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RENORMALIZED ENTROPY SOLUTIONS FOR DEGENERATE NONLINEAR EVOLUTION PROBLEMS

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ABSTRACT. We study the degenerate differential equation

$$b(v)_t - \operatorname{div} a(v, \nabla g(v)) = f \quad \text{on } Q := (0,T) \times \Omega$$

with the initial condition $b(v(0,\cdot)) = b(v_0)$ on Ω and boundary condition $v = u$ on some part of the boundary $\Sigma := (0,T) \times \partial\Omega$ with $g(u) \equiv 0$ a.e. on Σ. The vector field a is assumed to satisfy the Leray-Lions conditions, and the functions b, g to be continuous, locally Lipschitz, nondecreasing and to satisfy the normalization condition $b(0) = g(0) = 0$ and the range condition $R(b+g) = \mathbb{R}$. We assume also that g has a flat region $[A_1, A_2]$ with $A_1 \leq 0 \leq A_2$. Using Kruzhkov's method of doubling variables, we prove an existence and comparison result for renormalized entropy solutions.

1. Introduction

Let Ω be a $C^{1,1}$ bounded open subset of $\mathbb{R}^N$ with regular boundary if $N > 1$ and let $p > 1$. We consider the initial-boundary value problem of parabolic-hyperbolic type: (Problem $P_{b,g}(v_0, u, f)$)

$$\begin{gathered} \frac{\partial b(v)}{\partial t} - \operatorname{div} a(v, \nabla g(v)) = f \quad \text{on } Q := (0,T) \times \Omega \\ v = u \quad \text{on a part of } \Sigma := (0,T) \times \partial\Omega \\ b(v)(0,\cdot) = v_0 := b(v_0) \quad \text{on } \Omega, \end{gathered} \tag{1.1}$$

where $b, g : \mathbb{R} \to \mathbb{R}$ are nondecreasing, locally Lipschitz continuous such that $b(0) = g(0) = 0$ and $R(b+g) = \mathbb{R}$. We assume also that:

- The function g has a flat region around 0; i.e., there exists $A_1 \leq 0 \leq A_2$ such that $g(x) = 0$ for all $x \in [A_1, A_2]$ and g is strictly increasing in $]-\infty, A_1[\cup]A_2, +\infty[$.
- The data $v_0 : \Omega \to \mathbb{R}$ and $f : Q \to \mathbb{R}$ are measurable functions with $b(v_0) \in L^1(\Omega)$ and $f \in L^1(Q)$. Moreover the boundary data $u : \Sigma \to \mathbb{R}$ is assumed to be continuous with $g(u) = 0$ a.e. in Σ.
- The vector field $a : \mathbb{R} \to \mathbb{R}^N$ is assumed to be continuous, to satisfy the growth condition

$$|a(r,\xi) - a(r,0)| \leq C(|r|)|\xi|^{p-1} \quad \text{for all } (r,\xi) \in \mathbb{R} \times \mathbb{R}^n \tag{1.2}$$

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with $C : \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing and the weak coerciveness condition

$$(a(r,\xi) - a(r,0)) \cdot \xi \geq \lambda(|r|)|\xi|^p \quad \text{for all } r \in \mathbb{R},\ \xi \in \mathbb{R}^N \tag{1.3}$$

where $\lambda : \mathbb{R}^+ \to]0,\infty[$ is a continuous function satisfying for all $k > 0$, $\lambda_k := \inf_{\{r;\,|b(r)|\leq k\}} \lambda(r) > 0$.

• To prove the uniqueness of a solution, we assume that a satisfies the additional condition

$$\begin{aligned}&(a(r,\xi) - a(s,\eta)) \cdot (\xi - \eta) + (B(g(r) - B(g(s)))(1 + |\xi|^p + |\eta|^p)\\ &\geq \Gamma(r,s) \cdot \xi + \tilde{\Gamma}(r,s) \cdot \eta,\end{aligned} \tag{1.4}$$

for all $r, s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$, for some continuous function $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is locally Lipschitz continuous on $\mathbb{R} \setminus [A_1, A_2]$ and some continuous fields $\Gamma, \tilde{\Gamma}$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$. In particular, Hypothesis (1.4) implies $\Gamma(r,r) = \tilde{\Gamma}(r,r) = 0$ for every $r \in \mathbb{R}$.

The above formulation involves a large class of problems such as Stefan problems, filtrations and flows through porous media, etc. Since b and g are not assumed to be strictly increasing, the problem can behave differently: it is of elliptic-parabolic type when g is not partially degenerate, purely hyperbolic when $g \equiv 0$ and the three aspects coexist when b and g degenerate partially on some regions of $\mathbb{R}$. Remark that the problem can not be totally degenerate thanks to the range condition on $b + g$. It is well known that in the elliptic-parabolic case, the boundary conditions are satisfied in the Dirichlet sense; i.e. pointwise. Existence and uniqueness results for this type problems are now well known (see [1, 13, 6]). It is not the case for the hyperbolic problems which can be over determined when we impose a condition on "all the boundary". A simple example which illustrates this ambiguity is the Burger's equation on an interval $[a, b] \subset \mathbb{R}$. In [8], the authors have given a "right formulation" of a solution for the Burger's equation on a bounded domain, where the boundary condition is read as an entropy condition on the boundary. However, their formulation involves the trace of the entropy solution which means that it is restricted to the BV-framework. This in turn implies some strong regularity on the flux and the boundary data itself. An other integral formulation of the entropy condition is given by Otto [28] and guarantees existence and uniqueness in the more general case where the flux is Lipschitz continuous and the data are in L^∞. These results were extended to the L^1 setting and for merely continuous flux (see among others [17], [10]). For hyperbolic problems with nonhomogeneous boundary conditions on the boundary, we refer to [7, 4, 20, 3, 5, 32, 33, 34], etc.

The boundary condition is not the unique difficulty when dealing with hyperbolic problems. Indeed, even when the problem is posed on the whole space $\mathbb{R}^N$, the usual variational formulation is ill-posed in the sense that a weak solution is usually not unique. In order to have a good theory of existence and uniqueness, Kruzhkov has introduced the first notion of entropy solution (see [22, 23]) which is obtained by comparison with particular test functions and which coincides with the physical solution obtained by regularization methods.

In the parabolic-hyperbolic case, the problem is more complicate because the two behaviour parabolic and hyperbolic coexist. This means that the boundary condition is satisfied in the Dirichlet sense in the regions where g is strictly increasing but has to be read as an entropy condition when g is degenerate.

Carrillo [14] has given a formulation which conciliates between the two aspects (hyperbolic and parabolic) for the problem $(P_{b,g}) : b(v)_t - \Delta g(v) + \operatorname{div} \Phi(v) = f$ with homogeneous boundary conditions. The author has also studied the related Cauchy problem from the point of view of semi-group theory. Under an extra condition on the flux Φ, known as the "structure condition", he has proved existence and uniqueness results for the stationary and the evolution problems.

In a recent work [26], the authors have studied the same problem $(P_{b,g})$ with nonhomogeneous conditions on the boundary in the particular case, where $b \equiv I_{\mathbb{R}}$. They have proved the existence and uniqueness of a "weak entropy solution" and consistency with viscosity approximations. The boundary condition is given by means of a limit expressed by "boundary layer" and can be viewed as a generalization of the condition proposed by Felix Otto in [28].

In an earlier work [2], we have studied the problem $P_{b,I_{\mathbb{R}}}$ with nonhomogeneous boundary conditions and without assuming the structure condition on the flux Φ. Using monotonicity and strong penalization methods, we have proved existence of a renormalized entropy solution and uniqueness results.

Here, using another method of approximation, we prove an existence and uniqueness result for the problem with "g-homogeneous" boundary conditions. The plan of the paper is as follows: In section 2, we introduce some notations and define the renormalized entropy solution of (1.1) in the general setting described in the introduction, then we announce our main results. Section 3 is devoted to the proof of the comparison principle and section 4 to the proof of the existence result.

2. Definitions, notation and main results

- For any $k, l \in \mathbb{R}$, for a.e. $(t,x) \in \Sigma$, let

$$\omega^+((t,x),k,l) := \max_{k \le r,s \le l \vee k} |(a(r,0) - a(s,0)) \cdot \vec{\eta}(x)|,$$
$$\omega^-((t,x),k,l) := \max_{l \wedge k \le r,s \le k} |(a(r,0) - a(s,0)) \cdot \vec{\eta}(x)|,$$

where $\vec{\eta}(x)$ denotes the unit outer normal to $\partial\Omega$ in x.

- For $k > 0$, T_k is the truncation function at level k; i.e.,

$$T_k(r) = \begin{cases} r \wedge k := \min(r,k) & \text{for } r \ge 0, \\ r \vee (-k) := \max(r,-k) & \text{for } r \le 0. \end{cases}$$

- By $T^1, T^{1,2}$ and T^2, we denote the truncation functions defined successively by

$$T^1(r) = r \wedge A_1, \quad T^{1,2}(r) = A_1 \vee r \wedge A_2, \quad T^2(r) = r \vee A_2.$$

- The operators H_δ, $\delta > 0$ and H_0 are defined by

$$H_\delta(s) := \min(\frac{s^+}{\delta}, 1), \quad H_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$$

- By sign^+, we denote the multivalued function

$$\overset{+}{\text{sign}}(r) = \begin{cases} 0 & \text{if } r < 0, \\ [0,1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

• The proof of the comparison result involves also sequences of mollifiers $(\rho_n)_n$ which are defined as follows:

$$\rho_n(t) = n\rho(nt) \text{ where } \rho \in \mathcal{C}_c^\infty(-1,0) \text{ such that } \int_{-1}^0 \rho(t)\,dt = 1. \tag{2.1}$$

Definition 2.1. A measurable function $v : Q \to \mathbb{R}$ is said to be a renormalized entropy solution of (1.1) if $b(v) \in L^1(Q)$ and for all $k > 0$, $g(T_k v) \in L^p(0,T,W^{1,p}(\Omega))$ with $g(T_k v) = 0$ in the sense of traces in $L^p(0,T,W^{1,p}(\Omega))$, and there exists some families of non-negative bounded measures $\mu_l := \mu_l(v)$ and $\nu_l = \nu_l(v)$ on $\overline{Q}$ such that

$$\|\mu_l\|,\ \|\nu_{-l}\| \to 0, \quad \text{as } l \to \infty,$$

and the following entropy inequalities are satisfied:

For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T)\times\mathbb{R}^N)$ such that $\xi \geq 0$ and $(g(u\wedge l)-g(k))^+\xi = 0$ a.e. on Σ, for all $l \geq k$,

$$\begin{aligned} &\int_\Sigma \omega^+((t,x),k,u\wedge l)\xi + \int_Q (b(v\wedge l)-b(k))^+\xi_t + \int_\Omega (b(v_0\wedge l)-b(k))^+\xi(0,\cdot) \\ &- \int_Q \chi_{\{v\wedge l>k\}}(a(v\wedge l,\nabla g(v\wedge l)) - a(k,0))\cdot\nabla\xi + \int_Q \chi_{\{v\vee l>k\}} f\xi \\ &\geq -\langle \mu_l,\xi\rangle \end{aligned} \tag{2.2}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T)\times\mathbb{R}^N)$ such that $\xi \geq 0$ and $(g(k)-g(u\vee l))^+\xi = 0$ a.e. on Σ, for $l \leq k$,

$$\begin{aligned} &\int_\Sigma \omega^-((t,x),k,u\vee l)\xi + \int_Q (b(k)-b(v\vee l))^+\xi_t + \int_\Omega (b(k)-b(v_0\vee l))^+\xi(0,\cdot) \\ &- \int_Q \chi_{\{k>v\vee l\}}(a(k,0) - a(v\vee l,\nabla g(v\wedge l))\cdot\nabla\xi - \int_Q \chi_{\{k>v\vee l\}} f\xi\} \\ &\geq -\langle \nu_l,\xi\rangle. \end{aligned} \tag{2.3}$$

Remark 2.2. **(i)** The notion of renormalized entropy solution in already introduced in [17] for the purely hyperbolic problem ($g \equiv 0$ and $b = I_{\mathbb{R}}$). It allows usually to prove existence and uniquenes results in the L^1-setting. Here, due to the degeneracy of b and g, the existence of a weak solution in $L^p(0,T,W_0^{1,p}(\Omega))$ can not be proved even in the case of L^∞-data (see for example [6]).

(ii) In the case where the function b is strictly increasing and the data $v_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$ and $u \in C(\Sigma)$, using integration by parts tools, it is easily proved that the renormalized entropy solution corresponding to L^∞-data is also in $L^\infty(Q)$. In this case, it can be equivalently characterized as follows: A function $v \in L^\infty(Q)$ is a weak entropy solution of (1.1) if

$$g(v) \in L^p(0,T,W^{1,p}(\Omega)) \text{ and } g(v) = g(0)$$

in the sense of traces in $L^p(0,T,W^{1,p}(\Omega))$, and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T)\times\mathbb{R}^N)$ such that $\xi \geq 0$ and $(-g(k))^+\xi = 0$ a.e. on Σ,

$$\begin{aligned} &\int_\Sigma \omega^+((t,x),k,a)\xi + \int_Q (b(v)-b(k))^+\xi_t + \int_\Omega (b(v_0)-b(k))^+\xi(0,\cdot) \\ &- \int_Q \chi_{\{v>k\}}(a(v,\nabla g(v)) - a(k,0))\cdot\nabla\xi + \int_Q \chi_{\{v>k\}} f\xi\} \geq 0 \end{aligned} \tag{2.4}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(g(k))^+\xi = 0$; a.e. on Σ,

$$\begin{aligned} &\int_\Sigma \omega^-((t,x),k,a)\xi + \int_Q \{(b(k)-b(v))^+\xi_t + \int_\Omega (b(k)-b(v_0))^+\xi(0,\cdot) \\ &- \int_Q \chi_{\{k>v\}}(a(k,0) - a(v,\nabla g(v)) \cdot \nabla\xi - \int_Q \chi_{\{k>v\}} f\xi\} \geq 0. \end{aligned} \tag{2.5}$$

(iii) If we penalize the problem (1.1) by a suitable strong perturbation ψ, then the renormalized entropy solution is also in this case in $L^\infty(Q)$, hence a weak entropy solution.

The main results of this paper are the following:

Theorem 2.3. *For any $(v_0, f) \in L^1(\Omega) \times L^1(Q)$ with $b(v_0) \in L^1(\Omega)$, for any $u : \Sigma \to [A_1, A_2]$ continuous, there exists a renormalized entropy solution of* (1.1).

Corollary 2.4. *For any $(v_0, f) \in L^\infty(\Omega) \times L^\infty(Q)$ with $b(v_0) \in L^\infty(\Omega)$, for any $u : \Sigma \to [A_1, A_2]$ continuous, there exists a unique $w \in L^\infty(Q)$ with $w = b(v)$ a.e. in Q and v is a weak entropy solution of* (1.1).

Remark 2.5. • *By approximation, the existence result holds true also for a measurable boundary data $u : \Sigma \to [A_1, A_2]$ satisfying $\overline{a}(u, 0) \in L^1(\Sigma)$ without any additional assumption on the regularity of u. Here, we define the function $\overline{a} : \mathbb{R} \times \partial\Omega \to \mathbb{R}$ by*

$$\overline{a}(s, x) := \sup\{|a(r, 0) \cdot \eta(x)|,\ r \in [-s^-, s^+]\}.$$

• *The uniqueness result follows as a consequence of a comparison theorem (see Theorem 3.1 below).*

3. Proofs of the comparison and uniqueness results

We first prove a comparison result in the L^∞-setting.

