Electronic Journal of Differential Equations, Vol. 2009(2009), No. 146, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

REMARKS ON THE PHRAGMÉN-LINDELÖF THEOREM FOR L^p-VISCOSITY SOLUTIONS OF FULLY NONLINEAR PDES WITH UNBOUNDED INGREDIENTS

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Abstract. The Phragmén-Lindelöf theorem for L^p-viscosity solutions of fully nonlinear second order elliptic partial differential equations with unbounded coefficients and inhomogeneous terms is established.

1. Introduction

The notion of L^p-viscosity solutions was introduced in [5] to study fully nonlinear second order elliptic partial differential equations (PDEs for short) with unbounded inhomogeneous terms. We refer to [3] (see also [4]) as a pioneering work for the regularity theory of viscosity solutions of fully nonlinear PDEs.

It turned out that the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle can be extended to L^p-viscosity solutions for fully nonlinear second order elliptic PDEs with unbounded coefficients and inhomogeneous terms in [14]. See also [17] for a generalization.

As an application of the ABP maximum principle in [14], it is known that the (boundary) weak Harnack inequality for L^p-viscosity solutions of the associated extremal PDEs in [15] (see also [16]) holds, which implies Hölder continuity for L^p-viscosity solutions of fully nonlinear elliptic PDEs with unbounded ingredients. We also refer to [19] for Hölder continuity estimates on L^p-viscosity solutions by a different approach.

On the other hand, qualitative properties of viscosity solutions of fully nonlinear elliptic PDEs have been investigated as generalizations for classical elliptic PDE theory. For instance, the ABP maximum principle in unbounded domains in [7] and [15], the Liouville property in [11, 6], the Hadamard principle in [6], and the Phragmén-Lindelöf theorem in [8]. We refer to references in [8, 11, 6] for these qualitative properties of strong/classical solutions.

Our aim here is to extend the Phragmén-Lindelöf theorem in [8] when PDEs have unbounded coefficients (i.e. μ in this paper). In view of the boundary weak Harnack inequality in [15], it is natural to relax the hypotheses on coefficients and inhomogeneous terms. However, for the weak Harnack inequality, we need

2000 *Mathematics Subject Classification.* 35B53, 35D40, 35B50.
Key words and phrases. Phragmén-Lindelöf theorem; L^p-viscosity solution; weak Harnack inequality.

Submitted August 28, 2009. Published November 20, 2009.

to suppose that the coefficient to the first derivatives is small enough in L^n-norm. When we work in bounded domains, this is not a restriction. Since we are concerned with unbounded domains, we will need a bit more delicate analysis than those in [8].

Since our argument is essential to treat domains of conical type (i.e. the case for $\eta > 0$ in our notation), we will mainly discuss this case. We will add corresponding results for domains of cylindrical type (i.e. the case for $\eta = 0$).

Our paper is organized as follows: section 2 is devoted to showing the definitions and known results. In section 3, we present the ABP type estimates on L^p-viscosity subsolutions of fully nonlinear PDEs with unbounded ingredients under appropriate geometric conditions. We show the Phragmén-Lindelöf theorem in our setting in section 4. In section 5, we give a proof of an elementary geometric property, which is needed in the proof of Lemma 3.2.

2. Preliminaries

We consider fully nonlinear second order PDEs in unbounded domains $\Omega \subset \mathbb{R}^n$:

$$G(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given measurable functions. We also suppose that $(r, p, M) \in \mathbb{R} \times \mathbb{R}^n \times S^n \to G(x, r, p, M)$ is continuous for almost all $x \in \Omega$. Here, S^n denotes the set of symmetric matrices of order n equipped with the standard order.

We will use the standard notation from [13]. We denote by $L^p_+(\Omega)$ the set of all nonnegative functions in $L^p(\Omega)$.

Throughout this paper, we assume that

$$p > \frac{n}{2}.$$

We recall two facts: if $u \in W^{2,p}_{\rm loc}(\Omega)$ for $p > \frac{n}{2}$, then we may identify u with a continuous function on Ω, and u is twice differentiable for almost all $x \in \Omega$.

First of all, we recall the definition of L^p-viscosity solutions of (2.1).

Definition 2.1. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (2.1) if

$$\operatorname*{ess\,lim\,inf}_{x\to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \le 0$$
$$\Big(\text{resp., } \operatorname*{ess\,lim\,sup}_{x\to x_0} \{G(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \ge 0\Big)$$

whenever $\phi \in W^{2,p}_{\rm loc}(\Omega)$ and $x_0 \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$. A function $u \in C(\Omega)$ is called an L^p-viscosity solution of (2.1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution of (2.1).

To make easier recalling the right inequality, we will often say that u is an L^p-viscosity solution of

$$G(x, u, Du, D^2u) \le f(x) \tag{2.2}$$
$$\big(\text{resp.,} \quad G(x, u, Du, D^2u) \ge f(x)\big), \tag{2.3}$$

if it is an L^p-viscosity subsolution (resp., supersolution) of (2.1).

Remark 2.2. If u is an L^p-viscosity subsolution (resp., supersolution) of (2.1), then it is also an L^q-viscosity subsolution (resp., supersolution) of (2.1) provided $q \geq p$.

In what follows, instead of (2.1), we mainly consider PDEs which do not depend on u-variable:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{2.4}$$

We will assume that F is (degenerate) elliptic:

$$\begin{gathered} F(x,p,M) \leq F(x,p,N) \\ \text{for } (x,p,M,N) \in \Omega \times \mathbb{R}^n \times S^n \times S^n \text{ provided } M \geq N. \end{gathered} \tag{2.5}$$

For fixed ellipticity constants $0 < \lambda \leq \Lambda$, we assume that

$$\begin{gathered} \text{there is } \mu \in L^q_+(\Omega) \text{ such that} \\ \mathcal{P}^-(M) - \mu(x)|p| \leq F(x,p,M) \quad \text{for } (x,p,M) \in \Omega \times \mathbb{R}^n \times S^n, \end{gathered} \tag{2.6}$$

where the Pucci operators $\mathcal{P}^{\pm} : S^n \to \mathbb{R}$ are defined by

$$\mathcal{P}^-(M) = \min\{-\mathrm{trace}(AM) : A \in S^n_{\lambda,\Lambda}\}, \quad \mathcal{P}^+(M) = -\mathcal{P}^-(-M).$$

Here, $S^n_{\lambda,\Lambda} := \{M \in S^n : \lambda I \leq M \leq \Lambda I\}$. We refer the reader to [8] for examples of PDEs which satisfy (2.5) and (2.6). We first recall a lemma concerning change of unknown functions.

