 \leq c_n \leq c, \quad \text{for } n \in \mathbb{N}(n_0), \tag{2.1}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \leq \frac{(1+c_n)}{M_2}, \tag{2.2}$$

then every solution of (1.1) *is oscillatory.*

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x_n > 0$ and $x_{n-\theta} > 0$ for $n \geq n_1 \in \mathbb{N}(n_0)$. We set

$$z_n = x_n + c_n x_{n-k} - \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m})$$

for $n \geq n_1 + \theta$, then

$$\begin{aligned} \Delta(a_n \Delta z_n) &= \Delta(a_n \Delta(x_n + c_n x_{n-k})) - q_n f(x_{n-m}) - q_{n-l+m} f(x_{n-l}) \\ &= -p_n f(x_{n-l}) + q_{n-l+m} f(x_{n-l}) \\ &= -(p_n - q_{n-l+m}) f(x_{n-l}) \leq -b M_1 x_{n-l}, \end{aligned} \tag{2.3}$$

for $n \geq n_1 + \theta$. Thus, we have $\{a_n \Delta z_n\}$ nonincreasing and $\Delta z_n \geq 0$ or $\Delta z_n < 0$, $n \geq N$ for some $N \geq n_1 + \theta$. We discuss the following two possible cases:

Case 1: $\Delta z_n \geq 0$ for all $n \geq N$. Summing (2.3) from N to n, we obtain

$$\infty > a_N \Delta z_N \geq -a_{n+1}\Delta z_{n+1} + a_N \Delta z_N \geq bM_1 \sum_{s=N}^{n} x_{s-l}$$

and therefore $\{x_n\}$ is summable for $n \in \mathbb{N}(N)$. Thus, from the condition (2.1), we have

$$y_n = x_n + c_n x_{n-k} \tag{2.4}$$

is also summable. Further, it is clear that for $n \geq N$,

$$\Delta y_n = \Delta(x_n + c_n x_{n-k}) = \Delta z_n + \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s f(x_{s-m}),$$

which implies that $\{y_n\}$ is nondecreasing. Therefore, $y_n \geq y_N$, $n \geq N$, which yields that y_n is not summable, a contradiction.

Case 2: $\Delta z_n < 0$ for all $n \geq N$. Summing $a_n \Delta z_n \leq a_N \Delta z_N < 0$, from N to $n-1$, we obtain

$$z_n \leq z_N + a_N\, z_N \sum_{s=N}^{n-1} \frac{1}{a_s}, \quad n \geq N,$$

and we see from (H1) that $\lim_{n\to\infty} z_n = -\infty$. We claim that $\{x_n\}$ is bounded from above. If this is not the case, then there exists an integer $N_1 \geq N+1$ such that

$$z_{N_1} < 0 \quad \text{and} \quad \max_{N \leq n \leq N_1} x_n = x_{N_1}. \tag{2.5}$$

Then, we have

$$\begin{aligned}
0 > z_{N_1} &= x_{N_1} + c_{N_1} x_{N_1-k} - \sum_{s=N}^{N_1-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m}) \\
&\geq \Big\{1 + c_{N_1} - M_2 \sum_{s=N}^{N_1-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t \Big\} x_{N_1} - k \\
&\geq \Big\{1 + c_{N_1} - M_2 \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \Big\} x_{N_1} - k \geq 0
\end{aligned}$$

which is a contradiction, so that $\{x_n\}$ is bounded from above. Hence for every $L > 0$, there exists an integer $N_2 \geq N_1$ such that $x_n \leq L$ for all $n \geq N_2$. We then have

$$z_n \geq -M_2 L \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \geq -L > -\infty, \quad n \geq N_2.$$

This contradicts the fact that $\lim_{n\to\infty} z_n = -\infty$. The proof is now complete. □

Next, we turn to the oscillation theorem for (1.2).

Theorem 2.2. *With respect to the difference equation* (1.2)*, assume* (H1)-(H5)*. If*

$$0 \leq c_n \leq c < 1, \tag{2.6}$$

and

$$c + M_2 \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \leq 1 \tag{2.7}$$

then every solution of (1.2) *oscillates or satisfies* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a non-oscillatory solution of (1.2). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \le n_1 \in \mathbb{N}(n_0)$. If we define

$$z_n = x_n - c_n x_{n-k} - \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m}) \tag{2.8}$$

then as in the proof of Theorem 2.1, we have

$$\Delta(a_n \Delta z_n) = -(p_n - q_{n-l+m}) f(x_{n-l}) \le -bM_1 x_{n-l} \tag{2.9}$$

for $n \ge n_1 + \theta$, and conclude that $\{\Delta z_n\}$ is eventually non-increasing. Therefore, $\Delta z_n < 0$ or $\Delta z_n \ge 0$ for all $n \ge N \ge n_1 + \theta$.