Theorem 3.1. *Let $(v_{0i}, f_i) \in L^\infty(\Omega) \times L^\infty(Q)$ and $u_i \in L^\infty(\Sigma)$ such that $g(u_i) = 0$ a.e. on Σ, $i = 1, 2$ with $u_1 \in C(\Sigma)$. Let $v_i \in L^\infty(Q)$ be a weak entropy solution of $P_{b,g}(v_{0i}, u_i, f_i)$. Then there exists $\kappa \in L^\infty(Q)$ with $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ a.e. in Q such that, for any $\xi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$, $\xi \geq 0$,*

$$\begin{aligned} -\int_\Sigma \omega^-((t,x),u_1,u_2)\xi \leq &\int_Q (b(v_1) - b(v_2))^+\xi_t + \int_Q \kappa(f_1 - f_2)\xi \\ &- \int_Q \chi_{\{v_1>v_2\}}(a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla\xi \\ &+ \int_\Omega (b(v_{01}) - b(v_{02}))^+\xi(0,\cdot). \end{aligned} \tag{3.1}$$

Remark 3.2. If $v_1 \in L^\infty(Q)$ satisfies the entropy inequalities (2.2) and (2.3) for the data f_1, v_{01}, u_1 and for b, g and a vector field a, then $-v_1$ satisfies the same inequalities with data $-f_1$, $-v_{01}$, $-u_1$ and for $-b(-.)$, $-g(-.)$, $-a(-., -.)$. Consequently under the assumptions of Theorem 3.1, one also has existence of

$\tilde\kappa \in$sign$^+(v_2 - v_1)$ such that for all $\xi \in \mathcal{D}([0,T)\times\mathbb{R}^N), \xi \geq 0$

$$\begin{aligned}
-\int_\Sigma \omega^-((t,x),u_1,u_2)\xi \leq \int_Q (b(v_2)-b(v_1))^+\xi_t + \int_Q \tilde\kappa(f_1-f_2)\xi \\
-\int_Q \chi_{\{v_2>v_1\}}(a(v_2,\nabla g(v_2)-a(v_1,\nabla g(v_1))\cdot\nabla\xi \\
+\int_\Omega (b(v_{02})-b(v_{01}))^+\xi(0,\cdot).
\end{aligned}\tag{3.2}$$

Summing up (3.1) and (3.2), it follows that

$$\begin{aligned}
&\int_\Omega |b(v_1)-b(v_2)|(t) \\
&\leq \int_0^t\int_\Omega |f_1-f_2| + \int_\Omega |b(v_{01})-b(v_{02})| \\
&\quad + \int_0^t\int_{\partial\Omega} \max_{\{\min(u_1,u_2)\leq r,s\leq\max(u_1,u_2)\}} |(a(r,0)-a(s,0))\cdot\vec\eta(x)|
\end{aligned}\tag{3.3}$$

a.e. in $[0,T)$.

The result being already known in the purely hyperbolic and parabolic case respectively, we will assume in the following that g has at least a flat region i.e. $A_1 \neq A_2$. In the proof, we need the following estimations:

Lemma 3.3. *Let $f \in L^\infty(Q)$, $v_0 \in L^\infty(\Omega)$, $u \in L^\infty(\Sigma)$ such that $g(u) = 0$ a.e. on Σ and $v \in L^\infty(Q)$ be a weak entropy solution of* (1.1)*. Then, there exists a positive constant C depending only on $\|f\|_{L^\infty(Q)}$ $\|v\|_{L^\infty(Q)}$ and $\|b(v_0)\|_{L^\infty(\Omega)}$ such that*

$$\int_{Q\cap\{0\leq g(v)\leq\delta\}} |\nabla g(v)|^p \leq \delta C, \tag{3.4}$$

$$\int_{Q\cap\{-\delta\leq g(v)\leq 0\}} |\nabla g(v)|^p \leq \delta C. \tag{3.5}$$

Proof. As v is a weak solution of (1.1), using $T_\delta(g(v))^+ \in L^p((0,T),W_0^{1,p}(\Omega))$ as test function, we get

$$\begin{aligned}
&\int_Q \tilde a(v,\nabla g(v))\cdot T_\delta(g(v))^+ \\
&\leq \|b(v_0)\|_{L^\infty(\Omega)} + |\int_Q |f||T_\delta(g(v))^+| + |\int_Q a(v,0)\cdot\nabla T_\delta(g(v))^+|.
\end{aligned}$$

Hence, by the usual chain rule argument (see for example [1] and [16]), applying the Green-Gauss formula and the growth condition (1.2), we obtain (3.4). The second estimation (3.5) can be proved in a similar way. □

Lemma 3.4. *Let* $(v_0,u,f)\in L^\infty(\Omega)\times C(\Sigma)\times L^\infty(Q)$ *and* v *be a weak solution of* $P_{b,g}(v_0,a,f)$ *. Then*

$$\begin{aligned} &\int_Q \chi_{\{v>k\}}(a(v,\nabla g(v))-a(k,0))\cdot\nabla\xi \\ &+\int_Q \chi_{\{v>k\}}\{(b(k)-b(v))\xi_t - f\xi\}\,dx\,dt - \int_\Omega (b(v_0)-b(k))^+\xi\,dx \\ &= -\lim_{\delta\to 0}\int_Q (a(v,\nabla g(v))-a(v,0))\cdot\nabla g(v)H'_\delta(g(v)-g(k))\xi\,dx\,dt \end{aligned} \tag{3.6}$$

for any $(k,\xi)\in\mathbb{R}\times\mathcal{D}^+([0,T)\times\mathbb{R}^N)$ *such that* $(-g(k))^+\xi=0$ *a.e. on* Σ.

Moreover,

$$\begin{aligned} &\int_Q \chi_{\{k>v\}}(a(v,\nabla g(v))-a(k,0))\cdot\nabla\xi \\ &+\int_Q \chi_{\{k>v\}}\{(b(k)-b(v))\xi_t - f\xi\}\,dx\,dt + \int_\Omega (b(k)-b(v_0))^+\xi\,dx \\ &= \lim_{\delta\to 0}\int_Q (a(v,\nabla g(v))-a(v,0))\cdot\nabla g(v)H'_\delta(g(k)-g(v))\xi\,dx\,dt \end{aligned} \tag{3.7}$$

for any $(k,\xi)\in\mathbb{R}\times\mathcal{D}^+([0,T)\times\mathbb{R}^N)$ *such that* $(g(k))^+\xi=0$ *on* Σ.

The proof of the above lemma follows the same lines as in the proof of [14, Lemma 5].

Proof of Theorem 3.1. Let $(B_i)_{i=0\dots m}$ be a covering of $\overline{\Omega}$ satisfying $B_0\cap\partial\Omega=\emptyset$, and such that, for each $i\ge 1$, B_i is a ball contained in some larger ball $\tilde B_i$ with $\tilde B_i\cap\partial\Omega$ is part of the graph of a Lipschitz function. Let $(\mathcal{P}_i)_{i=0\dots m}$ denote a partition of unity subordinate to the covering $(B_i)_i$ and denote by ξ an arbitrary Function in $\mathcal{D}((0,T)\times\mathbb{R}^N)$, $\xi\ge 0$.

As usual, we use Kruzhkov's technique of doubling variables in order to prove the comparison result (see [22], [23], [17], [3], etc): We choose two pairs of variables (t,x) and (s,x) and consider v_1 as a function of (s,x) and v_2 as a function of $(t,x)\in Q$. Define the test function $\xi^i_{m,n}:(t,x,s,y)\mapsto\mathcal{P}_i(x)\xi(t,x)\varrho_n(x-y)\rho_m(t-s)$, where $(\varrho_n)_n$ is a sequence of mollifiers in $\mathbb{R}^N$ such that $x\mapsto\varrho_n(x-y)\in\mathcal{D}(\Omega)$, for all $y\in B_i$, $\sigma_n(x)=\int_\Omega\varrho_n(x-y)\,dy$ is an increasing sequence for all $x\in B_i$, and $\sigma_n(x)=1$ for all $x\in B_i$ with $d(x,\mathbb{R}^N\setminus\Omega)>c/N$ for some $c=c(i)$ depending on B_i. Then, for m,n sufficiently large,

$$\begin{gathered} (s,y)\mapsto\xi^i_{m,n}(t,x,s,y)\in\mathcal{D}(]0,T[\times\mathbb{R}^N),\quad\text{for any }(t,x)\in Q,\\ (t,x)\mapsto\xi^i_{m,n}(t,x,s,y)\in\mathcal{D}([0,T[\times\Omega),\quad\text{for any }(s,y)\in Q\\ \operatorname{supp}_y(\xi^i_{m,n}(t,s,x,.))\subset B_i,\ \text{for any }(t,s,x)\in[0,T)^2\times\operatorname{supp}(\mathcal{P}_i). \end{gathered}$$

For convenience, we sometimes omit the index i and simply set $\mathcal{P}=\mathcal{P}_i$, $B=B_i$ and $\xi^i_{m,n}=\xi_{m,n}$.

The main idea of our proof is to compare locally two solutions on each sufficiently small ball $\mathcal{B}((t,x),r)$ such that $\mathcal{B}((t,x),r)\cap\Sigma\neq\emptyset$ and $\max_{\Sigma\cap\mathcal{B}((t,x),r)}u-\min_{\Sigma\cap\mathcal{B}((t,x),r)}u\le\varepsilon$. To this end, for all $\eta>0$, let

$$(\mathcal{B}^\eta_j:=\mathcal{B}((t_j,x_j),\eta))_{j=0,\dots,p_\eta}\ \text{be a finite covering of }[0,T)\times\overline{\Omega} \tag{3.8}$$

such that $([0,T) \times \Omega) \subset \cup_j \mathcal{B}_j^\eta =: O_\eta$ and $|\overline{O_\eta} \setminus ([0,T) \times \Omega)| \leq c\eta$, for a positive constant c independent of η. Let

$$(\varphi_j^\eta)_{j=0,\dots,p_\eta} \text{ be a partition of unity subordinate to } (\mathcal{B}_j^\eta)_j \tag{3.9}$$

and define

$$\zeta_{m,n} := \zeta_{m,n}^{j,\eta} : (t,x,s,y) \mapsto \xi_{m,n}(t,x,s,y)\varphi_j^\eta(t,x).$$

Obviously, $\zeta_{m,n}$ satisfies the following properties: for m and n sufficiently large,

$$\begin{gathered}(s,y) \mapsto \zeta_{m,n}(t,x,s,y) \in \mathcal{D}(]0,T[\times \mathbb{R}^N), \quad \text{for any } (t,x) \in Q,\\ (t,x) \mapsto \zeta_{m,n}(t,x,s,y) \in \mathcal{D}([0,T[\times \Omega \cap \mathcal{B}_j^\eta), \quad \text{for any } (s,y) \in Q\\ \operatorname{supp}_y(\zeta_{m,n}(t,s,x,.)) \subset B, \quad \text{for any } (t,s,x) \in [0,T)^2 \times \operatorname{supp}(\mathcal{P}).\end{gathered}$$

Moreover, the function

$$\begin{aligned}\hat{\zeta}_n(t,x) &:= \int_Q \xi_{m,n}(t,x,s,y)d(s,y)\\ &= \xi(t,x)\mathcal{P}(x)\varphi_j^\eta(t,x)\int_{\mathbb{R}} \varrho_n(x-y)d\,y \int_0^T \rho_m(t-s)ds\\ &= \xi(t,x)\mathcal{P}(x)\varphi_j^\eta(t,x)\int_{\mathbb{R}} \varrho_n(x-y)dy\\ &= \xi(t,x)\mathcal{P}(x)\varphi_j^\eta(t,x)\sigma_n(x)\end{aligned} \tag{3.10}$$

satisfies $\hat{\zeta}_n \in \mathcal{D}([0,T[\times\Omega \cap \mathcal{B}_j^\eta)$, $0 \leq \hat{\zeta}_n \leq \xi$, for all $n \in \mathbb{N}$. Let

$$Q_1 := \{(s,y) \in Q / v_1(s,y) \in [A_1,A_2]\}, \quad Q_2 := \{(t,x) \in Q / v_2(t,x) \in [A_1,A_2]\}.$$

Then, $\nabla_y g(v_1) = 0$ a.e in Q_1 and $\nabla_x g(v_2) = 0$ a.e in Q_2 . Moreover, $H_0(v_1 - v_2) = H_0(g(v_1) - g(v_2))$ a.e in $(Q \setminus Q_1) \times Q \cup Q \times (Q \setminus Q_2)$.

First inequality: From now on, we denote by $\tilde{a} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ the vector field defined by:

$$\tilde{a}(r,\xi) = a(r,\xi) - a(r,0) \tag{3.11}$$

Let $k_j^\eta := \max_{\{\mathcal{B}_j^\eta \cap \Sigma\}} u_1$. We first prove the following inequality

$$\begin{aligned}0 \leq &\int_Q (b(v_1 \vee k_j^\eta) - b(v_2 \vee k_j^\eta))^+ (\xi\varphi_j^\eta)_t \mathcal{P}\\ &- \int_Q \chi_{\{v_1 \vee k_j^\eta > v_2 \vee k_j^\eta\}} (a(v_1 \vee k_j^\eta, \nabla g(v_1 \vee k_j^\eta))\\ &- a(v_2 \vee k_j^\eta, \nabla g(v_2 \vee k_j^\eta)) \cdot \nabla_x(\xi\varphi_j^\eta \mathcal{P})\\ &+ \int_Q \kappa_1 \chi_{\{v_1 > k_j^\eta\}} (f_1 - \chi_{\{v_2 \geq k_j^\eta\}} f_2)\xi\varphi_j^\eta \mathcal{P}\\ &+ \int_\Omega (b(v_{01} \vee k_j^\eta) - b(v_{02} \vee k_j^\eta))^+ \xi(0,x)\varphi_j^\eta(t,x)\mathcal{P}(t,x)\\ &+ \lim_{n\to\infty} \mathcal{L}_{k_j^\eta}(\xi\varphi_j^\eta \mathcal{P}\sigma_n),\end{aligned} \tag{3.12}$$

where $\kappa_1 \in L^\infty(Q)$, $\kappa_1 \in \operatorname{sign}^+(v_1 - v_2 \vee k_j^\eta)$ and $\mathcal{L}_\alpha$, $\alpha > 0$ is a linear operator which will be defined later (see (3.24)).