Lemma 2.3 ([8, Lemma 1])**.** *Assume* (2.5) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$ *for* $q > n$. *Then, there exist constants* $h_j > 0$ $(j = 1, 2)$ *satisfying the following property: if* $\xi \in C^2(\Omega)$ *satisfies*

$$\xi(x) > 0, \quad \frac{|D\xi|}{\xi}(x) \leq k_1(x), \quad \frac{|D^2\xi|}{\xi}(x) \leq k_2(x) \quad \text{for } x \in \Omega$$

with some functions $k_j \in C(\Omega)$ $(j = 1, 2)$*, then for* L^p*-viscosity subsolution* $w \in C(\Omega)$ *of* (2.4) *with* $f \in L^p_+(\Omega)$, $u := \frac{w}{\xi}$ *is an* L^p*-viscosity solution of*

$$\mathcal{P}^-(D^2u) - \gamma_1(x)|Du| - \gamma_2(x)u \leq \frac{f(x)}{\xi(x)} \quad \text{in } \Omega[u], \tag{2.7}$$

where $\Omega[u] = \{x \in \Omega \mid u(x) > 0\}$, $\gamma_1(x) = h_1k_1(x) + \mu(x)$ *and* $\gamma_2(x) = h_2k_2(x) + k_1(x)\mu(x)$.

We will use the constant $p_0 = p_0(n, \lambda, \Lambda) \in [\frac{n}{2}, n)$, for which we refer to [12]. It is known that for $p > p_0$, and $f \in L^p(B_r(z))$, where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$, there exists a (unique) strong solution $u \in C(\overline{B}_r(z)) \cap W^{2,p}_{\mathrm{loc}}(B_r(z))$ of

$$\mathcal{P}^-(D^2v(x)) = f(x) \quad \text{a.e. in } B_r(z)$$

under $v(x) = 0$ for $x \in \partial B_r(z)$ with estimates:

$$-C\|f^-\|_{L^p(B_r(z))} \leq v(x) \leq C\|f^+\|_{L^p(B_r(z))} \quad \text{in } B_r(z),$$

where $C = C(n, \lambda, \Lambda, p) > 0$ is a constant, and for $0 < s < r$,

$$\|v\|_{W^{2,p}(B_s(z))} \leq C'\|f\|_{L^p(B_r(z))},$$

where $C' = C'(n, \lambda, \Lambda, p, r - s) > 0$.

We remark that to prove the ABP maximum principle [14, Theorem 2.9], which implies the boundary weak Harnack inequality [15, Theorem 6.1], it suffices to obtain the existence of strong solutions of the above extremal equation only in balls although this fact is not clearly mentioned in [14, 15]. In fact, this existence result holds with local $W^{2,p}$-estimates for more general domains satisfying the uniform exterior cone property but the $p_0 \in [\frac{n}{2}, n)$ associated with general domains might be bigger than the above. We also notice that we may replace cubes by balls in the (boundary) weak Harnack inequality in [15] by Cabré's covering argument, which we will see in the proof of Lemma 3.2 below.

Fix $R > 0$ and $z \in \mathbb{R}^n$. Let $T,\ T' \subset B_R(z)$ be domains such that

$$\overline{T} \subset T', \quad \text{and} \quad \theta_0 \leq \frac{|T|}{|T'|} \leq 1 \quad \text{for some } \theta_0 > 0.$$

When we apply our weak Harnack inequality below, our choice of T and T' always satisfies the above condition.

For a given domain $A \subset \mathbb{R}^n$ and a function $v \in C(A)$, we define $v^-_{T',A}$ on $T' \cup A$ by

$$v^-_{T',A}(x) = \begin{cases} \min\{v(x), m\} & \text{if } x \in A, \\ m & \text{if } x \in T' \setminus A, \end{cases}$$

where

$$m = \liminf_{x \to T' \cap \partial A} v(x).$$

We note that if $T' \cap \partial A \neq \emptyset$, then $v^-_{T',A}$ is a real-valued function and that if $T' \cap \partial A \neq \emptyset$, v is a nonnegative L^p-viscosity supersolution of (2.4) and $f \leq 0$ in $T' \cap A$, then $v^-_{T',A}$ is a nonnegative L^p-viscosity supersolution of (2.4) in T'.

Next, we recall the boundary weak Harnack inequality when PDEs have unbounded coefficients and inhomogeneous terms.

Lemma 2.4 ([15, Theorem 6.1]). *Let T, T', A be as above. Assume that $T \cap A \neq \emptyset$ and $T' \setminus A \neq \emptyset$ and that*

$$q > n, \quad q \geq p > p_0. \tag{2.8}$$

Then, there exist constants $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$, $r = r(n, \lambda, \Lambda, p, q) > 0$ and $C_0 = C_0(n, \lambda, \Lambda, p, q) > 0$ satisfying the following property: if $\mu \in L^q_+(T' \cap A)$, $f \in L^p_+(T' \cap A)$, a nonnegative L^p-viscosity solution $w \in C(T' \cap A)$ of

$$\mathcal{P}^+(D^2 w) + \mu(x)|Dw| \geq -f(x) \quad \text{in } T' \cap A,$$

and

$$\|\mu\|_{L^n(T' \cap A)} \leq \varepsilon_0, \tag{2.9}$$

then it follows that

$$\Big(\frac{1}{|T|} \int_T (w^-_{T',A})^r \, dx\Big)^{1/r} \leq C_0 \Big(\inf_T w^-_{T',A} + R\|f\|_{L^n(T' \cap A)} \Big)$$

provided that $q > n$ and $q \geq p \geq n$, and

$$\begin{aligned} &\Big(\frac{1}{|T|} \int_T (w^-_{T',A})^r dx\Big)^{1/r} \\ &\leq C_0 \Big(\inf_T w^-_{T',A} + R^{2-\frac{n}{p}} \|f\|_{L^p(T' \cap A)} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k} \|\mu\|^k_{L^q(T' \cap A)} \Big) \end{aligned}$$

provided that $q > n > p > p_0$, where $M = M(n, p, q) \geq 1$ is an integer.