Case 1: $\Delta z_n < 0$ for all $n \ge N$. Then $\lim_{n\to\infty} z_n = -\infty$. We claim that $\{x_n\}$ is bounded from above. If it is not the case, there exists an integer $N_1 > N$ such that $z_{N_1} < 0$ and $\max_{N \le n \le N_1} x_n = x_{N_1}$. Then, we have

$$\begin{aligned} 0 > z_{N_1} &= x_{N_1} - c_{N_1} x_{N_1-k} - \sum_{s=N}^{N_1-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m}) \\ &\ge \Big\{1 - c - M_2 \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n-l+m}^{n-1} q_s \Big\} x_{N_1} \ge 0 \end{aligned}$$

which is a contradiction, so that $\{x_n\}$ is bounded from above. From (2.6)-(2.8) we see that $\{z_n\}$ is bounded which contradicts the fact that $\lim_{n\to\infty} z_n = -\infty$.

Case 2: $\Delta z_n \ge 0$ for all $n \ge n_1$. In this case, we see that L is a nonnegative constant, where $L = \lim_{n\to\infty} a_n \Delta z_n$. Considering (H4) and summing (2.9) from n_1 to ∞ we obtain

$$\begin{aligned} \infty > a_{n_1} \Delta z_{n_1} - L &= \sum_{n=n_1}^{\infty} (p_n - q_{n-l+m}) f(x_{n-l}) \\ &\ge M_1 \sum_{n=n_1}^{\infty} (p_n - q_{n-l+m}) x_{n-l} \ge M_1 b \sum_{n=n_1}^{\infty} x_{n-l} \end{aligned}$$

which implies that $\{x_n\}$ is summable, and thus $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

3. Oscillation Results for (1.1) and (1.2) With Forcing Terms

In this section, we consider (1.1) and (1.2) with forcing terms of the form

$$\Delta(a_n \Delta(x_n + c_n x_{n-k})) + p_n f(x_{n-l}) - q_n f(x_{n-m}) = e_n, n \in \mathbb{N}(n_0) \tag{3.1}$$

$$\Delta(a_n \Delta(x_n - c_n x_{n-k})) + p_n f(x_{n-l}) - q_n f(x_{n-m}) = e_n, n \in \mathbb{N}(n_0) \tag{3.2}$$

where $\{e_n\}$ is a sequence of real numbers.

Theorem 3.1. *With respect to the difference equation* (3.1), *assume* (H1)-(H5), (2.1) *and* (2.2). *If there exists a sequence* $\{E_n\}$ *such that*

$$\lim_{n\to\infty} E_n \text{ is finite and } \Delta(a_n \, \Delta E_n) = e_n \text{ for all } n \in \mathbb{N}(n_0), \tag{3.3}$$

then every solution of (3.1) *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (3.1) such that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. If we denote

$$B_n = x_n + c_n x_{n-k} - \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m}) - E_n + A + 1 \tag{3.4}$$

where $\lim_{n\to\infty} E_n = A$, then from (3.1) we obtain

$$\Delta(a_n \Delta B_n) \leq -bM_1 x_{n-l} \leq 0, \quad n \geq n_1 + \theta. \tag{3.5}$$

By (3.5), there exists an integer $n_2 \geq n_1 + \theta$ such that $\Delta B_n \geq 0$ or $\Delta B_n < 0$ for $n \geq n_2$. By hypotheses there exists sufficiently large integer n_3 such that $-E_n + A + 1 > 0$ for all $n \geq n_3$. Let $N = \max\{n_2, n_3\}$.

Let $\Delta B_n < 0$ for $n \geq N$. Then from (H1) and (3.5), we have $\lim_{n\to\infty} B_n = -\infty$. First we show that $\{x_n\}$ is bounded. If this is not the case, there exists an integer $N_1 > N$ satisfying $B_{N_1} < 0$ and $\max_{N \leq n \leq N_1} x_n = x_{N_1}$. Then, we have

$$\begin{aligned} 0 > B_{N_1} &= x_{N_1} + c_{N_1} x_{N_1-k} - \sum_{s=n_1}^{N_1-1} \frac{1}{a_s} \sum_{t=s-l+m}^{s-1} q_t f(x_{t-m}) - E_{N_1} + A + 1 \\ &\geq \Big\{1 + c_{N_1} - M_2 \sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{t=n-l+m}^{n-1} q_t \Big\} x_{N_1} - k \geq 0. \end{aligned}$$

This contradiction shows that $\{x_n\}$ must be bounded. Then there exists constant $L > 0$ such that $x_n \leq L$ for all $n \leq N$. It follows from (2.2) and (3.4) that $\{B_n\}$ is bounded, which contradicts the fact that $\lim_{n\to\infty} B_n = -\infty$.