As v_1 satisfies (2.4) and (3.6) (with $v = v_1,\ v_0 = v_{01}, f = f_1$), choosing $k = v_2(t,x) \vee k_j^\eta$ and $\xi(s,y) = \zeta_{m,n}(t,x,s,y)$, integrating (2.4) in (t,x) over Q_2 and (3.6), over $Q \setminus Q_2$, using the same arguments as in [3] and [14], we find

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde a(v_1,\nabla_y g(v_1))\cdot\nabla_y g(v_1) H_\delta'(g(v_1)-g(v_2\vee k_j^\eta))\zeta_{m,n}\\
&=\lim_{\delta\to 0}\int_{Q\times\{Q\setminus Q_2\}} \tilde a(v_1,\nabla_y g(v_1))\cdot\nabla_y g(v_1) H_\delta'(g(v_1)-g(v_2\vee k_j^\eta))\zeta_{m,n}\\
&\le \int_{Q\times Q}(b(v_1)-b(v_2\vee k_j^\eta))^+(\zeta_{m,n})_s\\
&\quad+\int_{Q\times Q}\chi_{\{v_1>v_2\vee k_j^\eta\}}(a(v_1\vee k_j^\eta,0)-a(v_2\vee k_j^\eta,0))\cdot\nabla_y\zeta_{m,n}\\
&\quad-\int_{Q\times Q}H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_y\zeta_{m,n}\\
&+\int_Q\chi_{\{v_1>v_2\vee k_j^\eta\}}f_1\zeta_{m,n}.
\end{aligned}\tag{3.13}$$

Now, since : $(t,x) \mapsto \zeta_{m,n}(t,x,s,y)H_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta)) \in \mathcal{D}([0,T) \times \Omega)$ for a.e. $(s,y) \in Q$, we have

$$\int_{Q\times Q}\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_x(H_\delta(g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\zeta_{m,n})\,dx\,dt=0.\tag{3.14}$$

Therefore, going to the limit on δ, we get

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}}\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_x g(v_2\vee k_j^\eta)\\
&\times H_\delta'(g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\zeta_{m,n}\\
&=\int_{Q\times Q}H_0(g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_x\zeta_{m,n}\\
&=\int_{Q\times Q}H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)\tilde a(v_1,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_x\zeta_{m,n}.
\end{aligned}\tag{3.15}$$

Arguing as in [14], inequality (3.13) can be written as follows

$$\begin{aligned}
&\int_{Q\times Q}\{-\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_{x+y}\zeta_{m,n}H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)\\
&+\int_{Q\times Q}\{(a(v_1\vee k_j^\eta,0)-a(v_2\vee k_j^\eta,0))\cdot\nabla_y\zeta_{m,n}\\
&+(b(v_1\vee k_j^\eta)-b(v_2\vee k_j^\eta))(\zeta_{m,n})_s+f_1\zeta_{m,n}\}H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)\\
&\ge\lim_{\delta\to 0}\int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}}\{\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_y g(v_1\vee k_j^\eta)\\
&\quad-\tilde a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))\cdot\nabla_x g(v_2\vee k_j^\eta)\\
&\quad\times H_\delta'(g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\zeta_{m,n}\}
\end{aligned}\tag{3.16}$$

with $\nabla_{x+y}(\cdot) := \nabla_x(\cdot)+\nabla_y(\cdot)$. Now, as v_2 is an entropy solution of $P_{b,g}(v_{02}, u_2, f_2)$, choosing $k = v_1(s,y) \vee k_j^\eta$, $\xi(t,x) = \zeta_{m,n}(t,x,s,y)$ in (2.5) and (3.7) (with $v = v_2$,

$v_0 = v_{02}, f = f_2$), integrating (2.5) in (s, y) over Q_1 and (3.7) over $Q \setminus Q_2$, we find

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\{Q\setminus Q_1\}\times Q} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2))\zeta_{m,n} \\
&= \lim_{\delta\to 0}\int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2))\zeta_{m,n} \\
&\le \int_{Q\times Q} (b(v_1 \vee k_j^\eta) - b(v_2))^+ (\zeta_{m,n})_t \\
&\quad - \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2\}} f_2 \zeta_{m,n} + \int_{\Omega\times Q} (b(v_1 \vee k_j^\eta) - b(v_{02}))^+ \zeta_{m,n}(0, x, s, y) \\
&\quad + \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2\}} (a(v_1 \vee k_j^\eta, 0) - a(v_2, \nabla_x g(v_2))) \cdot \nabla_x \zeta_{m,n}
\end{aligned} \tag{3.17}$$

It is easily verified that

$$\begin{aligned}
&\int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2))\zeta_{m,n} \\
&= \int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta)) \cdot \nabla_x g(v_2 \vee k_j^\eta) \\
&\quad \times H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\zeta_{m,n} \\
&\quad + \int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2 \wedge k_j^\eta, \nabla_x g(v_2 \wedge k_j^\eta)) \cdot \nabla_x g(v_2 \wedge k_j^\eta) \\
&\quad \times H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \wedge k_j^\eta))\zeta_{m,n}
\end{aligned} \tag{3.18}$$

Moreover, the right hand side of (3.17) is equal to

$$\begin{aligned}
&\int_{Q\times Q} (b(v_1 \vee k_j^\eta) - b(v_2 \vee k_j^\eta))^+ (\zeta_{m,n})_t \\
&- \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2 \vee k_j^\eta\}} \chi_{\{v_2 \ge k_j^\eta\}} f_2 \zeta_{m,n} \\
&- \int_{Q\times Q} H_0(v_1 \vee k_j^\eta - v_2 \vee k_j^\eta) \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta)) \cdot \nabla_x \zeta_{m,n} \\
&+ \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2 \vee k_j^\eta\}} (a_g(g(v_1 \vee k_j^\eta), 0) - a_g(g(v_2 \vee k_j^\eta), 0)) \cdot \nabla_x \zeta_{m,n} \\
&+ \int_{Q\times\Omega} (b(v_1 \vee k_j^\eta) - b(v_{02} \vee k_j^\eta))^+ \zeta_{m,n}(0, x, s, y) \\
&+ \int_{Q\times Q} (b(k_j^\eta) - b(v_2))^+ (\zeta_{m,n})_t - \int_{Q\times Q} \chi_{\{k_j^\eta > v_2\}} f_2 \nabla_y g(v_1 \vee k_j^\eta) \cdot \zeta_{m,n} \\
&+ \int_{\Omega} (b(k_j^\eta) - b(v_{02}))^+ \zeta_{m,n}(0, x, s, y), \\
&+ \int_{Q\times Q} \chi_{\{k_j^\eta > v_2\}} (a(k_j^\eta, 0) - a(v_2, \nabla_x g(v_2))) \cdot \nabla_x \zeta_{m,n}.
\end{aligned} \tag{3.19}$$

Since $(s,y) \mapsto \zeta_{m,n}(t,x,s,y)H_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))) \in \mathcal{D}([0,T) \times \Omega)$ for a.e. $(t,x) \in Q$, we have

$$\int_{Q\times Q} \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta)) \cdot \nabla_y (H_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\zeta_{m,n}) = 0. \tag{3.20}$$

Therefore,

$$\begin{aligned}
&-\lim_{\delta\to 0} \int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta))) \cdot \nabla_y g(v_1 \vee k_j^\eta) H'_\delta(g(v_1 \vee k_j^\eta) \\
&- g(v_2 \vee k_j^\eta))\zeta_{m,n} \\
&= \int_{Q\times Q} H_0(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta))\nabla_y \zeta_{m,n}.
\end{aligned}$$

The second term in the right hand side of (3.18) being nonnegative, inequality (3.17) can be equivalently written as

$$\begin{aligned}
&\int_{Q\times Q} (b(v_1 \vee k_j^\eta) - b(v_2 \vee k_j^\eta))^+ (\zeta_{m,n})_t - \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2 \vee k_j^\eta\}} \chi_{\{v_2 \geq k_j^\eta\}} f_2 \zeta_{m,n} \\
&- \int_{Q\times Q} H_0(v_1 \vee k_j^\eta - v_2 \vee k_j^\eta) \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta)) \cdot (\nabla_y \zeta_{m,n} + \nabla_x \zeta_{m,n}) \\
&+ \int_{Q\times Q} (a(v_1 \vee k_j^\eta, 0) - a(v_2 \vee k_j^\eta, 0)) \cdot \nabla_x \zeta_{m,n} H_0(v_1 \vee k_j^\eta - v_2 \vee k_j^\eta) \\
&+ \int_{Q\times Q} (b(v_1 \vee k_j^\eta) - b(v_{02} \vee k_j^\eta))^+ \zeta_{m,n}(0,x,s,y) + \int_{Q\times Q} (b(k_j^\eta) - b(v_2))^+ (\zeta_{m,n})_t \\
&+ \int_{Q\times Q} \chi_{\{k_j^\eta > v_2\}} f_2 \zeta_{m,n} + \int_\Omega (b(k_j^\eta) - b(v_{02}))^+ \zeta_{m,n}(0,x,s,y), \\
&+ \int_{Q\times Q} \chi_{\{k_j^\eta > v_2\}} (a(k_j^\eta, 0) - a(v_2, \nabla_x g(v_2))) \cdot \nabla_x \zeta_{m,n} \\
&\geq \lim_{\delta\to 0} \int_{\{Q\setminus Q_1\}\times\{Q\setminus Q_2\}} \tilde{a}(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta)) \cdot (\nabla_x g(v_2 \vee k_j^\eta) - \nabla_y g(v_1 \vee k_j^\eta)) \\
&\quad \times H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\zeta_{m,n}.
\end{aligned} \tag{3.21}$$

Summing up inequalities (3.16) and (3.21), we get

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} (a(v_1 \vee k_j^\eta, \nabla_y g(v_1 \vee k_j^\eta) - a(v_2 \vee k_j^\eta, \nabla_x g(v_2 \vee k_j^\eta))) \\
&\times (\nabla_y g(v_1 \vee k_j^\eta) - \nabla_x g(v_2 \vee k_j^\eta)) H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\zeta_{m,n} \\
&- \lim_{\delta\to 0} \int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} (a_g(g(v_1 \vee k_j^\eta), 0) - a(g(v_2 \vee k_j^\eta), 0)) \\
&\times (\nabla_y g(v_1 \vee k_j^\eta) - \nabla_x g(v_2 \vee k_j^\eta)) H'_\delta(g(v_1 \vee k_j^\eta) - g(v_2 \vee k_j^\eta))\zeta_{m,n} \\
&\leq \int_{Q\times Q} (b(v_1 \vee k_j^\eta) - b(v_2 \vee k_j^\eta))^+ (\xi \varphi_j^\eta)_t \mathcal{P} \varrho_n \rho_m \\
&\quad + \int_{Q\times Q} \chi_{\{v_1\vee k_j^\eta > v_2\vee k_j^\eta\}} \chi_{\{v_1 > k_j^\eta\}} (f_1 - \chi_{\{v_2 \geq k_j^\eta\}} f_2)\zeta_{m,n} \\
&\quad + \int_Q \int_\Omega (b(v_1 \vee k_j^\eta) - b(v_{02} \vee k_j^\eta))^+ \zeta_{m,n}(0,x,s,y)
\end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&-\int_{Q\times Q}(a(v_1\vee k_j^\eta,\nabla_y g(v_1\vee k_j^\eta))-a(v_2\vee k_j^\eta,\nabla_x g(v_2\vee k_j^\eta)))\\
&\times(\nabla_{x+y}\zeta_{m,n})H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)+\int_{Q\times Q}(b(k_j^\eta)-b(v_2))^+(\zeta_{m,n})_t\\
&-\int_{Q\times Q}\chi_{\{k_j^\eta>v_2\}}f_2\zeta_{m,n}+\int_Q\int_\Omega(b(k_j^\eta)-b(v_{02}))^+\zeta_{m,n}(0,x,s,y)\\
&-\int_{Q\times Q}\chi_{\{k_j^\eta>v_2\}}(a(k_j^\eta,0)-a(v_2,\nabla_x g(v_2))\cdot\nabla_x\zeta_{m,n}.
\end{aligned}
$$

Denote the integrals on the right hand side of (3.22) by $I_1,\dots,I_8$ successively. Using similar estimations as in [3] and [5], going to the limit with m and n respectively, one gets

$$
\begin{gathered}
\lim_{m,n\to\infty}I_1=\int_Q(b(v_1\vee k_j^\eta)-b(v_2\vee k_j^\eta))^+(\xi\mathcal{P}\varphi_j^\eta)_t,\\
\limsup_{m,n\to\infty}I_2\le\int_Q\kappa_1\chi_{\{v_1>k_j^\eta\}}(f_1-\chi_{\{v_2\ge k_j^\eta\}}f_2)\xi(t,x)\mathcal{P}\varphi_j^\eta(t,x)
\end{gathered}
$$

for some

$$
\kappa_1\in L^\infty(Q)\text{ with }\kappa_1\in\overset{+}{\operatorname{sign}}(v_1-v_2\vee k_j^\eta)\quad\text{a.e. in }Q,\tag{3.23}
$$

$$
\begin{aligned}
\limsup_{m,n\to+\infty}I_3&\le\int_\Omega(b(v_{01}\vee k_j^\eta)-b(k_j^\eta\vee v_{02}))^+\xi(0,x)\varphi_j^\eta(t,x)\mathcal{P},\\
\limsup_{m,n\to+\infty}I_4&=\int_Q H_0(v_1\vee k_j^\eta-v_2\vee k_j^\eta)(a(v_1,\nabla g(v_1\vee k_j^\eta))\\
&\quad-a(v_2,\nabla g(v_2\vee k_j^\eta)))\cdot\nabla(\xi\varphi_j^\eta\mathcal{P}).
\end{aligned}
$$

Next, applying Fubini's theorem and taking into account (3.10), we find

$$
\begin{aligned}
I_5+I_6+I_7+I_8&=\int_Q(b(k_j^\eta)-b(v_2))^+(\hat\zeta_n)_t-\int_Q\chi_{\{k_j^\eta>v_2\}}f_2\hat\zeta_n d(t,x)\\
&\quad-\int_Q\chi_{\{k_j^\eta>v_2\}}(a(k_j^\eta,0)-a(v_2,\nabla_x(g(v_2)))\cdot\nabla_x(\hat\zeta_n)\\
&\quad+\int_\Omega(b(k_j^\eta)-b(v_{02}))^+\hat\zeta_n(0,x)dx.
\end{aligned}
$$

Following [14], we define the functional $\mathcal{L}_{k_j^\eta}$ on $\mathcal{D}([0,T[\times\Omega)$ by

$$
\begin{aligned}
\mathcal{L}_{k_j^\eta}(\zeta)&=\int_Q(b(k_j^\eta)-b(v_2))^+\zeta_t-\chi_{\{k_j^\eta>v_2\}}f_2\zeta)\\
&\quad-\int_Q\chi_{\{k_j^\eta>v_2\}}(a(k_j^\eta,0)-a(v_2,\nabla_x g(v_2)))\cdot\nabla_x\zeta\\
&\quad+\int_\Omega(b(k_j^\eta)-b(v_{02}))^+\zeta(0,x)dx.
\end{aligned}\tag{3.24}
$$

As v_2 is an entropy solution, we have $\mathcal{L}_{k_j^\eta}(\zeta)+\int_\Sigma\omega^-(x,k_j^\eta,u_2)\zeta\ge0$ for all $\zeta\in\mathcal{D}([0,T)\times\Omega)$, $\zeta\ge0$, a.e. $\mathcal{L}_{k_j^\eta}$ is the sum of the positive linear functional $\zeta\in\mathcal{D}([0,T)\times\mathbb{R}^N)\mapsto\mathcal{L}_{k_j^\eta}(\zeta)+\int_\Sigma\omega^-(x,k_j^\eta,u_2)\zeta$ and the linear functional: $\zeta\mapsto\int_\Sigma\omega^-(x,k_j^\eta,u_2)\zeta$. Since $(\hat\zeta)_n=(\xi\sigma_n\varphi_j^\eta)_n\subset\mathcal{D}([0,T)\times\Omega)$ is an increasing sequence satisfying $0 \le \xi\sigma_n\varphi_j^\eta\mathcal{P} \le \xi\varphi_j^\eta\mathcal{P}$, $(\mathcal{L}_{k_j^\eta}(\hat\zeta_n) + \int_\Sigma \omega^-(x,k_j^\eta,u_2)\hat\zeta_n)$ is a bounded and increasing sequence and thus converges and $(-\int_\Sigma \omega^-(x,k_j^\eta,u_2)\hat\zeta_n$ converges also. As a consequence, $I_5+I_6+I_7+I_8 = \mathcal{L}_{k_j^\eta}(\xi\mathcal{P}\sigma_n\varphi_j^\eta)$ converges as $n\to\infty$.