Remark 2.5. We refer to [16] for the (boundary) weak Harnack inequality for L^p-viscosity supersolutions of fully nonlinear PDEs with superlinear growth in the gradient and unbounded ingredients.

In the next section, we will establish some local and global ABP type estimates on L^p-viscosity subsolutions for (2.4). To this end, we recall the notations concerning the shape of domains from [8].

Definition 2.6 (Local geometric condition)**.** Let $\sigma, \tau \in (0,1)$. We call $y \in \Omega$ a $G_{\sigma,\tau}$ point in Ω if there exist $R = R_y > 0$ and $z = z_y \in \mathbb{R}^n$ such that

$$y \in B_R(z), \quad \text{and} \quad |B_R(z) \backslash \Omega_{y,B_R(z),\tau}| \geq \sigma |B_R(z)|, \tag{2.10}$$

where $\Omega_{y,B_R(z),\tau}$ is the connected component of $B_{\frac{R}{\tau}}(z) \cap \Omega$ containing y. For $\sigma, \tau \in (0,1)$, and $R_0 > 0$, $\eta \geq 0$, we call $y \in \Omega$ a $G^{R_0,\eta}_{\sigma,\tau}$ point in Ω if y is a $G_{\sigma,\tau}$ point in Ω with $R = R_y > 0$ and $z = z_y$ satisfying

$$R \leq R_0 + \eta |y|. \tag{2.11}$$

Remark 2.7. For the sake of simplicity of notations, for a $G_{\sigma,\tau}$ point $y \in \Omega$, we will write B_y for $B_{\frac{R_y}{\tau}}(z_y)$, where $R_y > 0$ and $z_y \in \mathbb{R}^n$ are from Definition 2.6.

Definition 2.8 (Global geometric condition)**.** We call Ω a $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ domain if any $y \in \Omega$ is a $G^{R_0,\eta}_{\sigma,\tau}$ point in Ω.

We refer the reader to [20] and [8] for examples of domains Ω satisfying $G^{R_0,\eta}_{\sigma,\tau}$. We also refer to [1] for a generalization.

3. ABP type estimates

We present pointwise estimates on L^p-viscosity subsolutions of (2.4), which is often referred as the Krylov-Safonov growth lemma.

In what follows, we fix $\sigma, \tau \in (0,1)$ and $R_0 > 0$. Let $y \in \Omega$ be a $G^{R_0,\eta}_{\sigma,\tau}$ point with $\eta \geq 0$. It is possible to apply our weak Harnack inequality in B_y, which is from Definition 2.6, if $\|\mu\|_{L^n(B_y \cap \Omega)} \leq \varepsilon_0$. Here and later, $\varepsilon_0 > 0$ is the constant from Lemma 2.4.

Even if $\|\mu\|_{L^n(B_y \cap \Omega)} > \varepsilon_0$, we may use Cabré's covering argument; we divide B_y into small pieces so that we may apply the weak Harnack inequality in each piece. We then obtain the weak Harnack inequality in B_y by combining all the inequalities for small pieces.

However, since we need the estimates uniform in $y \in \Omega$, this argument does not work immediately because of unboundedness of $\{R_y\}_{y \in \Omega}$ when $\eta > 0$.

To avoid this difficulty, we will suppose a decay rate of μ: $\|\mu\|_{L^q(\Omega \backslash B_t(0))} = o(t^{-(1-\frac{n}{q})})$. More precisely, for fixed $q > n$, we suppose that for all $\delta > 0$ there is $T_\delta > 0$ such that

$$\|\mu\|_{L^q(\Omega \backslash B_t(0))} \leq \delta t^{-(1-\frac{n}{q})} \quad \text{for } t \geq T_\delta. \tag{3.1}$$

Remark 3.1. It is assumed in [8] that $\mu(x) = O(|x|^{-1})$ as $|x| \to \infty$, which only implies $\|\mu\|_{L^q(\Omega \backslash B_t(0))} = O(t^{-(1-\frac{n}{q})})$.

Of course, if $\eta = 0$ (hence $R_y \leq R_0$), then we can apply directly Cabré's argument.

Lemma 3.2. *Assume that (2.5), (2.8) and (2.6) hold with $\mu \in L^q_+(\Omega)$. Let $\eta > 0$ and $y \in \Omega$ be a $G^{R_0,\eta}_{\sigma,\tau}$ point in Ω with $R = R_y > 0$ and $z = z_y \in \mathbb{R}^n$. Then, there exist $\kappa = \kappa(n, \lambda, \Lambda, \sigma, \tau, R_0, \eta) \in (0, 1)$ and $\varepsilon = \varepsilon(n, \sigma, \eta) > 0$ satisfying the following property: if $w \in C(\Omega)$ is an L^p-viscosity subsolution of (2.4) with $f \in L^p_+(\Omega)$, then we have the following properties: (i) Assume that $|y| \le R_0$. (a) If $p \ge n$, then*

$$w(y) \le \kappa \sup_{B_y\cap\Omega} w^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} w^+ + R_0\|f\|_{L^n(B_y\cap\Omega)}.$$

(b) If $p_0 < p < n$, then

$$\begin{aligned} w(y) \le{}& \kappa \sup_{B_y\cap\Omega} w^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} w^+ \\ &+ R_0^{2-\frac{n}{p}}\|f\|_{L^p(B_y\cap\Omega)} \sum_{k=0}^{M} R_0^{(1-\frac{n}{q})k}\|\mu\|^k_{L^q(B_y\cap\Omega)}. \end{aligned}$$

(ii) Assume that (3.1) is satisfied and that $|y| > R_0$. (a) If $p \ge n$, then

$$w(y) \le \kappa \sup_{B_y\cap\Omega} w^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} w^+ + R\|f\|_{L^n(B_y\cap\Omega\setminus B_{\varepsilon R}(0))}.$$

(b) If $p_0 < p < n$, then

$$\begin{aligned} w(y) \le{}& \kappa \sup_{B_y\cap\Omega} w^+ + (1-\kappa) \limsup_{x\to B_y\cap\partial\Omega} w^+ \\ &+ R^{2-\frac{n}{p}}\|f\|_{L^p(B_y\cap\Omega\setminus B_{\varepsilon R}(0))} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k}\|\mu\|^k_{L^q(B_y\cap\Omega\setminus B_{\varepsilon R}(0))}. \end{aligned}$$

Here $M = M(n, p, q) \ge 1$ is the integer in Lemma 2.4.