Let $\Delta B_n \geq 0$ for $n \geq N$. Summing (3.5), we have

$$\infty > a_N \Delta B_N \geq a_N \Delta B_N - a_n \Delta B_n \geq bM_1 \sum_{n=N}^{\infty} x_{n-l}$$

which implies that $\{x_n\}$ is summable, and thus $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Theorem 3.2. *With respect to the difference equation* (3.2)*, assume* (H1)-(H5), (2.6) *and* (2.7)*. If* (3.3) *holds, then every solution of* (3.2) *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose that $\{x_n\}$ is nonoscillatory solution of (3.2) such that $x_n > 0$ and $x_{n-\theta} > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. Let us denote with

$$W_n = z_n - E_n + A + 1 \tag{3.6}$$

where z_n is defined by (2.8). Then, we have

$$\Delta(a_n \Delta W_n) \leq -bM_1 x_{n-l} \leq 0, \quad n \geq n_1 + \theta. \tag{3.7}$$

Therefore, we have the following two cases: $\Delta W_n < 0$ for $n \geq N \geq n_1 + \theta$ which implies that $\lim_{n\to\infty} W_n = -\infty$. It is not hard to prove that $\Delta W_n < 0$ is not possible by following the arguments as in the proof of Theorem 3.1.

Therefore, $\Delta W_n \geq 0$ for all $n \geq N$. From (3.7), we obtain $\{x_n\}$ is summable, and thus $\lim_{n\to\infty} x_n = 0$. The proof is now complete. □

4. Examples

In this section, we present some examples to illustrate the results obtained in the pervious sections.

Example 4.1. Consider the difference equation

$$\begin{aligned} &\Delta(n\Delta(x_n+2x_{n-1}))+\big(6n+3+(\frac{2}{3^{n+2}})\big)\frac{x_{n-4}(1+x_{n-4}^2)}{(2+x_{n-4}^2)}\\ &-(\frac{2}{3^{n+2}})\frac{x_{n-2}(1+x_{n-2}^2)}{(2+x_{n-2}^2)}=0,\quad n\ge 1\,. \end{aligned} \tag{4.1}$$

Here $a_n=n$, $c_n=2$, $l=4$, $m=2$, $p_n=6n+3+2(\frac{1}{3^{n+2}})$, $k=1$, $q_n=2(\frac{1}{3^{n+2}})$, and $f(u)=\frac{u(1+u^2)}{2+u^2}$. With $M_1=\frac{1}{2}$ and $M_1=1$, all conditions (H1)-(H5) hold. Further, we see that

$$\sum_1^\infty \frac{1}{a_n}=\sum_1^\infty\frac{1}{n}=\infty,$$

and

$$\sum_{n=1}^\infty\frac{1}{n}\sum_{s=n-2}^{n-1}2\Big(\frac{1}{3^{s+2}}\Big)=2\sum_1^\infty\frac{1}{n}\Big(\frac{1}{3^n}+\frac{1}{3^{n+1}}\Big)<\frac{8}{3}\sum_1^\infty\frac{1}{3^n}=\frac{4}{3}<3.$$

Hence by Theorem 2.1, all solutions of equation (4.1) are oscillatory. In fact $\{x_n\}=\{(-1)^n\}$ is one such solution of equation (4.1).

Example 4.2. Consider the difference equation

$$\begin{aligned} &\Delta(n\Delta(x_n-\frac{1}{2}x_{n-2}))+(\frac{3}{2}(2n+1)+\frac{1}{3^{n+6}})\frac{(x_{n-3}+x_{n-3}^3)}{(2+x_{n-3}^2)}\\ &-\frac{1}{3^{n+6}}\frac{(x_{n-1}+x_{n-1}^3)}{(2+x_{n-1}^2)}=0,\quad n\ge 1\,. \end{aligned} \tag{4.2}$$