To estimate the terms in the left hand side of inequality (3.22), we use our assumption on the diffusion function g: As $\tilde a(r,0)=0$, for small $\delta>0$, we have

$$
\begin{aligned}
&\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} (\tilde a(v_1\vee k_j^\eta, \nabla_y g(v_1\vee k_j^\eta)) - \tilde a(v_2\vee k_j^\eta, \nabla_x g(v_2\vee k_j^\eta)))\\
&\quad\times(\nabla_y g(v_1\vee k_j^\eta) - \nabla_x g(v_2\vee k_j^\eta)) H_\delta'(g(v_1\vee k_j^\eta) - g(v_2\vee k_j^\eta))\zeta_{m,n}\\
&= \frac{1}{\delta}\int_{(Q\times Q)} (\tilde a(v_1\vee k_j^\eta, \nabla_y g(T^2(v_1\vee k_j^\eta))) - \tilde a(v_2\vee k_j^\eta, \nabla_x g(T^2(v_2\vee k_j^\eta))))\\
&\quad\times(\nabla_y g(T^2(v_1\vee k_j^\eta))\\
&\quad - \nabla_x g(T^2(v_2\vee k_j^\eta)))\chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta\in]A_2,+\infty(,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n}\\
&\quad + \frac{1}{\delta}\int_{(Q\times Q)} (\tilde a(v_1\vee k_j^\eta, \nabla_y g(T^2(v_1\vee k_j^\eta))) - \tilde a(T^1(v_2\vee k_j^\eta), \nabla_x g(T^1(v_2\vee k_j^\eta))))\\
&\quad\times(\nabla_y g(T^2(v_1\vee k_j^\eta))\\
&\quad - \nabla_x g(T^1(v_2\vee k_j^\eta)))\chi_{\{v_1\vee k_j^\eta\in]A_2,+\infty(, v_2\vee k_j^\eta\in]-\infty,A_1(,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n}\\
&:= \mathcal{S}_1+\mathcal{S}_2.
\end{aligned}
$$

We estimate $\mathcal{S}_2$ and split this term as follows:

$$
\begin{aligned}
\mathcal{S}_2 &:= \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^2(v_1\vee k_j^\eta), \nabla_y g(T^2(v_1\vee k_j^\eta)))\zeta_{m,n}\\
&\quad\times\nabla_y g(T^2(v_1\vee k_j^\eta))\chi_{\{0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\le\delta\}}\chi_{\{v_2\vee k_j^\eta\in]-\infty,A_1(\}}\zeta_{m,n}\\
&\quad - \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^2(v_1\vee k_j^\eta), \nabla_y g(T^2(v_1\vee k_j^\eta)))\zeta_{m,n}\\
&\quad\times\nabla_x g(T^1(v_2\vee k_j^\eta))\chi_{\{0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\le\delta\}}\zeta_{m,n}\\
&\quad + \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^1(v_2\vee k_j^\eta), \nabla_x g(T^1(v_2\vee k_j^\eta)))\zeta_{m,n}\\
&\quad\times\nabla_x g(T^1(v_2\vee k_j^\eta))\chi_{\{0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\le\delta\}}\chi_{\{v_1\vee k_j^\eta\in]A_2,+\infty(\}}\zeta_{m,n}\\
&\quad - \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^1(v_2\vee k_j^\eta), \nabla_x g(T^1(v_2\vee k_j^\eta)))\zeta_{m,n}\\
&\quad\times\nabla_y g(T^2(v_1\vee k_j^\eta))\chi_{\{0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta))\le\delta\}}\zeta_{m,n}\\
&:= \mathcal{S}_2^1+\mathcal{S}_2^2+\mathcal{S}_2^3+\mathcal{S}_2^4.
\end{aligned}
$$

By the weak coerciveness condition (1.3),

$$
\begin{aligned}
\mathcal{S}_2^1 &\le \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^2(v_1\vee k_j^\eta), \nabla_y(g(T^2(v_1\vee k_j^\eta)))\\
&\quad\times\nabla_y T_\delta(g(T^2(v_1\vee k_j^\eta))^+\zeta_{m,n}\chi_{\{0<-g(v_2)\le\delta\}}\\
&\le \frac{1}{\delta}\int_{(Q\times Q)} \tilde a(T^2(v_1), \nabla_y g(T^2 v_1))\cdot\nabla_y T_\delta(g(T^2(v_1))))^+\zeta_{m,n}\chi_{\{0<-g(v_2)\le\delta\}}
\end{aligned}
$$

$$\le \delta C(\|f_1\|_{L^\infty(Q)}, \|b(v_{01})\|_{L^\infty(\Omega)}, \|\tilde{u}_1\|_{L^\infty(Q)}, \|v_1\|_{L^\infty(Q)}) \int_Q \chi_{\{0<-g(v_2)\le\delta\}}.$$

In the above inequality we use (3.4) and (3.5). Hence, $\lim_{\delta\to 0} \mathcal{S}_2^1 = 0$ and $\mathcal{S}_2^i$, $i = 2, 3, 4$ can be estimated in the same way. Dealing with $\mathcal{S}_1$, we use the additional hypothesis (1.4) on the vector field a to get

$$\begin{aligned}
\mathcal{S}_1 \ge & -\frac{1}{\delta}\int_{(Q\times Q)} \chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty(,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n} \\
& \times B(g(T^2(v_1\vee k_j^\eta)) - g(T^2(v_2\vee k_j^\eta))) \\
& \times (1 + |\nabla_y g(T^2(v_1\vee k_j^\eta))|^p + |\nabla_x g(T^2(v_2\vee k_j^\eta))|^p) \\
& + \frac{1}{\delta}\int_{(Q\times Q)} \Gamma(T^2(v_1\vee k_j^\eta), T^2(v_2\vee k_j^\eta)) \cdot \nabla_y g(T^2(v_1\vee k_j^\eta)) \\
& \times \chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty(,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n} \\
& + \frac{1}{\delta}\int_{(Q\times Q)} \tilde{\Gamma}(T^2(v_1\vee k_j^\eta), T^2(v_2\vee k_j^\eta)) \cdot \nabla_x g(T^2(v_2\vee k_j^\eta)) \\
& \times \chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty[,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n} \\
& - \frac{1}{\delta}\int_{(Q\times Q)} (a(T^2(v_1\vee k_j^\eta),0) - a(T^2(v_2\vee k_j^\eta),0)) \\
& \times \nabla_y g(T^2(v_1\vee k_j^\eta))\chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty[,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n} \\
& \times -\frac{1}{\delta}\int_{(Q\times Q)} (a(T^2(v_1\vee k_j^\eta),0) - a(T^2(v_2\vee k_j^\eta),0)) \\
& \times \nabla_x g(T^2(v_2\vee k_j^\eta))\chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty[,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n} \\
:= & \mathcal{S}_1^1 + \mathcal{S}_1^2 + \mathcal{S}_1^3 + \mathcal{S}_1^4 + \mathcal{S}_1^5.
\end{aligned}$$

Applying the divergence theorem, we get

$$\begin{aligned}
\mathcal{S}_1^4 = \int_{Q\times Q} \Big(\int_0^{\gamma(v_1,v_2)} & (a_g(g(T^2(v_2\vee k_j^\eta)) + \delta r, 0) \\
& - a_g(g(T^2(v_2\vee k_j^\eta)),0))\, dr\Big) \nabla_y \zeta_{m,n}
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
\mathcal{S}_1^5 = -\int_{Q\times Q} \Big(\int_0^{\gamma(v_1,v_2)} & (a_g(g(T^2(v_1\vee k_j^\eta)),0) \\
& - a_g(g(T^2(v_1\vee k_j^\eta)) - \delta r, 0))\, dr\Big) \nabla_x \zeta_{m,n}
\end{aligned} \tag{3.26}$$

where $a_g(r,\xi) = a(g^{-1}(r),\xi)$ and

$$\gamma(v_1,v_2) := \inf\,(g(T^2(v_1\vee k_j^\eta)) - g(T^2(v_2\vee k_j^\eta)))^+/\delta, 1).$$

Due to the continuity of $a_g(r,\xi)$ in $r \in]A_2,+\infty[$, it follows that

$$\lim_{\delta\to 0} \mathcal{S}_1^4 = \lim_{\delta\to 0} \mathcal{S}_1^5 = 0. \tag{3.27}$$

The terms $\mathcal{S}_1^2$ and $\mathcal{S}_1^3$ can be estimated in a similar way, finally as B is locally Lipschitz in $g(\overline{]A_2,+\infty[})$,

$$\lim_{\delta\to 0} \mathcal{S}_1^1 \ge c \lim_{\delta\to 0} \int_{(Q\times Q)} \chi_{\{v_1\vee k_j^\eta, v_2\vee k_j^\eta \in]A_2,+\infty[,0<g(v_1\vee k_j^\eta)-g(v_2\vee k_j^\eta)\le\delta\}}\zeta_{m,n}$$

$$\times B(g(T^2(v_1 \vee k_j^\eta) - g(T^2(v_2 \vee k_j^\eta)))$$
$$\times (1 + |\nabla_y g(T^2(v_1 \vee k_j^\eta))|^p + |\nabla_x g(T^2(v_2 \vee k_j^\eta))|^p) = 0$$

for some constant c depending on $\|v_1\|_{L^\infty(Q)}$, $\|v_2\|_{L^\infty(Q)}$ and $\|u\|_{L^\infty(\Sigma)}$. Combining all the estimates, we get

$$\begin{aligned}
0 \le & \int_Q (b(v_1 \vee k_j^\eta) - b(v_2 \vee k_j^\eta))^+ (\xi \varphi_j^\eta)_t \mathcal{P} \\
& - \int_Q (a(v_1 \vee k_j^\eta, \nabla_x g(v_1 \vee k_j^\eta)) - a(v_2 \vee k_j^\eta, \nabla g(v_2 \vee k_j^\eta))) \\
& \times \nabla_x (\xi \varphi_j^\eta \mathcal{P}) \chi_{\{v_1 \vee k_j^\eta > v_2 \vee k_j^\eta\}} \\
& + \int_Q \kappa_1 \chi_{\{v_1 > k_j^\eta\}} (f_1 - \chi_{\{v_2 \ge k_j^\eta\}} f_2) \xi \varphi_j^\eta \mathcal{P} \\
& + \int_\Omega (b(v_{01} \vee k_j^\eta) - b(v_{02} \vee k_j^\eta))^+ \xi(0,x) \varphi_j^\eta(t,x) \mathcal{P}(t,x) \\
& + \lim_{n\to\infty} \mathcal{L}_{k_j^\eta} (\xi \varphi_j^\eta \mathcal{P} \sigma_n).
\end{aligned} \tag{3.28}$$

Second inequality: We are going to prove the inequality

$$\begin{aligned}
& - \int_0^T \int_{\partial\Omega \cap B} \omega^-((t,x), k_j^\eta, u_2) \xi \varphi_j^\eta \\
& \le \int_Q (b(v_1 \wedge k_j^\eta) - b(v_2))^+ (\xi \varphi_j^\eta)_t \mathcal{P} \\
& - \int_Q \chi_{\{v_1 \wedge k_j^\eta \ge v_2\}} (a(v_1 \wedge k_j^\eta, \nabla_x g(v_1 \wedge k_j^\eta)) - a(v_2, \nabla_x g(v_2))) \cdot \nabla_x (\xi \varphi_j^\eta \mathcal{P}) \\
& + \int_\Omega (b(v_{01} \wedge k_j^\eta) - b(v_{02}))^+ \xi(0,x) \varphi_j^\eta(t,x) \mathcal{P} \\
& + \int_Q \kappa_2 \chi_{\{v_2 < k_j^\eta\}} (\chi_{\{v_1 \le k_j^\eta\}} f_1 - f_2) \xi \varphi_j^\eta \mathcal{P} \\
& + \int_\Omega (b(v_{01} \wedge k_j^\eta) - b(v_{02}))^+ \xi(0,x) \varphi_j^\eta(t,x) \mathcal{P}
\end{aligned} \tag{3.29}$$

for some $\kappa_2 \in \operatorname{sign}^+(v_1 \wedge k_j^\eta - v_2)$. Then, summing up (3.29) and (3.12), we find in the final step the desired comparison result. To this aim, we choose as a test function

$$\zeta_{m,n} := \zeta_{m,n}^{i,j} : (t,x,s,y) \mapsto \mathcal{P}_i(y) \xi(s,y) \varrho_n(y-x) \rho_m(s-t) \varphi_j^\eta(s,y).$$

Then, for m, n sufficiently large,

$$\begin{gathered}
(s,y) \mapsto \zeta_{m,n}^{i,}(t,x,s,y) \in \mathcal{D}(]0,T[\times\Omega), \quad \text{for any } (t,x) \in Q, \\
(t,x) \mapsto \zeta_{m,n}^{i,j}(t,x,s,y) \in \mathcal{D}([0,T[\times\mathbb{R}^N), \quad \text{for any } (s,y) \in Q \\
\operatorname*{supp}_x(\zeta_{m,n}^{i,j}(t,s,y,.)) \subset B_i, \quad \text{for any } (t,s,y) \in [0,T)^2 \times \operatorname{supp}(\mathcal{P}_i).
\end{gathered}$$

As $v_1 = v_1(s,y)$ satisfies (2.4) and (3.6), choosing $k = v_2(t,x) \wedge k_j^\eta$ and $\xi = \zeta_{m,n}(t,x,\cdot,\cdot)$, integrating (2.4) in (t,x) over Q_2 and (3.6) over $Q \backslash Q_2$ (note that, due to the new choice of the test function, this choice is admissible), for a.e. $(t,x) \in Q$, we get

$$\begin{aligned}
&-\lim_{\delta\to 0}\int_Q \tilde a(v_1,\nabla_y g(v_1))\cdot\nabla_y g(v_1)H'_\delta(g(v_1)-g(v_2\wedge k_j^\eta))\zeta_{m,n}\\
&\le \int_Q (b(v_1)-b(v_2\wedge k_j^\eta))^+(\zeta_{m,n})_s\\
&\quad+\int_Q (a(v_1,0)-a(v_2\wedge k_j^\eta,0))\cdot\nabla_y\zeta_{m,n}\chi_{\{v_1>v_2\wedge k_j^\eta\}}\\
&\quad+\int_Q \chi_{\{v_1>v_2\wedge k_j^\eta\}}f_1\zeta_{m,n}+\int_\Omega (b(v_{01})-b(v_2\wedge k_j^\eta))^+\zeta_{m,n}(t,x,0,y)\\
&\quad-\int_Q \chi_{\{v_1>v_2\wedge k_j^\eta\}}\tilde a(v_1,\nabla_y g(v_1)\cdot\nabla_y\zeta_{m,n}\\
&=\int_Q (b(v_1\wedge k_j^\eta)-b(v_2\wedge k_j^\eta))^+(\zeta_{m,n})_s\\
&\quad-\int_Q (a(v_1\wedge k_j^\eta,0)-a(v_2\wedge k_j^\eta,0))\cdot\nabla_y\zeta_{m,n}\chi_{\{v_1\wedge k_j^\eta>v_2\wedge k_j^\eta\}}\\
&\quad+\int_Q \chi_{\{v_1\wedge k_j^\eta>v_2\wedge k_j^\eta\}}\chi_{\{v_1\le k_j^\eta\}}f_1\zeta_{m,n}\\
&\quad+\int_\Omega (b(v_{01}\wedge k_j^\eta)-b(v_2\wedge k_j^\eta))^+\zeta_{m,n}(t,x,0,y)\\
&\quad+\int_Q (b(v_1)-b(k_j^\eta))^+(\zeta_{m,n})_s\\
&\quad-\int_Q \chi_{\{v_1>k_j^\eta\}}\{(a(v_1,\nabla_y g(v_1))-a(k_j^\eta,0))\cdot\nabla_y\zeta_{m,n}+f_1\zeta_{m,n}\}\\
&\quad+\int_\Omega (b(v_{01})-b(k_j^\eta))^+\zeta_{m,n}(t,x,0,y),
\end{aligned}$$

where for the last equality we have used the fact that $(r-s\wedge k)^+=(r\wedge k-s\wedge k)^++(r-k)^+$, $\chi_{\{r>s\wedge k\}}=\chi_{\{r\wedge k>s\wedge k\}}\chi_{\{r\le k\}}+\chi_{\{r>k\}}$, for all $r,s,k\in\mathbb{R}$. Let η be sufficiently small so that for all $(s,y),(t,x)\in\Sigma$ with $d((s,y),(t,x))\le\eta$, $u_1(s,y)-u_1(t,x)|\le\frac{\varepsilon}{2}$.