Remark 3.3. To get the weak maximum principle (Lemma 4.1 below), it is important to have the term $\|f\|_{L^p(B_y\cap\Omega\setminus B_{\varepsilon R}(0))}$ instead of $\|f\|_{L^p(B_y\cap\Omega)}$ in the estimates of the assertion (ii) above.

Proof. First of all, we shall omit giving the proof in the case of $\|\mu\|_{L^q(\Omega)} = 0$ because it is easy to do it, and we suppose that $\|\mu\|_{L^q(\Omega)} > 0$.

It is enough to show the assertion when $\hat{C} := \limsup_{x\to B_y\cap\partial\Omega} w^+(x) = 0$. In fact, after having established the assertion when $\hat{C} = 0$, we may apply the result to $w - \hat{C}$ to prove the assertion in the general case.

Due to (2.6), w is an L^p-viscosity solution of

$$\mathcal{P}^-(D^2w) - \mu(x)|Dw| \le f(x) \quad \text{in } \Omega.$$

Setting $C_w = \sup_{B_y\cap\Omega} w^+$, we immediately see that $v(x) := C_w - w(x)$ is an L^p-viscosity solution of

$$\mathcal{P}^+(D^2v) + \mu(x)|Dv| \ge -f(x) \quad \text{in } \Omega.$$

We shall first prove (ii).

Case (ii) $|y| > R_0$**:** Fix $\varepsilon \in (0, \frac{1}{2}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{\frac{1}{n}}\})$. Note that $2\varepsilon < 1/(1+\eta)$ and $(2\varepsilon)^n < \sigma/4$. We set $T = B_R(z) \setminus \overline{B}_{2\varepsilon R}(0)$ and $T' = B_y \setminus \overline{B}_{\varepsilon R}(0)$. Observe that

$$2\varepsilon R < \frac{R}{1+\eta} \le \frac{R_0 + \eta|y|}{1+\eta} < |y|$$

and consequently $y \in T = B_R(z) \setminus \overline{B}_{2\varepsilon R}(0)$. Let A be the connected component of $T' \cap \Omega$ which contains y. We have

$$\begin{aligned} |T\setminus A| &\geq |T\setminus \Omega_{y,B_R(z),\tau}| \\ &\geq |B_R(z)\setminus \Omega_{y,B_R(z),\tau}| - |B_{2\varepsilon R}(0)| \\ &\geq \sigma |B_R(0)| - (2\varepsilon)^n |B_R(0)| \\ &\geq \frac{\sigma}{2}|B_R(0)| \\ &\geq \frac{\sigma}{2}|T|. \end{aligned}$$

Since

$$T' \cap \partial A \subset T' \cap \partial(T' \cap \Omega) \subset T' \cap (\partial T' \cup \partial\Omega) = T' \cap \partial\Omega, \tag{3.2}$$

in view of $\hat{C} \leq 0$, we have

$$\liminf_{x\to T'\cap\partial A} v(x) = C_w - \limsup_{x\to T'\cap\partial A} w(x) \geq C_w. \tag{3.3}$$

Now, we verify (2.9). By (3.1), we can choose $T_\varepsilon > 0$ such that

$$\|\mu\|_{L^q(\Omega\setminus B_t(0))} \leq \frac{\varepsilon_0}{|B_1(0)|^{\frac{1}{n}(1-\frac{n}{q})}} \Big(\frac{\tau\varepsilon}{t}\Big)^{1-\frac{n}{q}} \quad \text{for } t \geq T_\varepsilon.$$

Assume $R \geq A_1 := T_\varepsilon \varepsilon^{-1}$. Using the above, we see

$$\|\mu\|_{L^n(T'\cap A)} \leq |B_1(0)|^{\frac{1}{n}(1-\frac{n}{q})} \Big(\frac{R}{\tau}\Big)^{1-\frac{n}{q}} \|\mu\|_{L^q(\Omega\setminus B_{\varepsilon R}(0))} \leq \varepsilon_0.$$

Setting $m = \liminf_{x\to T'\cap\partial A} v(x)$, we use (3.3) to show for any $r > 0$,

$$\Big(\frac{\sigma}{2}\Big)^{1/r} C_w \leq \Big(\frac{|T\setminus A|}{|T|}\Big)^{1/r} C_w \leq \Big(\frac{1}{|T|}\int_{T\setminus A} m^r dx\Big)^{1/r} \leq \Big(\frac{1}{|T|}\int_T (v^-_{T',A})^r dx\Big)^{1/r}.$$

Since $y \in A$, we have

$$\inf_T v^-_{T',A} \leq v(y) = C_w - w(y). \tag{3.4}$$

Thus, letting $r > 0$ be the constant from Lemma 2.4, we have

$$\Big(\frac{\sigma}{2}\Big)^{1/r} C_w \leq C_0 \Big(\inf_T v^-_{T',A} + R\|f\|_{L^n(T'\cap A)}\Big) \leq C_0 \big(C_w - w(y) + R\|f\|_{L^n(T'\cap\Omega)}\big)$$

if $p \geq n$, and

$$\Big(\frac{\sigma}{2}\Big)^{1/r} C_w \leq C_0\Big(C_w - w(y) + \|f\|_{L^p(T'\cap\Omega)} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k+2-\frac{n}{p}} \|\mu\|^k_{L^q(T'\cap\Omega)}\Big)$$

if $p \in (p_0, n)$. Therefore, we conclude that the assertion (ii) holds with $\kappa = 1 - (\frac{\sigma}{2})^{1/r} \min\{C_0^{-1}, 1\} > 0$ in the case where $R \geq A_1$.