Here $a_n=n$, $c_n=\frac{1}{2}$, $l=3$, $m=1$, $p_n=\frac{3}{2}(2n+1)+\frac{1}{3^{n+6}}$, $q_n=\frac{1}{3^{n+6}}$, and $f(u)=\frac{u(1+u^2)}{2+u^2}$. With $M_1=1/2$ and $M_1=1$, it is easy to check that conditions (H1)-(H5) hold. Further, we see that

$$\sum_1^\infty \frac{1}{a_n}=\sum_1^\infty\frac{1}{n}=\infty,$$

and

$$\begin{aligned} c+\sum_{n=1}^\infty\frac{1}{a_n}\sum_{s=n-2}^{n-1}q_s&=\frac{1}{2}+\sum_1^\infty\frac{1}{n}\sum_{s=n-2}^{n-1}\frac{1}{2}(\frac{1}{3^{s+6}})\\ &=\frac{1}{2}+\sum_1^\infty\frac{1}{n}(\frac{1}{3^{n+4}}+\frac{1}{3^{n+5}})\\ &<\frac{1}{2}+\frac{1}{2}(\frac{1}{3^4}+\frac{1}{3^5})<1. \end{aligned}$$

Hence by Theorem 2.2, all solution of equation (4.2) are oscillatory. In fact $\{x_n\}=\{(-1)^n\}$ is one such solution of equation (4.2).

Example 4.3. Consider the difference equation

$$\begin{aligned}&\Delta^2(x_n+2x_{n-2})+(\frac{n}{n+1})\frac{x_{n-3}(1+|x_{n-3}|)}{2+|x_{n-3}|}-\frac{1}{2^{n+3}}\frac{x_{n-1}(1+|x_{n-1}|)}{2+|x_{n-1}|}\\&=\frac{1}{2^{(n+1)(n+2)(n+3)}}+\frac{1}{2^{n+2}},\quad n\geq 1.\end{aligned}\tag{4.3}$$

For this equation, we see that $a_n = 1$, $c_n = 2$, $l = 3$, $m = 1$, $k = 2$, $p_n = n/(n+1)$, $q_n = \frac{1}{2^{n+2}}$, $e_n = \frac{1}{2^{(n+1)(n+2)(n+3)}} + \frac{1}{2^{n+2}}$ and $f(u) = \frac{u(1+|u|)}{2+|u|}$. We may set $M_1 = \frac{1}{2}$ and $M_2 = 1$, we may have $p_n - q_{n+2} = \frac{n}{n+1} - \frac{1}{2^{n+4}} > \frac{15}{32} > 0$ and $E_n = \frac{1}{n+1} - \frac{1}{2^n} \to 0$ as $n \to \infty$. It is not hard to see that

$$\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{s=n-2}^{n-1}q_s=\sum_{1}^{\infty}\sum_{s=n-2}^{n-1}\frac{1}{2^{s+3}}=\frac{3}{4}<3.$$

Therefore, all conditions of Theorem 3.1 are satisfied, and hence every solution of equation (4.3) are either oscillatory or tends to zero at infinity.

Example 4.4. Consider the difference equation

$$\begin{aligned}&\Delta(n\Delta(x_n-\frac{1}{4}x_{n-2}))+(\frac{n^2}{n^2+1})\frac{x_{n-4}(1+|x_{n-4}|)}{2+|x_{n-4}|}-\frac{1}{4^{n+2}}\frac{x_{n-2}(1+|x_{n-2}|)}{2+|x_{n-2}|}\\&=\frac{n-1}{2^{n+2}},\ n\geq 1.\end{aligned}\tag{4.4}$$

For this equation, $a_n = n$, $c_n = 1/4$, $l = 4$, $m = 2$, $p_n = \frac{n^2}{n^2+1}$, $q_n = \frac{1}{4^{n+2}}$, $e_n = \frac{n-1}{2^{n+2}}$ and $f(u) = \frac{u(1+|u|)}{2+|u|}$. We may set $M_1 = \frac{1}{2}$ and $M_2 = 1$, we may have $p_n - q_{n+2} = \frac{n^2}{n^2+1} - \frac{1}{4^{n+4}} > \frac{1}{4} > 0$ and $E_n = \frac{n}{2^n} \to 0$ as $n \to \infty$. It is easy to see that

$$\begin{aligned}c+\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{s=n-2}^{n-1}q_s&=\frac{1}{4}+\sum_{1}^{\infty}\frac{1}{n}\sum_{s=n-2}^{n-1}\frac{1}{4^{s+2}}\\&=\frac{1}{4}+\sum_{1}^{\infty}\frac{1}{n}\big(\frac{1}{4^n}+\frac{1}{4^{n+1}}\big)<\frac{2}{3}<1.\end{aligned}$$

Therefore, all conditions of Theorem 3.2 are satisfied, and hence every solution of equation (4.4) are oscillatory or tends to zero at infinity.

Note that the results in [6, 8] cannot be applied to (4.1), (4.4).

Acknowledgements. The authors want to thank the anonymous referee for his or her suggestions which improve the content of this article.

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