As $v_2=v_2(t,x)$ is an entropy solution, choosing $k=v_1(s,y)\wedge k_j^\eta$, $\xi=\zeta_{m,n}$ (this choice is admissible because $g(k_j^\eta)=0$), integrating (2.5) over Q_1 and (3.7) over $Q\setminus Q_1$, we obtain

$$\begin{aligned}
&-\lim_{\delta\to 0}\int_{(Q\setminus Q_2)\times Q}(a(v_2,\nabla g(v_2))-a(v_2,0))\cdot\nabla g(v_2)H'_\delta(g(v_1\wedge k_j^\eta)-g(v_2)\\
&-\int_\Sigma \omega^-((t,x),v_1(s,y)\wedge k_j^\eta,u_2)\zeta_{m,n}\\
&\le\int_Q (b(v_1\wedge k_j^\eta)-b(v_2))^+(\zeta_{m,n})_t\\
&\quad+\int_Q \chi_{\{v_1\wedge k_j^\eta>v_2\}}\{(a(v_1\wedge k_j^\eta,0)-a(v_2,\nabla_x g(v_2))\cdot\nabla_x\zeta_{m,n}-\int_Q f_2\zeta_{m,n}\}
\end{aligned}$$

for a.e. (s,y) in Q. The integral on the left is $\ge-\int_\Sigma\omega^-((t,x),k_j^\eta,u_2)\zeta_{m,n}$. Moreover, obviously, $(r\wedge k-s)^+=(r\wedge k-s\wedge k)^+$ for all $r,s,k\in\mathbb{R}$. Therefore, integrating the preceding inequalities in (t,x) resp. (s,y) over Q, summing up, using the same type of arguments as above, passing to the limit with $m,n\to\infty$ successively, for some $\kappa_2\in L^\infty(Q)$ with $\kappa_2\in\operatorname{sign}^+(v_1\wedge k_j^\eta-v_2)$, we obtain

$$
\begin{aligned}
&-\int_\Sigma \omega^-((t,x),k_j^\eta,u_2)\xi\mathcal{P}_i\\
&\le \int_Q (b(v_1\wedge k_j^\eta)-b(v_2\wedge k_j^\eta))^+\xi_t\mathcal{P}_i\varphi_j^\eta\\
&\quad -\int_Q \chi_{\{v_1\wedge k_j^\eta\ge v_2\wedge k_j^\eta\}}(a(v_1\wedge k_j^\eta,\nabla g(v_1\wedge k_j^\eta))\\
&\quad -a(v_2\wedge k_j^\eta,\nabla g(v_2\wedge k_j^\eta)))\cdot\nabla_x(\xi\mathcal{P}_i\varphi_j^\eta)\\
&\quad +\int_Q \kappa_2\chi_{\{v_2<k_j^\eta\}}(\chi_{\{v_1\le k_j^\eta\}}f_1-f_2)\xi\mathcal{P}_i\varphi_j^\eta\\
&\quad +\int_\Omega (b(v_{01}\wedge k_j^\eta)-b(v_{02}\wedge k_j^\eta))^+\xi(0,x)\mathcal{P}_i(x)\varphi_j^\eta+\lim_{n\to\infty}\tilde{\mathcal{L}}(\xi\sigma_n\mathcal{P}_i\varphi_j^\eta),
\end{aligned}
\tag{3.30}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{L}}_k:=\tilde{\mathcal{L}}_k(v_1):\zeta\in\mathcal{D}([0,T[\times\mathbb{R}^N)\mapsto &\int_Q (b(v_1)-b(k))^+\zeta_s\\
&+\int_Q \chi_{\{v_1>k\}}\{(a(v_1,\nabla g(v_1))-a(k,0))\cdot\nabla_y\zeta+f_1\zeta\}\\
&+\int_\Omega (b(v_{01})-b(k))^+\zeta(0,y).
\end{aligned}
$$

Using the same arguments as above, we can prove that $\tilde{\mathcal{L}}_k(\xi\sigma_n\mathcal{P}_i\varphi_j^\eta))$ converges (as $\mathcal{L}_k(\xi\sigma_n\mathcal{P}_i\varphi_j^\eta))$ with n. Note also that $(r\vee k-s\vee k)^+ + (r\wedge k-s\wedge k)^+=(r-s)^+$, for all $r,s,k\in\mathbb{R}$. Moreover, if we define $\kappa:=\kappa_1\chi_{\{v_1>k_j^\eta\}}+\kappa_2\chi_{\{v_2<k_j^\eta\}}\chi_{\{v_1\le k_j^\eta\}}$, then $\kappa=\kappa_1\chi_{\{v_2\ge k_j^\eta\}}\chi_{\{v_1>k_j^\eta\}}+\kappa_2\chi_{\{v_2<k_j^\eta\}}\in\operatorname{sign}^+(v_1-v_2)$. Therefore, summation of (3.28) and (3.30) yields

$$
\begin{aligned}
&-\int_\Sigma \omega^-((t,x),k_j^\eta,u_2)\xi\mathcal{P}_i\varphi_j^\eta\\
&\le \int_Q (b(v_1)-b(v_2))^+\xi_t\mathcal{P}_i\varphi_j^\eta\\
&\quad -\int_Q \chi_{\{v_1\ge v_2\}}(a(v_1,\nabla g(v_1))-a(v_2,\nabla g(v_2)))\cdot\nabla_x(\xi\mathcal{P}_i\varphi_j^\eta)\\
&\quad +\int_Q \kappa(f_1-f_2)\xi\mathcal{P}_i\varphi_j^\eta+\int_\Omega (b(v_{01})-b(v_{02}))^+\xi(0,x)\mathcal{P}_i\varphi_j^\eta(x)\\
&\quad +\lim_{n\to\infty}\mathcal{L}(\xi\mathcal{P}_i\varphi_j^\eta\sigma_n)+\lim_{n\to\infty}\tilde{\mathcal{L}}(\xi\mathcal{P}_i\varphi_j^\eta\sigma_n),
\end{aligned}
\tag{3.31}
$$

for any $\xi\in\mathcal{D}([0,T[\times\mathbb{R}^N)$, $\xi\ge 0$, for all $i\in\{1,\dots,p\}$, $j\in\{1,\dots,p_\eta\}$.

Remark 3.5. For $\xi\in\mathcal{D}([0,T)\times\Omega)$, the method of doubling variables allows to prove the following local comparison result:

There exists $\kappa \in L^\infty(Q)$ with $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ a.e. in Q such that, for any $\zeta \in \mathcal{D}([0,T[\times\Omega)$, $\zeta \geq 0$,

$$\begin{aligned}
0 \leq & \int_Q (b(v_1) - b(v_2))^+ \zeta + \int_Q \kappa(f_1 - f_2)\zeta \\
& - \int_Q \chi_{\{v_1 > v_2\}}(a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2)) \cdot \nabla\zeta \\
& + \int_\Omega (b(v_{01}) - b(v_{02}))^+ \zeta(0, \cdot).
\end{aligned} \tag{3.32}$$

The proof in this case is easier as the global comparison result. Indeed, as $\xi = 0$ on Σ, we can choose $k = v_2(t,x)$ (resp $k = v_1(s,x)$) in (2.4) (resp in 2.5) and we have only to add the obtained inequalities, then to go to the limit on m, n in order to get (3.32).

As $\xi = \xi(1-\sigma_m) + \xi\sigma_m$ and $\xi\sigma_m \in \mathcal{D}([0,T[\times\Omega)$ for m sufficiently large, applying the local comparison principle (3.32) with $\zeta = \xi\sigma_m$, the global estimate (3.31) with $\xi(1-\sigma_m)$, we obtain

$$\begin{aligned}
& \int_Q (b(v_1) - b(v_2))^+ \xi_t \mathcal{P}_i \varphi_j^\eta - \chi_{\{v_1 \geq v_2\}}(a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x(\xi \mathcal{P}_i \varphi_j^\eta) \\
& + \int_Q \kappa(f_1 - f_2)\xi \mathcal{P}_i \varphi_j^\eta + \int_\Omega (b(v_{01}) - b(v_{02}))^+ \xi(0,x) \mathcal{P}_i \varphi_j^\eta(x) \\
& \geq \int_Q (b(v_1) - b(v_2))^+ (\xi(1-\sigma_m)\varphi_j^\eta)_t \mathcal{P}_i \\
& \quad - \int_Q \chi_{\{v_1 \geq v_2\}}(a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x(\xi(1-\sigma_m)\mathcal{P}_i \varphi_j^\eta) \\
& \quad + \int_Q \kappa(f_1 - f_2)\xi(1-\sigma_m)\mathcal{P}_i \varphi_j^\eta + \int_\Omega (b(v_{01}) - b(v_{02}))^+ \xi(0,x)(1-\sigma_m)\mathcal{P}_i \varphi_j^\eta(x) \\
& \geq - \int_\Sigma \omega^-((t,x), k_j^\eta, u_2)\xi \mathcal{P}_i \varphi_j^\eta (1-\sigma_m) - \lim_{n\to\infty} \mathcal{L}(\xi \mathcal{P}_i \varphi_j^\eta (1-\sigma_m)\sigma_n) \\
& \quad - \lim_{n\to\infty} \tilde{\mathcal{L}}(\xi \mathcal{P}_i \varphi_j^\eta (1-\sigma_m)\sigma_n) \\
& = - \int_\Sigma \omega^-((t,x), k_j^\eta, u_2)\xi \mathcal{P}_i \varphi_j^\eta - \lim_{n\to\infty} \mathcal{L}(\xi \mathcal{P}_i \varphi_j^\eta (\sigma_n - \sigma_m\sigma_n)) \\
& \quad - \lim_{n\to\infty} \tilde{\mathcal{L}}(\xi \mathcal{P}_i \varphi_j^\eta (\sigma_n - \sigma_m\sigma_n)).
\end{aligned}$$

Note that $\mathcal{P}_i \varphi_j^\eta \sigma_n \sigma_m = \mathcal{P}_i \varphi_j^\eta \sigma_m$ for n sufficiently large. Therefore,

$$\lim_{m\to\infty}\lim_{n\to\infty} \mathcal{L}(\xi \mathcal{P}_i \varphi_j^\eta (\sigma_n - \sigma_m\sigma_n)) = \lim_{m\to\infty}\lim_{n\to\infty} \tilde{\mathcal{L}}(\xi \mathcal{P}_i \varphi_j^\eta (\sigma_n - \sigma_m\sigma_n)) = 0,$$

and thus, passing to the limit with $m \to \infty$ in the preceding inequality yields

$$\begin{aligned}
& \int_Q (b(v_1) - b(v_2))^+ (\xi\varphi_j^\eta)_t \mathcal{P}_i - \chi_{\{v_1 \geq v_2\}}(a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x(\xi \mathcal{P}_i \varphi_j^\eta) \\
& \quad + \int_Q \kappa(f_1 - f_2)\xi \mathcal{P}_i \varphi_j^\eta + \int_\Omega (b(v_{01}) - b(v_{02}))^+ \xi(0,x) \mathcal{P}_i \varphi_j^\eta(x) \\
& \geq - \int_\Sigma \omega^-((t,x), k_j^\eta, u_2)\xi \mathcal{P}_i \varphi_j^\eta
\end{aligned}$$

for all $j = 1, \dots, p_\eta$. Summing over $j = 0, \dots, p_\eta$, we find

$$\begin{aligned}
&\int_Q (b(v_1) - b(v_2))^+ \xi_t \mathcal{P}_i - \chi_{\{v_1 \geq v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x (\xi \mathcal{P}_i) \\
&\quad + \int_Q \kappa(f_1 - f_2) \xi \mathcal{P}_i + \int_\Omega (b(v_{01}) - b(v_{02}))^+ \xi(0, x) \mathcal{P}_i \\
&\geq - \sum_{j=1}^{p_\eta} \int_\Sigma \omega^-((t, x), k_j^\eta, u_2) \xi \mathcal{P}_i \\
&\geq - \sum_{j=1}^{p_\eta} \int_\Sigma \omega^-((t, x), u_1 + \frac{\varepsilon}{2}, u_2) \xi \mathcal{P}_i \\
&= - \sum_{j=1}^{p_\eta} \int_{\Sigma \cap \partial B_i} \omega^-((t, x), u_1 + \frac{\varepsilon}{2}, u_2) \xi
\end{aligned}$$

By continuity of ω , letting $\varepsilon \to 0$, and after summation over i, we get (3.1).