Next assume that $R < A_1$. We can choose constants

$$\rho_0 = \rho_0(n, q, \tau, \varepsilon_0, \varepsilon, A_1, \|\mu\|_{L^q(\Omega)}),$$

$\mu_0 = \mu_0(n, q, \tau, \varepsilon_0, \varepsilon, A_1, \|\mu\|_{L^q(\Omega)}) \in (0, 1)$, $N_0 = N_0(n, q, \tau, \varepsilon_0, \varepsilon, A_1, \|\mu\|_{L^q(\Omega)}) \in \mathbb{N}$ and a finite sequence $\{x_i\}_{i=1}^{N_0} \subset T'$ such that

$$\overline{T} \subset \cup_{i=1}^{N_0} B_{\rho_0 R}(x_i) \subset \cup_{i=1}^{N_0} \overline{B}_{2\rho_0 R}(x_i) \subset T', \tag{3.5}$$

$$|B_{\rho_0 R}(x_i) \cap B_{\rho_0 R}(x_{i+1})| \geq \mu_0 |B_{\rho_0 R}(0)|, \tag{3.6}$$

where $B_{\rho_0 R}(x_{N_0+1}) = B_{\rho_0 R}(x_1)$, and

$$\rho_0 \le \frac{1}{A_1 |B_1(0)|^{1/n}} \Big(\frac{\varepsilon_0}{\|\mu\|_{L^q(\Omega)}} \Big)^{\frac{q}{q-n}}. \tag{3.7}$$

We see that

$$\|\mu\|_{L^n(B_{\rho_0 R}(x_i))} \le |B_{\rho_0 R}(x_i)|^{\frac{1}{n}-\frac{1}{q}} \|\mu\|_{L^q(B_y\cap\Omega)} \le \varepsilon_0.$$

For the reader's convenience, we recall Cabré's covering argument when $p \ge n$. Since $v^-_{T',A}$ is a nonnegative L^p-viscosity supersolution of $\mathcal{P}^+(D^2u) + \mu(x)|Du| \ge -f(x)$ in T', in view of Lemma 2.4, we have

$$\|v^-_{T',A}\|_{L^r(B_{\rho_0 R}(x_i))} \le |B_{\rho_0 R}(x_i)|^{1/r} C_0 \Big(\inf_{B_{\rho_0 R}(x_i)} v^-_{T',A} + \rho_0 R \|f\|_{L^n(A)} \Big)$$

for $i = 1, 2, \dots, N_0$, where $r, C_0 > 0$ are from Lemma 2.4. Furthermore, for $i \in \{1, 2, \dots, N_0\}$, setting $B_i = B_{\rho_0 R}(x_i)$, we have

$$\begin{aligned} \inf_{B_i} v^-_{T',A} &\le \inf_{B_i\cap B_{i+1}} v^-_{T',A} \\ &\le \Big(\frac{1}{|B_i\cap B_{i+1}|} \int_{B_i\cap B_{i+1}} (v^-_{T',A})^r dx \Big)^{1/r} \\ &\le C_1 \Big(\inf_{B_{i+1}} v^-_{T',A} + R\|f\|_{L^n(A)} \Big) \end{aligned}$$

for some $C_1 \ge 1$. Thus, repeating this argument, for $1 \le i < N_0$, we have

$$\inf_{B_i} v^-_{T',A} \le C_1^{N_0-1} \Big(\inf_{B_{N_0}} v^-_{T',A} + N_0 R \|f\|_{L^n(A)} \Big).$$

Since we may assume that $\inf_T v^-_{T',A} = \inf_{B_{N_0}} v^-_{T',A}$, there is $C_2 > 0$ such that

$$\|v^-_{T',A}\|_{L^r(T)} \le \sum_{i=1}^{N_0} \|v^-_{T',A}\|_{L^r(B_i)} \le R^{\frac{n}{r}} C_2 \left(\inf_T v^-_{T',A} + R\|f\|_{L^n(A)} \right).$$

When $p_0 < p < n$, we can easily apply the above argument to show that

$$\|v^-_{T',A}\|_{L^r(T)} \le R^{\frac{n}{r}} C_2 \Big(\inf_T v^-_{T',A} + R^{2-\frac{n}{p}} \|f\|_{L^p(A)} \sum_{k=0}^{M} R^{(1-\frac{n}{q})k} \|\mu\|^k_{L^q(A)} \Big).$$

What remains of the proof follows the same argument as in the case of $R \ge A_1$.

Case (i) $|y| \le R_0$**:** Since we have $R \le (1+\eta)R_0$ in this case, we may regard Ω as a bounded domain. Thus, we can use the standard covering argument by Cabré without using (3.1). Setting $T = B_R(z)$, $T' = B_{\frac{R}{\tau}}(z)$ and $A = \Omega_{y,B_R(z),\tau}$, we have

$$|T\backslash A| = |B_R(z) \setminus \Omega_{y,B_R(z),\tau}| \ge \sigma |B_R(z)| \ge \frac{\sigma}{2}|T|.$$

We shall only give a proof when $\|\mu\|_{L^n(T'\cap A)} \le \varepsilon_0$.

Following the same argument as in case (ii) with the above inequality, and new A, T, T', we have

$$\Big(\frac{\sigma}{2}\Big)^{1/r} C_w \le C_0 \left(\inf_T v^-_{T',A} + R_0 \|f\|_{L^n(B_y\cap\Omega)} \right) \le C_0 \left(C_w - w(y) + R_0 \|f\|_{L^n(B_y\cap\Omega)} \right)$$

provided that $p \ge n$, and

$$\Big(\frac{\sigma}{2}\Big)^{1/r} C_w \le C_0 \Big(C_w - w(y) + \|f\|_{L^p(B_y\cap\Omega)} \sum_{k=0}^{M} R_0^{(1-\frac{n}{q})k+2-\frac{n}{p}} \|\mu\|^k_{L^q(B_y\cap\Omega)} \Big)$$

provided that $p \in (p_0, n)$. Therefore, we conclude that the assertion holds with the same $\kappa \in (0,1)$ as in case (ii). □

Remark 3.4. The above proof clearly shows that ε can be any constant satisfying $0 < \varepsilon < \frac{1}{2}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$. In the above proof, we have stated that N_0 can be chosen independently of z and R, which may not be trivial. We will give a proof of this fact in Appendix.

The corresponding result for $\eta = 0$ is as follows.

Corollary 3.5. *Assume that* (2.5), (2.8) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$. *Let* $y \in \Omega$ *be a* $G^{R_0,0}_{\sigma,\tau}$ *point in* Ω *with* $R = R_y > 0$ *and* $z = z_y \in \mathbb{R}^n$. *Then, there exist* $\kappa = \kappa(n, \lambda, \Lambda, \sigma, \tau, R_0) \in (0,1)$ *and* $\varepsilon = \varepsilon(n,\sigma) > 0$ *satisfying the following property: if* $w \in C(\Omega)$ *is an* L^p*-viscosity subsolution of* (2.4) *with* $f \in L^p_+(\Omega)$, *then the same estimates as in Lemma 3.2 (i) hold.*

In the case of $\eta = 0$, we always have $|y| \le R_0$ unlike Lemma 3.2. For the proof of the above corollary, we just follow the steps in the proof of Lemma 3.2 (i).

When $\Omega \subset \mathbb{R}^n$ is a $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ domain, we derive the ABP maximum principle for L^p-viscosity subsolutions bounded from above of (2.4).

Theorem 3.6 (ABP maximum principle in unbounded domains)**.** *Assume* (2.8), (2.5) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$ *satisfying* (3.1). *Let* $\eta > 0$ *and* $\Omega \subset \mathbb{R}^n$ *be a* $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ *domain. Assume also*

$$
\begin{gathered}
\sup_{y\in\Omega, |y|>R_0} R_y \|f\|_{L^n(A_y\cap\Omega)} < \infty \quad \text{if } p \ge n, \\
\sup_{y\in\Omega, |y|>R_0} R_y^{2-\frac{n}{p}} \|f\|_{L^p(A_y\cap\Omega)} < \infty \quad \text{if } p_0 < p < n.
\end{gathered} \tag{3.8}
$$

Let $0 < \varepsilon < \min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$. *Then, there exists*

$$
C = C(n, \lambda, \Lambda, p, q, \varepsilon, \sigma, \tau, R_0, \eta) > 0
$$

satisfying the following properties: if $w \in C(\Omega)$ *is an* L^p*-viscosity subsolution bounded from above of* (2.4) *with* $f \in L^p_+(\Omega)$, *then it follows that*

$$
\begin{aligned}
\sup_\Omega w \le \limsup_{x\to\partial\Omega} w^+(x) &+ C \sup_{y\in\Omega,|y|>R_0} R_y \|f\|_{L^n(A_y\cap\Omega)} \\
&+ C R_0 \sup_{y\in\Omega,|y|\le R_0} \|f\|_{L^n(B_y\cap\Omega)},
\end{aligned} \tag{3.9}
$$

provided that $p \ge n$, *and*

$$
\begin{aligned}
\sup_\Omega w \le \limsup_{x\to\partial\Omega} w^+(x) &+ C \sup_{y\in\Omega,|y|>R_0} R_y^{2-\frac{n}{p}} \|f\|_{L^p(A_y\cap\Omega)} \sum_{k=0}^M R_y^{(1-\frac{n}{q})k} \|\mu\|^k_{L^q(A_y\cap\Omega)} \\
&+ C R_0^{2-\frac{n}{p}} \sup_{y\in\Omega,|y|\le R_0} \|f\|_{L^p(B_y\cap\Omega)} \sum_{k=0}^M R_0^{(1-\frac{n}{q})k} \|\mu\|^k_{L^q(B_y\cap\Omega)}
\end{aligned} \tag{3.10}
$$

provided that $p \in (p_0, n)$. *Here,* $A_y = B_{\frac{R_y}{\tau}}(z_y) \setminus B_{\varepsilon R_y}(0)$ *and* $B_y = B_{\frac{R_y}{\tau}}(z_y)$.

Proof. We take the supremum over $y \in \Omega$ with the estimates in Lemma 3.2 to conclude the inequalities (3.9) and (3.10). □

Remark 3.7. By following our proof of Lemma 3.2 (ii), it is easy to show that (3.1) implies

$$\sup_{y\in\Omega,|y|>R_0} R_y^{1-\frac{n}{q}}\|\mu\|_{L^q(A_y\cap\Omega)}<\infty. \tag{3.11}$$

To show the ABP maximum principle in unbounded domains corresponding to the case $\eta=0$, we do not need to assume (3.8) since $R_y\le R_0$.

Corollary 3.8. *Assume* (2.8), (2.5) *and* (2.6) *with* $\mu\in L^q_+(\Omega)$. *Let* $\Omega\subset\mathbb{R}^n$ *be a* $\hat{G}^{R_0,0}_{\sigma,\tau}$ *domain. Then, there exists* $C=C(n,\lambda,\Lambda,p,q,\varepsilon,\sigma,\tau,R_0)>0$ *satisfying the following properties: if* $w\in C(\Omega)$ *is an* L^p*-viscosity subsolution bounded from above of* (2.4) *with* $f\in L^p_+(\Omega)$, *then it follows that* (3.9) *holds provided* $p\ge n$, *and that* (3.10) *holds provided* $p\in(p_0,n)$.

4. Phragmén-Lindelöf theorem

In this section, we show that the weak maximum principle holds for PDEs with zero-order terms. As before, assuming that Ω is a $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ domain, for each $y\in\Omega$, we use the notations $R_y>0$ and $z_y\in\mathbb{R}^n$. Also, B_y and A_y, respectively, denote $B_{\frac{R_y}{\tau}}(z_y)$ and $B_{\frac{R_y}{\tau}}(z_y)\setminus B_{\varepsilon R_y}(0)$ for $\varepsilon\in(0,\frac12\min\{\frac{1}{1+\eta},(\frac{\sigma}{4})^{1/n}\})$.

Lemma 4.1. *Assume* (2.5), (2.8) *and* (2.6) *with* $\mu\in L^q_+(\Omega)$ *satisfying* (3.1). *Let* $\eta>0$ *and* Ω *be a* $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ *domain. Then, there exists* $c_0=c_0(n,\lambda,\Lambda,p,q,\sigma,\tau,R_0,\eta)>0$ *satisfying the following property: if* $c\in L^n_+(\Omega)$, $w\in C(\Omega)$ *is an* L^p*-viscosity solution bounded from above of*

$$F(x,Dw,D^2w)-c(x)w^+\le 0 \quad \text{in } \Omega \tag{4.1}$$

such that

$$\limsup_{x\to\partial\Omega} w(x)\le 0, \tag{4.2}$$

and

$$K_0:=\max\Big\{\sup_{y\in\Omega,|y|>R_0}\|\hat{c}\|_{L^n(A_y\cap\Omega)},\ \sup_{y\in\Omega,|y|\le R_0}\|c\|_{L^n(B_y\cap\Omega)}\Big\}\le c_0, \tag{4.3}$$

where $\hat{c}(x)=(1+|x|^2)^{1/2}c(x)$, *then* $w\le 0$ *in* Ω.