Remark 3.6. **(i)** In the proof of Theorem 3.1, we have only used the fact that v_1 verifies the following "local entropy inequalities": For all $\xi \in \mathcal{D}^+([0, T) \times \mathbb{R}^N)$ and all $k \in \mathbb{R}$ with $k > \max_{\Sigma \cap \operatorname{supp}(\xi)} u_1$,

$$\begin{aligned}
0 \leq &\int_Q (b(v_1) - b(k))^+ \xi_t + \int_\Omega (b(v_{01}) - b(k))^+ \xi(0, x) \\
&+ \int_Q \chi_{\{v_1 > k\}} (f_1 \xi - (a(v_1, \nabla g(v_1)) - a(k, 0)) \cdot \nabla \xi
\end{aligned} \tag{3.33}$$

and for all $k \in \mathbb{R}$ with $k < \min_{\Sigma \cap \operatorname{supp}(\xi)} u_1$,

$$\begin{aligned}
0 \leq &\int_Q (b(k) - b(v_1))^+ \xi_t + \int_\Omega (b(k) - b(v_{01}))^+ \xi(0, x) \\
&- \int_Q \chi_{\{k > v_1\}} (f_1 \xi + (a(v_1, \nabla g(v_1)) - a(k, 0)) \cdot \nabla \xi.
\end{aligned} \tag{3.34}$$

4. Existence of an entropy solution

Let $\tilde{\Omega}$ denote some Lipschitz domain strictly larger than Ω, $\tilde{Q} = (0, T) \times \tilde{\Omega}$. We define the trivial extension by 0 of the data u_0, f on the larger domain

$$\tilde{u}_0 := \begin{cases} u_0 & \text{on } \Omega \\ 0 & \text{on } \tilde{\Omega} \backslash \Omega, \end{cases} \quad \tilde{f} := \begin{cases} f & \text{on } Q \\ 0 & \text{on } \tilde{Q} \backslash Q. \end{cases}$$

Let $p, q \in \mathbb{N}$ (the penalization parameters) and define the penalization term $\beta_{p,q}(t, x, r) := \chi_{\tilde{Q} \backslash Q} (p(r - \tilde{u}(t, x))^+ - q(\tilde{u}(t, x) - r)^+$, $\forall r \in \mathbb{R}$, a.e. $(t, x) \in \tilde{Q}$ where $\tilde{u}$ is some extension of u on $\tilde{Q} \setminus Q$ with $g(\tilde{u}) \in L^p(0, T, W^{1,p}(\Omega))$. Multiplying $\tilde{u}$ if necessary by a smooth function χ such that $\chi \equiv 0$ on the boundary $\tilde{\Sigma}$ and $\chi \equiv 1$ on Σ we can assume that $\tilde{u} = 0$ on $\tilde{\Sigma} := (0, T) \times \partial \tilde{\Omega}$. Note that $\beta_{p,q}$ is Lipschitz continuous in r, uniformly in (t, x). Moreover, for all $p, p', q, q' \in \mathbb{N}$ with $q \leq q'$, $p \leq p'$,

$$\beta_{p,q}(t, x, r) - \beta_{p,q'}(t, x, r) \geq 0, \quad \beta_{p,q}(t, x, r) - \beta_{p',q}(t, x, r) \leq 0,$$

for all $r \in \mathbb{R}$, a.e. $(t,x) \in \tilde{Q}$, and

$$\lim_{m,n\to\infty} \beta_{p,q}(t,x,r) = \begin{cases} 0 & \text{if } r \in \mathbb{R}, (t,x) \in Q \\ \mathbb{R} & \text{if } r = \tilde{u}(t,x), (t,x) \in \tilde{Q}\backslash Q \\ \emptyset & \text{otherwise.} \end{cases}$$

The proof of the existence result consists of three steps: In the first step, we prove existence of a bounded entropy solution of the doubly penalized problem with homogeneous boundary conditions and L^∞-data v_0 and f: (Problem $P^{p,q}_{b_\alpha,g}(v_0,0,f,\psi)$)

$$\begin{gathered} b_\alpha(v)_t - \operatorname{div} a(v,\nabla g(v)) + \psi(v) + \beta_{p,q}(v) = \tilde{f} \quad \text{on } \tilde{Q} \\ \text{“}v = 0'' \quad \text{on some part of } \tilde{\Sigma}, \\ b_\alpha(v(0,.)) = b(\tilde{v}_0) \quad \text{in } \tilde{\Omega}, \end{gathered}$$

where $b_\alpha(r) = b_\alpha(r)+\alpha r$, $\psi : \mathbb{R} \to \mathbb{R}$ is increasing, Lipschitz continuous with $\psi(0) = 0$ and $\psi(\mathbb{R}) = \mathbb{R}$. This is done via approximation with the non-degenerate evolution problems with homogeneous boundary conditions: (problem $P^{p,q}_{b_\alpha,g_\varepsilon}(v_0,0,f,\psi)$)

$$\begin{gathered} b_\alpha(v)_t - \operatorname{div} a(v,\nabla g_\varepsilon(v)) + \psi(v) + \beta_{p,q}(v) = \tilde{f} \quad \text{on } \tilde{Q} \\ v = 0 \quad \text{on } \tilde{\Sigma}, \\ b_\alpha(v)(0,.) = b_\alpha(\tilde{v_0}) \quad \text{in } \tilde{\Omega}, \end{gathered}$$

where $g_\varepsilon(r) = g(r) + \varepsilon r$. In the second step, we let $p, q \to \infty$ and prove the existence of an entropy solution of the degenerate problem $P_{b_\alpha,g}(v_0,u,f,\psi)$. Then in a third step, we let $\alpha \to 0$. Finally, in the last step we pass to the limit with the perturbation term ψ to 0 and prove the existence result for L^1-data. We only give the details of the proof for the first and the second steps which are crucial. In the last steps, we use similar arguments as in [6] and [3].

4.1. **First step.** In this step, we assume that $u : \Sigma \to [A_1, A_2]$ is continuous with on Σ, $\tilde{u} \in C(\tilde{Q})$, $\tilde{u} = u$ on Σ and $b_\alpha(\tilde{u})_t \in L^1(\tilde{Q})$. The existence of a unique weak solution v of $P^{0,0}_{b_\alpha,g_\varepsilon}(v_0,0,f,\psi)$ is rather a classical result. Indeed the problem can be equivalently formulated as follows: (Problem (EP)(v_0,f,ψ,ε))

$$\begin{gathered} ((b_\alpha \circ g_\varepsilon^{-1})(v))_t - \operatorname{div} a(g_\varepsilon^{-1}(v),\nabla v) + \psi(g_\varepsilon^{-1}(v)) = \tilde{f} \quad \text{in } \tilde{Q} \\ v = 0 \quad \text{on } \Sigma \\ b_\alpha(v(0,.)) = b_\alpha(\tilde{v}_0) \quad \text{in } \tilde{\Omega} \end{gathered}$$

As $(r,\xi) \mapsto a(g_\varepsilon^{-1}(v),\xi)$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ satisfies the same hypothesis as the vector field a thanks to the strict monotonicity of g_ε, the classical theory of Leray-Lions applies and one can prove as in [14] that the weak solution satisfies the entropy inequalities:

For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(-g_\varepsilon(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq &\int_{\tilde{Q}} \{(b_\alpha(v_\varepsilon) - b_\alpha(k))^+\xi_t + \int_\Omega (b_\alpha(\tilde{v}_0) - b_\alpha(k))^+\xi(0,\cdot) \\ &+ \int_{\tilde{Q}} \chi_{\{v_\varepsilon>k\}}((\tilde{f} - \psi(v_\varepsilon)) - (a(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon)) - a(k,0)) \cdot \nabla\xi \end{aligned} \tag{4.1}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(g_\varepsilon(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq & \int_{\tilde{Q}} \{(b_\alpha(k) - b_\alpha(v_\varepsilon))^+ \xi_t + \int_{\tilde{\Omega}} (b_\alpha(k) - b_\alpha(\tilde{v}_0))^+ \xi(0,\cdot) \\ & - \int_{\tilde{Q}} \chi_{\{k > v_\varepsilon\}} ((\tilde{f} - \psi(v_\varepsilon))\xi - (a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - a(k,0)) \cdot \nabla \xi). \end{aligned} \tag{4.2}$$

Now, we want to go the limit with $\varepsilon \to 0$. Let $\tilde{f} \in L^\infty(\tilde{Q})$, $\tilde{v}_0 \in L^\infty(\tilde{\Omega})$ and $v \in L^\infty(\tilde{Q})$ be the unique entropy solution of $P^{0,0}_{b_\alpha, g_\varepsilon}(\tilde{v}_0, 0, \tilde{f}, \psi)$. Due to the Lipschitz continuity of $\beta_{p,q}$, using Banach's fixed point theorem, applying the comparison result Theorem 3.1, there exists a unique entropy solution v_ε of the penalized problem (problem $\tilde{P}^{p,q}_{b_\alpha, g_\varepsilon}(\tilde{v}_0, 0, \tilde{f}, \psi)$)

$$\begin{gathered} \frac{\partial b_\alpha(v)}{\partial t} - \operatorname{div} a(v, \nabla g_\varepsilon(v)) + \psi(v) = \phi \quad \text{on } \tilde{Q} \\ v = 0 \quad \text{on } \tilde{\Sigma} \\ v(0,\cdot) = \tilde{v_0} \quad \text{on } \tilde{\Omega} \end{gathered}$$

with right hand side $\phi = \tilde{f} - \beta_{p,q}(v_\varepsilon)$ a.e. $v_\varepsilon \in L^\infty(\tilde{Q})$, $g_\varepsilon(v_\varepsilon) \in L^p(0,T,W_0^{1,p}(\tilde{\Omega}))$ and v_ε satisfies the following entropy inequalities:

For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(-g_\varepsilon(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq & \int_{\tilde{Q}} \{(b_\alpha(v_\varepsilon) - b_\alpha(k))^+ \xi_t + \int_{\Omega} (b_\alpha(\tilde{v}_0) - b_\alpha(k))^+ \xi(0,\cdot) \\ & + \int_{\tilde{Q}} \chi_{\{v_\varepsilon > k\}} ((\tilde{f} - \psi(v_\varepsilon) - \beta_{p,q}(v_\varepsilon)) - (a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - a(k,0)) \cdot \nabla \xi \end{aligned} \tag{4.3}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(g_\varepsilon(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq & \int_{\tilde{Q}} \{(b_\alpha(k) - b_\alpha(v_\varepsilon))^+ \xi_t + \int_{\tilde{\Omega}} (b_\alpha(k) - b_\alpha(\tilde{v}_0))^+ \xi(0,\cdot) \\ & - \int_{\tilde{Q}} \chi_{\{k > v_\varepsilon\}} ((\tilde{f} - \psi(v_\varepsilon) - \beta_{p,q}(v_\varepsilon))\xi - (a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) - a(k,0)) \cdot \nabla \xi). \end{aligned} \tag{4.4}$$

By a particular choice of test functions, one can prove (see [6]) that $(v_\varepsilon)_\varepsilon$ and $(|\nabla g_\varepsilon(v_\varepsilon)|)_\varepsilon$ are uniformly bounded in $L^\infty(\tilde{Q})$ and $L^p(\tilde{Q})$ respectively. Thanks to the growth condition (1.2) on a, it follows that $(a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)))_\varepsilon$ is bounded in $L^{p'}(\tilde{Q})^N$ as well. Following classical arguments, extracting a subsequence if necessary, we can prove that as $\varepsilon \to 0$,

$$g_\varepsilon(v_\varepsilon) \text{ converges weakly to some } w \in L^\infty(\tilde{Q}) \cap L^p(0,T,W_0^{1,p}(\tilde{\Omega})) \tag{4.5}$$

in $L^p(0,T,W_0^{1,p}(\tilde{\Omega}))$ and

$$a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \text{ converges weakly in } L^{p'}(\tilde{Q})^N \text{ to some } \chi \in L^{p'}(\tilde{Q})^N. \tag{4.6}$$

Now, we use the L^∞ uniform bound on (v_ε) in order to deduce the weak-$*$ convergence of (v_ε) to a function $\overline{v}$. Then, going to the limit in the approximate entropy inequalities, we prove that $\overline{v}$ is an entropy process solution of $P^{p,q}_{b,g}(v_0, 0, f, \psi)$ (see

Definition 4.3 below). Finally using a "stronger" principle of uniqueness, we show that $\overline{v}$ is the entropy solution of $P^{p,q}_{b,g}(v_0, 0, f, \psi)$ and that the convergence is strong.

Definition 4.1. Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$, (u_n) be a bounded sequence of $L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0,1))$. The sequence (u_n) converges towards u in the "nonlinear weak-$*$ sense" if

$$\int_\Omega \phi(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega \phi(u(x,\mu))\psi(x)\,dx\,d\mu, \quad \text{as } n \to \infty, \tag{4.7}$$

for all $\psi \in L^1(\Omega)$, for all $\phi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

Lemma 4.2. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 1)$ and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-$*$ sense.*

For a proof of the above lemma, see [21] or [18]). According to Lemma 4.2, the sequence (v_ε) is convergent in the nonlinear weak-$*$ sense to some $v \in L^\infty(Q \times (0,1))$. We will prove that the weak-$*$ limit is a weak entropy process solution of $P^{p,q}_{b,g}(\tilde{v}_0, 0, \tilde{f}, \psi)$ in the following sense.

Definition 4.3. Let $v_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$. A function $u \in L^\infty(\tilde{Q} \times (0,1))$ is a weak entropy process solution of $P^{p,q}_{b,g}(v_0, f, \psi)$ if

• $g(u) \in L^p(0,T,W^{1,p}_0(\Omega))$, $g(u(t,x,\mu)) = g(u(t,x))$ a.e. in $\tilde{Q} \times (0,1)$;

bullet for all $k \in \mathbb{R}$, for all $\xi \in C^\infty_0([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(-g(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq \int_0^1 (\int_{\tilde{Q}} \{(b(u) - b(k))^+ \xi_t)\,d\mu + \int_\Omega (b(v_0) - b(k))^+ \xi(0,\cdot) \\ + \int_0^1 (\int_{\tilde{Q}} \chi_{\{u>k\}}((f - \psi(u) - \beta_{p,q}(u))\xi - (a(u, \nabla g(u)) - a(k,0)) \cdot \nabla \xi)\,d\mu \end{aligned} \tag{4.8}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C^\infty_0([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(g(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq \int_0^1 (\int_{\tilde{Q}} \{(b(k) - b(u))^+ \xi_t)\,d\mu + \int_{\tilde{\Omega}} (b(k) - b(v_0))^+ \xi(0,\cdot) \\ - \int_0^1 (\int_{\tilde{Q}} \chi_{\{k>u\}}((f - \psi(u) - \beta_{p,q}(u))\xi - (a(u, \nabla g(u)) - a(k,0)) \cdot \nabla \xi)\,d\mu). \end{aligned} \tag{4.9}$$

Obviously, the sequence $a(v_\varepsilon, 0)_\varepsilon$ also converges in the weak$_*$ sense, to $a(v,0)$. To prove the strong compactness of $g(v_\varepsilon)$ in $L^1(Q)$, we estimate $\int_0^{T-h}\int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)|$ for all h small enough: Let $L_{q\circ b_\alpha^{-1}} > 0$ be a Lipschitz constant of $g \circ b_\alpha^{-1}$ on $[-L, L]$ with $L \geq \|v_\varepsilon\|_{L^\infty(Q)}$ for all $\varepsilon > 0$. Then, for all $r, s \in [-L, L]$

$$|g(r) - g(s)|^2 \leq \frac{1}{L_{g\circ b_\alpha^{-1}}} |b_\alpha(r) - b_\alpha(s)||g(r) - g(s)|.$$

It follows that

$$\int_0^{T-h} \int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)|$$

$$\begin{aligned}
&\le |\tilde{Q}|^{1/2}\Big(\int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)|^2\Big)^{1/2} \\
&\le \frac{1}{L_{g\circ b_\alpha^{-1}}} |\tilde{Q}|^{1/2}\Big(\int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)||b_\alpha(v_\varepsilon)(t+h,x) \\
&\quad - b_\alpha(v_\varepsilon)(t,x)|\Big)^{1/2}.
\end{aligned}$$

Now, as v_ε is a weak solution of $P^{p,q}_{b_\alpha,g}(v_0,f,\psi)$, taking $g(v_\varepsilon)(t+h,.) - g(v_\varepsilon)(t,.) \in W^{1,p}_0(\tilde{\Omega})$ as test function, we get

$$\begin{aligned}
&\int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)||b_\alpha(v_\varepsilon)(t+h,x) - b_\alpha(v_\varepsilon)(t,x)| \\
&\le \int_0^{T-h}\int_t^{t+h}\int_{\tilde{\Omega}} a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))\cdot\nabla(g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)) \\
&\quad + \int_0^{T-h}\int_t^{t+h}\int_{\tilde{\Omega}} (\tilde{f} - \psi(v_\varepsilon) - \beta_{p,q}(v_\varepsilon))(g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)) \\
&\le hM
\end{aligned}$$

for some $M > 0$ independent of ε. It follows that

$$\int_0^{T-h}\int_{\tilde{\Omega}} |g(v_\varepsilon)(t+h,x) - g(v_\varepsilon)(t,x)| \to 0 \text{ when } h \to 0, \tag{4.10}$$

uniformly with respect to ε.

Taking into account (4.5) and (4.10), as $v_\varepsilon \to v$ weak$-*$, it follows that up to a sequence

$$g_\varepsilon(v_\varepsilon) \text{ converges strongly and a.e. to } g(v) \in L^1(\tilde{Q}) \tag{4.11}$$

$$(v_\varepsilon\chi_{\{g(v)\neq 0\}}) \text{ converges strongly in } L^1(Q) \text{ and a.e. to } v\chi_{\{g(v)\neq 0\}}, \tag{4.12}$$

$$g_\varepsilon(v_\varepsilon) \text{ converges weakly to } g(v) \in L^\infty(\tilde{Q}) \cap L^p(0,T,W^{1,p}_0(\tilde{\Omega})) \tag{4.13}$$

in $L^p(0,T,W^{1,p}_0(\tilde{\Omega}))$. In particular, it follows that $g(v)$ is independent of μ. Now, in view of the above estimates, it remains only to identify χ with the weak limit of $a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))$ in $L^p(\tilde{Q})^N$. Let us note first that

$$\begin{aligned}
&\int_{\{v\in[A_1,A_2]\}} |\tilde{a}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))| \\
&= \int_{\{v\in[A_1,A_2], v_\varepsilon\in[A_1,A_2]\}} |\tilde{a}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))| \\
&\quad + \int_{\{v\in[A_1,A_2]\}} |\tilde{a}(v_\varepsilon, \nabla T_\delta(g_\varepsilon(A_1) - g(v_\varepsilon))^+))| + \int_{\{v\in[A_1,A_2]\}} |\tilde{a}(v_\varepsilon, \nabla T_\delta(g(v_\varepsilon))| \\
&\quad + \int_{\{v\in[A_1,A_2], g_\varepsilon(v_\varepsilon)<-\delta\}} |\tilde{a}(v_\varepsilon, \nabla T_\delta g_\varepsilon(v_\varepsilon)| + \int_{\{v\in[A_1,A_2], g_\varepsilon(v_\varepsilon)>\delta\}} |\tilde{a}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)|.
\end{aligned}$$

The first term in the right hand side can be estimated as follows:

$$\begin{aligned}
&\int_{\{v\in[A_1,A_2], v_\varepsilon\in[A_1,A_2]\}} |\tilde{a}(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon))| \\
&\le \int_{\{\varepsilon A_1 \le g_\varepsilon(v_\varepsilon) \le \varepsilon A_2\}} |\tilde{a}(v_\varepsilon, \nabla(T_{\varepsilon A_2} g(v_\varepsilon) - T_{\varepsilon A_1} g(v_\varepsilon))|
\end{aligned}$$

$$\le C(|A_1| + |A_2|) \int_{\{\varepsilon A_1 \le g_\varepsilon(v_\varepsilon) \le \varepsilon A_2\}} |\nabla(T_{\varepsilon A_2} g(v_\varepsilon) - T_{\varepsilon A_1} g(v_\varepsilon))|^{p-1}.$$

As v_ε is a weak solution of $P^{p,q}_{b_\alpha,g}(\tilde{v}_0, 0, \tilde{f}, \psi)$, taking $T_{\varepsilon A_2}(g(v_\varepsilon)) - T_{\varepsilon A_1}(g(v_\varepsilon))$ as test function, we get

$$\int_{\tilde{Q}} |\nabla(T_{\varepsilon A_2}(g(v_\varepsilon)) - T_{\varepsilon A_1}(g(v_\varepsilon))|^p \le \varepsilon |A_2 - A_1| c(\|\tilde{f}\|_{L^\infty(\tilde{Q})} + \|b(\tilde{v}_0)\|_{L^\infty(\tilde{\Omega})}).$$

Passing to the limit with $\varepsilon \to 0$ and $\delta \to 0$ successively, the second and third term can be estimated as in Lemma 3.3. The two last terms go also to 0 with $\varepsilon \to 0$ for fix δ thanks to the a.e. convergence of $g_\varepsilon(v_\varepsilon)$ to $g(v)$. Combining all the estimates, it follows that

$$\chi = 0 \quad \text{a.e. on the set } \{\nabla g(v) = 0\}. \tag{4.14}$$

Now, we use the regularization method of Landes [24] in order to prove that $\chi = \tilde{a}(v, \nabla g(v))$. For $\nu \in \mathbb{N}$, we define the regularization in time of the function $g(v(t,x))$ by

$$(g(v))_\nu(t,x) := \int_{-\infty}^{t} e^{\nu(s-t)} g(v)(s,x)\, ds$$

for a.e. $(t,x) \in \overline{Q}$ and for $s < 0$, we extend $v(t,x)$ by a function $\tilde{v}_0^\nu(x) \in L^\infty(\tilde{\Omega}) \cap W_0^{1,p}(\tilde{\Omega})$ with $\|b(\tilde{v}_0) - b(\tilde{v}_0^\nu))\|_1 \le \nu^{-1}$. Observe that $(g(v))_\nu \in L^p(0,T; W_0^{1,p}(\tilde{\Omega})) \cap L^\infty(\tilde{Q})$ and, moreover, is differentiable for a.e. $t \in (0,T)$ with $\frac{\partial}{\partial t}(g(v))_\nu = \nu(g(v) - (g(v))_\nu) \in L^p(0,T; W_0^{1,p}(\tilde{\Omega})) \cap L^\infty(\tilde{Q})$, $(g(v))_\nu(0,x) = g(\tilde{v}_0^\nu)(x)$ a.e. in $\tilde{\Omega}$, and $(g(v))_\nu \to g(v)$ in $L^p(0,T; W_0^{1,p}(\tilde{\Omega}))$ as $\nu \to \infty$. Let $\sigma \in \mathcal{D}^+([0,T))$. The main estimate is:

$$\liminf_{\nu \to \infty} \lim_{\varepsilon \to 0} \langle b(v_\varepsilon)_t, \sigma(g(T^2 v_\varepsilon) - (g(T^2 v))_\nu)\rangle \ge 0,$$

where $\langle .,. \rangle$ is the duality pairing between $L^{p'}(0,T; W^{-1,p'}(\tilde{\Omega}))$. Indeed, by the integration-by parts-formula, (see [3]),

$$\begin{aligned}
&\langle b(v_\varepsilon)_t, \sigma(g_\varepsilon(T^2 v_\varepsilon) - (g(T^2 v))_\nu)\rangle \\
&= -\int_{\tilde{Q}} \sigma_t \int_{A_2}^{v_\varepsilon} g_\varepsilon(T^2 r) db_\alpha(r) - \int \sigma(0) \int_{A_2}^{\tilde{v_0}} g_\varepsilon(T^2 r)\, db_\alpha(r) \\
&\quad + \int_{\tilde{Q}} \sigma_t b_\alpha(v_\varepsilon)(g(T^2 v))_\nu + \int_{\tilde{Q}} \sigma\nu(g(T^2 v) - (g(T^2 v))_\nu) b_\alpha(v_\varepsilon) \\
&\quad + \int_{\tilde{\Omega}} b_\alpha(\tilde{v_0})\sigma(0) g(T^2 v_0^\nu) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

It is not difficult to pass to limit with $\varepsilon \to 0$ in I_i, $i = 1,2,3,5$. Dealing with the term I_4, we have

$$\begin{aligned}
I_4 &= \int_{\tilde{Q}} \sigma\nu(g(T^2 v) - (g(T^2 v))_\nu)(b_\alpha g^{-1}(g(T^2 v_\varepsilon))) - b_\alpha(g^{-1}((g(T^2 v))_\nu))) \\
&\quad + \int_{\tilde{Q}} \sigma\nu(g(T^2 v) - g(T^2 v_\varepsilon))(b_\alpha g^{-1}(g(T^2 v_\varepsilon))) - b_\alpha(g^{-1}((g(T^2 v))_\nu))) \\
&\quad + \int_{\tilde{Q}} \sigma\nu(g(T^2 v) - (g(T^2 v))_\nu) b_\alpha(g^{-1}((g(T^2 v))_\nu))
\end{aligned}$$

$$\begin{aligned}&+\int_{\tilde{Q}}\sigma\nu(g(T^2v_\varepsilon)-(g(T^2v))_\nu)(b_\alpha(v_\varepsilon)-b_\alpha(T^2v_\varepsilon)))\\&=: I_4^1+I_4^2+I_4^3+I_4^4.\end{aligned}$$

It is clear that $I_4^1\geq 0$ and $I_4^4=0$. Moreover,

$$\begin{aligned}I_4^3&=\int_{\tilde{Q}}\sigma\nu(g(T^2v)-(g(T^2v))_\nu)b_\alpha(g^{-1}((g(T^2v))_\nu))\\&=\int_{\tilde{Q}}\sigma\frac{\partial}{\partial t}(g(T^2v))_\nu b_\alpha g^{-1}((g(T^2v))_\nu)\\&\quad\times\int_{\tilde{Q}}\sigma\frac{\partial}{\partial t}\int_0^{(g(T^2v))_\nu}(b_\alpha(\circ g^{-1})(r)\,dr\\&=-\int_{\tilde{Q}}\sigma_t\int_0^{(g(T^2v))_\nu}(b_\alpha\circ g^{-1})(r)\,dr-\int_{\tilde{\Omega}}\sigma(0)\int_0^{(g(T^2v_0))_\nu}b_\alpha\circ g^{-1})(r)\,dr\\&=-\int_{\tilde{Q}}\sigma_t\int_{A_2}^{g^{-1}((g(T^2v))_\nu}b_\alpha(r)dg(r)-\int_{\tilde{\Omega}}\sigma(0)\int_{A_2}^{g^{-1}(g(T^2\tilde{v_0}^{\nu})}b_\alpha(r)\,dg(r)\,dr.\end{aligned}$$

Thus, as $\lim_{\varepsilon\to 0,\alpha\to 0}I_4^2=0$,

$$\begin{aligned}&\liminf_{\nu\to\infty}\lim_{\varepsilon\to 0}\langle b_\alpha v_\varepsilon)_t,\sigma(g(T^2v_\varepsilon)-(g(T^2v))_\nu)\rangle\\&\geq-\int_{\tilde{Q}}\sigma_t\int_{A_2}^{T^2v}g(r)\,db_\alpha(r)-\int\sigma(0)\int_{A_2}^{T^2v_0}g(r)\,db_\alpha(r)\\&\quad+\int_{\tilde{Q}}\sigma_t b_\alpha(v)g(T^2v)-\int_{\tilde{Q}}\sigma_t\int_{A_2}^{T^2v}b_\alpha(r)\,dg(r)\\&\quad-\int_{\tilde{\Omega}}\sigma(0)\int_{A_2}^{T^2v_0}b_\alpha(r)\,dg(r)+\int_{\tilde{\Omega}}b_\alpha(v_0)\sigma(0)g(T^2\tilde{v_0})\end{aligned}$$

Next, note that

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to 0}\int_{\tilde{Q}}(\psi(v_\varepsilon)+\beta_{p,q}(v_\varepsilon))\sigma(g_\varepsilon(T^2v_\varepsilon)-(g(T^2v))_\nu)=0.$$

Therefore, using $\sigma(g(T^2v_\varepsilon)-(g(T^2v))_\nu)$ as a test function in the differential equality, we obtain

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to\infty}\int_{\tilde{Q}}\sigma a(v_\varepsilon,\nabla g_\varepsilon(T^2v_\varepsilon))\cdot\nabla(g(T^2v_\varepsilon)-(g(T^2v))_\nu)\leq 0. \tag{4.15}$$

As

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to\infty}\int_{\tilde{Q}}\sigma a(v_\varepsilon,0)\cdot\nabla(g(T^2v_\varepsilon)-(g(T^2v))_\nu)=0, \tag{4.16}$$

by the divergence theorem, it follows that

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to\infty}\int_{\tilde{Q}}\sigma\tilde{a}(v_\varepsilon,\nabla g_\varepsilon(T^2v_\varepsilon))\cdot\nabla(g(T^2v_\varepsilon)-(g(T^2v))_\nu)\leq 0. \tag{4.17}$$

This in turn implies (by the same arguments as in the proof of (4.14)) that

$$\limsup_{\nu\to\infty}\limsup_{\varepsilon\to\infty}\int_{\tilde{Q}}\sigma\tilde{a}(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla(g(v_\varepsilon)-(g(v))_\nu)\leq 0. \tag{4.18}$$

By the pseudo-monotonicity argument, we deduce that

$$\operatorname{div}\chi = \operatorname{div}\tilde a(v,\nabla g(v)) \quad \text{in } \mathcal{D}'(\tilde Q) \text{ for all } k>0. \tag{4.19}$$

Indeed, for $\xi\in\mathcal{D}(\tilde\Omega)$, $\xi\ge 0$, $\alpha\in\mathbb{R}$, we have

$$\begin{aligned}
&\alpha\int_{\tilde Q}\sigma\chi\nabla\xi\\
&=\lim_{\varepsilon\to0}\int_{\tilde Q}\alpha\sigma\tilde a(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla\xi\\
&\ge\limsup_{\nu\to\infty}\limsup_{\varepsilon\to0}\int_{\tilde Q}\sigma\tilde a(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))\cdot\nabla(g_\varepsilon(v_\varepsilon)-(g(v))_\nu+\alpha\xi)\\
&\ge\limsup_{\nu\to\infty}\limsup_{\varepsilon\to\infty}\int_{\tilde Q}\sigma\tilde a(v_\varepsilon,\nabla((g(v))_\nu-\alpha\xi))\cdot\nabla(g_\varepsilon(v_\varepsilon)-(g(v))_\nu+\alpha\xi)\\
&\ge\int_{\tilde Q}\alpha\sigma\tilde a(v,\nabla(g(v)-\alpha\xi))\cdot\nabla\xi.
\end{aligned}$$

Dividing by $\alpha>0$ resp. <0, passing to the limit with $\alpha\to0$, we obtain (4.19). As a further consequence of (4.15),

$$\lim_{\nu\to\infty}\lim_{\varepsilon\to0}\int_{\tilde Q}\sigma(\tilde a(v_\varepsilon,\nabla g_\varepsilon(v_\varepsilon))-\tilde a(v_\varepsilon,\nabla(g_\varepsilon(v_\varepsilon))_\nu))\cdot\nabla(g_\varepsilon(v_\varepsilon)-(g(v))_\nu)=0.$$

By the diagonal principle, there exists a sequence $\varepsilon(\nu)$ such that the (non-negative) function

$$\sigma(a(v_{\varepsilon(\nu)},\nabla g_\varepsilon(v_{\varepsilon(\nu)}))-a(v_{\varepsilon(\nu)},\nabla(g(v))_\nu))\cdot(\nabla g_\varepsilon(v_{\varepsilon(\nu)})-\nabla(g(v))_\nu)\to0$$

strongly in $L^1(\tilde Q)$ as $\nu\to\infty$.