Remark 4.2. Instead of (4.3), it is assumed in [8] that

$$c(x)\le\frac{c_0}{1+|x|^2}\quad \text{for } x\in\Omega. \tag{4.4}$$

Set $c(x)=\frac{1}{1+|x|^2}$. We easily see by following an argument in the proof of Lemma 2.4 (ii) that the K_0 associated with this c is finite.

Proof. Note that by (2.6) together with Remark 2.2, w is an L^n-viscosity solution of

$$\mathcal{P}^-(D^2w)-\mu(x)|Dw|-c(x)w^+\le 0.$$

We apply Theorem 3.6 with $f=cw^+$ to obtain that when $|y|\le R_0$,

$$R_0\|cw^+\|_{L^n(B_y\cap\Omega)}\le R_0\sup_\Omega w^+\|c\|_{L^n(B_y\cap\Omega)}\le R_0K_0\sup_\Omega w^+.$$

On the other hand, when $|y|>R_0$, we have

$$R_y\|cw^+\|_{L^n(A_y\cap\Omega)}\le\frac{R_y}{\sqrt{1+(\varepsilon R_y)^2}}\sup_\Omega w^+\|\hat{c}\|_{L^n(A_y\cap\Omega)}\le\frac{K_0}{\varepsilon}\sup_\Omega w^+. \tag{4.5}$$

Choosing $\varepsilon_1 = \frac{1}{4}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$ for instance, we have

$$\sup_\Omega w \le C_3 \max\big\{R_0, \frac{1}{\varepsilon_1}\big\} c_0 \sup_\Omega w^+$$

for some constant $C_3 > 0$. Taking $c_0 < 1/(C_3\max\{R_0, \frac{1}{\varepsilon_1}\})$, we conclude the proof. $\square$

The next Corollary can be proved exactly same as above by using Corollary 3.8 instead of Theorem 3.6.

Corollary 4.3. *Assume* (2.5), (2.8) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$. *Let* Ω *be a* $\hat{G}^{R_0,0}_{\sigma,\tau}$ *domain. Then, there exists* $c_0 = c_0(n,\lambda,\Lambda,p,q,\sigma,\tau,R_0) > 0$ *satisfying the following property: if* $c \in L^n_+(\Omega)$ *and* $w \in C(\Omega)$ *is an* L^p*-viscosity solution bounded from above of* (4.1) *such that* (4.2) *and* (4.3) *hold, then* $w \le 0$ *in* Ω.

Theorem 4.4 (Phragmén-Lindelöf theorem)**.** *Assume* (2.5), (2.8) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$ *satisfying* (3.1). *Let* $\eta > 0$ *and* Ω *be a* $\hat{G}^{R_0,\eta}_{\sigma,\tau}$ *domain. If* $w \in C(\Omega)$ *is an* L^p*-viscosity solution of*

$$F(x, Dw, D^2w) \le 0 \quad \text{in } \Omega \tag{4.6}$$

such that (4.2) *holds and*

$$w^+(x) = O(\log|x|) \quad \text{as } |x| \to \infty, \tag{4.7}$$

then $w \le 0$ *in* Ω.

Remark 4.5. In [8], it is assumed that $w^+(x) = O(|x|^\alpha)$ with a constant $\alpha > 0$ as $|x| \to \infty$, which is weaker than (4.7). In fact, to deal with unbounded coefficients (i.e. μ), we will have to use a different function ξ to apply Lemma 2.3. This is the reason why we suppose a restrictive growth rate (4.7) in comparison with that in [8].

Proof of Theorem 4.4. Define a positive smooth function

$$\xi(x) = \log(1 + (1+|x|^2)^{\beta/2}),$$

where $\beta > 0$ will be fixed later, and set $u = w/\xi$, which is bounded from above. A straightforward calculation shows that

$$\begin{aligned}
\frac{|D\xi|}{\xi}(x) &\le \frac{\beta}{(1+|x|^2)^{1/2}\log(1+(1+|x|^2)^{\beta/2})} =: k_1(x),\\
\frac{|D^2\xi|}{\xi}(x) &\le \frac{\beta C_4}{(1+|x|^2)\log(1+(1+|x|^2)^{\beta/2})} =: k_2(x)
\end{aligned}$$

for some $C_4 > 0$. Thus, in view of Lemma 2.3, we see that u is an L^n-viscosity solution of

$$\mathcal{P}^-(D^2u) - \gamma_1(x)|Du| - \gamma_2(x)u^+ \le 0 \quad \text{in } \Omega,$$

where

$$\begin{aligned}
\gamma_1(x) &= \frac{h_1\beta}{(1+|x|^2)^{1/2}\log(1+(1+|x|^2)^{\beta/2})} + \mu(x) =: \gamma_{11}(x) + \gamma_{12}(x)\\
\gamma_2(x) &= \frac{h_2\beta C_4}{(1+|x|^2)\log(1+(1+|x|^2)^{\beta/2})} + \frac{\beta\mu(x)}{(\log 2)(1+|x|^2)^{1/2}}\\
&=: \gamma_{21}(x) + \gamma_{22}(x)
\end{aligned}$$

We first show that γ_1 satisfies (3.1). Note that we only need to show that γ_{11} satisfies (3.1). Setting $g(x) = (|x| \log |x|)^{-1}$ for $|x| > 1$, we easily show $\|g\|_{L^q(B_t^c(0))} = o(t^{-(1-\frac{n}{q})})$ as $t \to \infty$, which implies that γ_{11} satisfies (3.1).

We next show that (4.3) holds for γ_2. We shall observe that

$$K_0' := \max\big\{ \sup_{y\in\Omega,|y|>R_0} \|\hat{\gamma}_2\|_{L^n(A_y\cap\Omega)}, \sup_{y\in\Omega,|y|\le R_0} \|\gamma_2\|_{L^n(B_y\cap\Omega)} \big\} \tag{4.8}$$

is small when $\beta \to 0$, where $\hat{\gamma}_2(x) = \sqrt{1+|x|^2}\gamma_2(x)$.

When $y \in \Omega$ satisfies $|y| \le R_0$, we see that $B_y \subset B_{R_0(2+\eta+\tau^{-1}(1+\eta))}(0)$. Thus, the second term in (4.8) can be small when $\beta > 0$ is small enough.

To estimate the first term of (4.8), we note that $A_y = B_y \setminus B_{\varepsilon R_y}(0) \subset B_{\varepsilon R_y}(0)^c$ provided $\varepsilon < \frac{1}{2(1+\eta)}$. Setting $\hat{\gamma}_{22}(x) = \sqrt{1+|x|^2}\gamma_{22}(x)$, by (3.1), we can choose $T_0 > 1$ such that

$$\|\hat{\gamma}_{22}\|_{L^q(\Omega\setminus B_t(0))} \le \beta t^{-(1-\frac{n}{q})} \quad \text{for } t \ge T_0.$$

Hence, for $R_y > A_2 := \frac{T_0}{\varepsilon}$, we have

$$\|\hat{\gamma}_{22}\|_{L^n(A_y\cap\Omega)} \le C_5 R_y^{1-\frac{n}{q}} \|\hat{\gamma}_{22}\|_{L^q(A_y\cap\Omega)} \le C_5 \frac{\beta}{\varepsilon_1^{1-\frac{n}{q}}}$$

for some $C_5 > 0$, where $\varepsilon_1 = \frac{1}{4}\min\{\frac{1}{1+\eta}, (\frac{\sigma}{4})^{1/n}\}$. If $R_y \le A_2$, then we have

$$\|\hat{\gamma}_{22}\|_{L^n(A_y\cap\Omega)} \le C_6 \beta R_y^{1-\frac{n}{q}} \|\mu\|_{L^q(\Omega)} \le C_6 \beta A_2^{1-\frac{n}{q}} \|\mu\|_{L^q(\Omega)}$$

for some $C_6 > 0$. Thus, in this case, we may suppose that $\|\hat{\gamma}_{22}\|_{L^n(A_y\cap\Omega)}$ is small by taking small $\beta > 0$.

The remaining case is to prove that $\sup_{y\in\Omega,|y|>R_0} \|\hat{\gamma}_{21}\|_{L^n(A_y\cap\Omega)}$ is small, where $\hat{\gamma}_{21}(x) = \sqrt{1+|x|^2}\gamma_{21}(x)$. To this end, we shall show that for any $c_0 > 0$, there is small $\beta > 0$ such that $\|\hat{\gamma}_{21}\|_{L^n(\mathbb{R}^n)} \le c_0$. Since

$$\int_t^\infty \frac{1}{r(\log r)^n} dr = \frac{1}{(n-1)(\log t)^{n-1}} \quad \text{for } t > 1,$$

we can choose $\hat{T} > 1$ independent of $\beta > 0$ such that $\|\hat{\gamma}_{21}\|_{L^n(B_{\hat{T}}(0)^c)} \le c_0/2$. For this fixed $\hat{T} > 0$, we can find small $\beta > 0$ such that $\|\hat{\gamma}_{21}\|_{L^n(B_{\hat{T}}(0))} \le c_0/2$. Therefore, using Lemma 4.1 with $\mu = \gamma_1$ and $c = \gamma_2$, we get $u \le 0$. This concludes the proof. $\square$

Our Phragmén-Lindelöf theorem for $\eta = 0$ is as follows.

Corollary 4.6 (Phragmén-Lindelöf theorem)**.** *Assume* (2.5)*,* (2.8) *and* (2.6) *with* $\mu \in L^q_+(\Omega)$*. Let* Ω *be a* $\hat{G}^{R_0,0}_{\sigma,\tau}$ *domain. If* $w \in C(\Omega)$ *is an* L^p*-viscosity solution of* (4.6) *such that* (4.2) *and* (4.7) *hold, then* $w \le 0$ *in* Ω*.*

Proof. The only difference from the proof of Theorem 4.4 is how to estimate $\hat{\gamma}_{22}$. However, since $R_y \le R_0$, we can show it immediately. $\square$

5. Appendix: A proof of an elementary geometric property

In the proof of Lemma 3.2, the integer N_0 might depend on $y \in \Omega$ such that $|y| > R_0$ and $R := R_y < A_1$. We shall show that the integer N_0 has an upper bound independent of such $y \in \Omega$. To this end, we recall our domains T and T' in this case: $T = B_R(z) \setminus \overline{B}_{2\varepsilon R}(0)$ and $T' = B_{\frac{R}{\tau}}(z) \setminus \overline{B}_{\varepsilon R}(0)$.

We note that the position of (T,T') varies depending on the distance of two centers; $|z|$.

For $t \in [0,1]$, we denote by (T_t, T'_t) the couple (T,T') when $|z| = (1-t)(\frac{1}{\tau} + 2\varepsilon)$. For instance, T_1 and T'_1 are annuli with the common center at $z = 0$ while $T_0 = B_R(z)$ and $T'_0 = B_{\frac{R}{\tau}}(z)$. All the possible positions of (T,T') can be found in $\{(T_t, T'_t) : t \in [0,1]\}$. For each (T_t, T'_t), it is easy to find an integer $N_{0,t} \in \mathbb{N}$ satisfying (3.5), (3.6), (3.7) with $N_0 = N_{0t}$.

For any fixed $t \in [0,1]$, we can choose $\{x_{i,t}\}_{i=1}^{N_{0,t}} \subset T'_t$ such that (3.5), (3.6), (3.7) with $N_0 = N_{0,t}$, $x_i = x_{i,t}$, $T = T_t$ and $T' = T'_t$. We can find $\delta_t > 0$ such that (3.5) holds for $T = T_s$ and $T' = T'_s$ for $s \in I_t := (t - \delta_t, t + \delta_t) \cap [0,1]$ because (T_t, T'_t) changes continuously in t. Since $[0,1] \subset \cup_{t\in[0,1]} I_t$, we can choose a finite set $\{t_k \in [0,1]\}_{k=1}^L$ such that $[0,1] \subset \cup_{k=1}^L I_{t_k}$. Therefore, we can take $\hat{N} := \max\{N_{0,t_k} : k = 1, 2, \dots, L\}$ as an upper bound for N_0.

Acknowledgements. The authors want to thank the anonymous referee for several suggestions and comments on the first draft of this article.

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