By standard arguments we first deduce that

$$\sigma(a(v_{\varepsilon(\nu)},\nabla g_\varepsilon(v_{\varepsilon(\nu)}))\cdot(\nabla g_\varepsilon(v_{\varepsilon(\nu)})-\nabla(g(v))_\nu)\to0$$

weakly in $L^1(\tilde Q)$ and then, using (4.19), that

$$\int_{\tilde Q}\sigma(a(v_{\varepsilon(\nu)},\nabla g_\varepsilon(v_{\varepsilon(\nu)}))\cdot\nabla g_\varepsilon(v_{\varepsilon(\nu)})\to\int_{\tilde Q}\sigma a(v,\nabla g(v))\cdot\nabla g(v)$$

as $\nu\to\infty$ for all σ in $L^\infty(\tilde Q)$.

Combining all estimates, for some appropriately chosen subsequence (still denoted by v_ε for simplicity) we can pass to the limit in the entropy inequalities; i.e., using $\xi\in\mathcal{D}(\tilde Q)$ as a test function in $P^{p,q}_{b,g_\varepsilon}(\tilde{v_0},0,\tilde f,\psi)$, passing to the limit in (4.1) and (4.2), we get for all $k\in\mathbb{R}$, for all $\xi\in C_0^\infty([0,T)\times\mathbb{R}^N)$ such that $\xi\ge0$ and $(-g(k))^+\xi=0$ a.e. on $\tilde\Sigma$,

$$\begin{aligned}
0\le&\int_0^1\int_{\tilde Q}(b_\alpha(v)-b_\alpha(k))^+\xi_t+\int_{\tilde\Omega}(b_\alpha(\tilde{v_0})-b_\alpha(k))^+\xi(0,\cdot)\\
&+\int_0^1\int_{\tilde Q}\chi_{\{v>k\}}(\tilde f-\psi(v)-\beta_{p,q}(v))\xi-(a(v,\nabla g(v))-a(k,0))\cdot\nabla\xi)
\end{aligned}\tag{4.20}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \geq 0$ and $(g(k))^+\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$\begin{aligned} 0 \leq & \int_0^1 \int_{\tilde{Q}} (b_\alpha(k) - b_\alpha(v))^+ \xi_t + \int_{\tilde{\Omega}} (b_\alpha(k) - b_\alpha(\tilde{v_0}))^+ \xi(0, \cdot) \\ & - \int_0^1 \int_{\tilde{Q}} \chi_{\{k>v\}} (\tilde{f} - \psi(v) - \beta_{p,q}(v))\xi - (a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi). \end{aligned} \tag{4.21}$$

Now, to prove that v is the week entropy solution of $P_{b,g}^{p,q}(\tilde{v_0}, 0, \tilde{f}, \psi)$, we use the following "reinforced" comparison principle.

Proposition 4.4. *Let $v_0^i \in L^\infty(\tilde{Q})$, $f_i \in L^\infty(\tilde{Q})$ and $u_i \in L^\infty(\tilde{Q} \times (0,1)$ be a weak entropy process of $P_{b,g}^{p,q}(\tilde{v}_0^i, 0, \tilde{f}_i, \psi)$ $i = 1, 2$. Then, there exists $\kappa \in L^\infty(\tilde{Q} \times (0,1))$ with $\kappa \in$ sign$^+(u_1 - u_2)$ a.e. in $\tilde{Q} \times (0,1)$ such that, for any $\xi \in \mathcal{D}([0,T[)$, $\xi \geq 0$,*

$$\begin{aligned} 0 \leq & \int_0^1 \int_{\tilde{Q}} (b(u_1(t,x,\alpha) - b(u_2(t,x,\mu)))^+ \xi_t \, dx \, dt \, d\alpha \, d\mu \\ & + \int_0^1 \int_{\tilde{Q}} \kappa(f_1 - f_2)\xi \, dx \, dt \, d\alpha \, d\mu \\ & + \int_{\tilde{\Omega}} (b(v_0^1) - b(v_0^2))^+ \xi(0,.) \, dx. \end{aligned}$$

In particular, in the case where $f_1 = f_2$ and $v_0^1 = v_0^2$, we have

$$u_1(t,x,\alpha) = u_2(t,x,\mu) \quad \text{a.e. } (t,x,\alpha,\mu) \in \tilde{Q} \times (0,1) \times (0,1).$$

Defining $w(t,x) = \int_0^1 u_1(t,x,\alpha) \, d\alpha$, we have $w(t,x) = u_1(t,x,\alpha) = u_2(t,x,\beta)$ a.e. $(t,x,\alpha,\beta) \in \tilde{Q} \times (0,1) \times (0,1)$.

The proof of Proposition 4.4 follows the same lines as those of Theorem 3.1 and is omitted here because it does not contain new ideas. The reader is referred to [18], in order to verify the technical tools needed to deal with measure-valued functions. By corollary 3.3 it follows that v is the unique weak entropy solution of $P_{b,g}^{p,q}(\tilde{v}_0, 0, \tilde{f}, \psi)$ and the first step of the proof is complete.

4.2. **Second step.** In the first step, we have proved the existence of an entropy solution $v_{p,q}$ corresponding to the problems $\tilde{P}_{b,g}^{p,q}(\tilde{v}_0, 0, \tilde{f}, \psi)$. By Theorem 3.1, a comparison principle holds. In particular, entropy solutions for different penalization parameters can be compared: for any $p, p', q \in \mathbb{N}$ with $p \leq p'$, there exists $\kappa \in L^\infty(\tilde{Q})$ with $\kappa \in \text{sign}^+(v_{p',q} - v_{p,q})$ a.e. on $\tilde{Q}$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned} & \int_{\tilde{\Omega}} (b(v_{p',q})(t,\cdot) - b(v_{p,q})(t,\cdot))^+ + \int_{\tilde{Q}} (\psi(v_{p',q}) - \psi(v_{p,q}))^+ \\ & \leq \int_0^t \int_{\tilde{\Omega}} \kappa((\tilde{f} - \beta_{p',q}(v_{p',q}) - (\tilde{f} - \beta_{p,q}(v_{p,q}))) \\ & = \int_0^t \int_{\tilde{\Omega}} \kappa(\beta_{p,q}(v_{p,q}) - \beta_{p,q}(v_{p',q}) - \chi_{\tilde{Q} \setminus Q}(p' - p)(v_{p',q} - \tilde{u})^+) \\ & \leq \int_0^t \int_{\tilde{\Omega}} \kappa(\beta_{p,q}(v_{p,q}) - \beta_{p,q}(v_{p',q}))^+ \leq 0. \end{aligned}$$

Consequently,

$$v_{p',q} \leq v_{p,q} \quad \text{a.e. } (t,x) \in \tilde{Q}.$$

In the same way, one can prove that, for all $p, q, q' \in \mathbb{N}$ with $n \le n'$,

$$v_{p,q} \le v_{p,q'} \quad \text{a.e. } (t,x) \in \tilde{Q}.$$

This comparison result already ensures the a.e. convergence of the solutions $v_{p,q}$ as, successively, $p \to \infty$ and $q \to \infty$.

To obtain an L^∞-bound on the approximate solutions, we compare $v_{p,q}$ with $C \in \mathbb{R}$ such that $\psi(C) = \|f\|_{L^\infty(\tilde{Q}} + \|b(\tilde{v_0})\|_{L^\infty(\tilde{\Omega})} + \|\tilde{u}\|_{L^\infty(\tilde{Q})} + 1$ as classical solution of $P_{b,g}(C, C, \psi(C) + \beta_{p,q}(C), \psi)$ (resp with $\tilde{C} \in \mathbb{R}$ such that $\psi(\tilde{C}) = -(\|f\|_{L^\infty(\tilde{Q})} + \|\tilde{v_0}\|_{L^\infty(\tilde{\Omega})} + \|\tilde{u}\|_{L^\infty(\tilde{Q})} + 1)$. Thanks to the strong perturbation ψ, we prove that $(v_{p,q})_{p,q}$ is uniformly bounded in $L^\infty(\tilde{Q})$. As a consequence, passing to a subsequence if necessary and using the diagonal principle, there exists a sequence $v_q = v_{p(q),q}$ which converges in $L^1(\tilde{Q})$ as $q \to \infty$ to some function $v \in L^\infty(\tilde{Q})$. In order to prove that v is a weak entropy solution of $P_{b,g}(\tilde{v_0}, u, f, \psi)$, it is sufficient to prove that v satisfies the family of entropy inequalities: for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \ge 0$ and $(-g(k))^+\xi = 0$ a.e. on Σ

$$\begin{aligned} -\int_\Sigma \omega^+((t,x), k, u)\xi \le \int_Q (b(v) - b(k))^+\xi_t + \int_\Omega (b(v_0) - b(k))^+\xi(0,\cdot) \\ + \int_Q \chi_{\{v>k\}}((f - \psi(v))\xi - (a(v, \nabla g(v)) - a(k,0)) \cdot \nabla\xi) \end{aligned} \tag{4.22}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)$ such that $\xi \ge 0$ and $(g(k))^+\xi = 0$ a.e. on Σ,

$$\begin{aligned} &-\int_\Sigma \omega^-((t,x), k, u)\xi \\ &\le \int_Q (b(k) - b(v))^+\xi_t + \int_\Omega (b(k) - b(v_0))^+\xi(0,\cdot) \\ &\quad - \int_Q \chi_{\{k>v\}}((f - \psi(v))\xi - (a(v, \nabla g(v)) - a(k,0)) \cdot \nabla\xi). \end{aligned} \tag{4.23}$$

Let us remark first that $\tilde{u}$ is a weak entropy solution of $\tilde{P}^{p,q}_{b,g}(\tilde{u}_0, 0, b(\tilde{u})_t - \operatorname{div} a(\tilde{u}, \nabla g(\tilde{u})) + \psi(\tilde{u}), \psi)$. Then, by the comparison principle, there exists $\kappa, \tilde{\kappa}$ in $L^\infty(\tilde{Q})$ with $\kappa \in \operatorname{sign}^+(v_{p,q} - \tilde{u})$, $\tilde{\kappa} \in \operatorname{sign}^+(\tilde{u} - v_{p,q})$ a.e. on $\tilde{Q}$ such that for a.e. $t \in (0,T)$,

$$\begin{aligned} &\int_{\tilde{Q}} (\psi(v_{p,q}) - \psi(\tilde{u}))^+\xi \\ &\le \int_{\tilde{Q}} (b(v_{p,q}) - b(\tilde{u}))^+\xi_t + \int_{\tilde{\Omega}} (b(\tilde{v_0}) - b(\tilde{u}_0))^+\xi(0,x) - \int_{\tilde{Q}\setminus Q} \chi_{\{v_{p,q}>\tilde{u}\}}\beta_{p,q}(v_{p,q})\xi \\ &\quad + \int_{\tilde{Q}} \kappa(\tilde{f} - \psi(\tilde{u}))\xi - b(\tilde{u})_t\xi + \operatorname{div} a(\tilde{u}, \nabla g(\tilde{u}))\xi) \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}&\int_{\tilde Q}(\psi(\tilde u)-\psi(v_{p,q}))^+\xi\\&\le\int_{\tilde\Omega}(b(\tilde u)-b(v_{p,q}))^+\xi_t+\int_{\tilde\Omega}(b(\tilde u_0)-b(\tilde v_0))^+\xi(0,x)+\int_{\tilde Q\setminus Q}\chi_{\{\tilde u>v_{p,q}\}}\beta_{p,q}(v_{p,q})\xi\\&\quad-\int_{\tilde Q}\tilde\kappa(\tilde f-\psi(\tilde u)\xi-b(\tilde u)_t\xi+\operatorname{div}a(\tilde u,\nabla g(\tilde u))\xi).\end{aligned}\tag{4.25}$$

This implies in particular that

$$\int_{\tilde Q\setminus Q}p(v_{p,q}-\tilde u)^+\le c \text{ and } \int_{\tilde Q\setminus Q}q(\tilde u-v_{p,q})^+\le c,$$

where $c:=c(f,\tilde u)$ is independent of p,q. Hence, by Fatou's Lemma,

$$\int_{\tilde Q\setminus Q}(v-\tilde u)^+\le 0,\quad \int_{\tilde Q\setminus Q}(\tilde u-v)^+\le 0.\tag{4.26}$$

Thus

$$v=\tilde u \text{ a.e.}\quad \text{on } \tilde Q\setminus Q.\tag{4.27}$$

Arguing as in [2], we prove that

$$g(v)=g(\tilde u)=0\quad\text{on }\Sigma\text{ in the sense of traces in }L^p([0,T),W^{1,p}(\Omega)).$$

Next, applying as in the first step the regularization method of Landes, we can prove that $a(v_{p,q},\nabla g(v_{p,q}))$ converges weakly in $(L^{p'}(Q))^N$ to $a(v,\nabla g(v))$.

Now, we prove that v is an entropy solution of $P_{b,g}(\tilde v_0,a,f,\psi)$: As $\tilde u$ is continuous, for all $\epsilon>0$, there exists $\delta>0$ such that $\max_{Q_\delta}\tilde u\le k+\frac{\epsilon}{2}$, where $Q_\delta:=\{(t,x)\in[0,T[\times\mathbb{R}^N;\operatorname{dist}((t,x),Q)\le\delta\}$. Let $(t,x)\in\tilde Q$, $0<r<\delta$, $\xi\in\mathcal{D}([0,T)\times\mathbb{R}^N)^+$ with $\operatorname{supp}(\xi)\subset B((t,x),r)\cap Q_\delta$. We apply the comparison principle Theorem 3.1 to $v_{p,q}$ solution of $\tilde P^{p,q}_{b,g}(\tilde v_0,0,\tilde f,\psi)$ and $k\ge\max_{\Sigma\cap B((t,x),r)}u+\epsilon$ as solution of $\tilde P^{p,q}_{b,g}(k,k,\psi(k)+\beta_{p,q}(k),\psi)$ on $\tilde Q$ to get for some $\kappa_{p,q}\in\operatorname{sign}^+(v_{p,q}-k)$

$$\begin{aligned}&-\int_{\tilde\Sigma}\omega^+((t,x),u,k)\xi+\int_{\tilde Q}(\psi(v_{p,q})-\psi(k))^+\xi\\&\le\int_{\tilde Q}(b(v_{p,q})-b(k)))^+\xi_t+\int_{\tilde Q}\kappa_{p,q}f\xi+\int_{\tilde\Omega}(b(\tilde v_0)-b(k))^+\xi(0,x)\\&\quad+\int_{\tilde Q}\chi_{\{v_{p,q}\ge k\}}(a(v_{p,q},\nabla g(v_{p,q}))-a(v_{p,q},0))\cdot\nabla\xi-\int_{\tilde Q\setminus Q}\chi_{\{v_{p,q}>k\}}\beta_{p,q}(v_{p,q})\xi\\&=\int_{Q}(b(v_{p,q})-b(k)))^+\xi_t+\int_{Q}\kappa_{p,q}f\xi+\int_{\Omega}(b(\tilde v_0)-b(k))^+\xi(0,x)\\&\quad+\int_{\tilde Q\cap Q_\delta\setminus Q}(b(v_{p,q})-b(k)))^+\xi_t+\int_{Q}\chi_{\{v_{p,q}\ge k\}}(a(v_{p,q},\nabla g(v_{p,q}))-a(v_{p,q},0))\cdot\nabla\xi\\&\quad+\int_{\tilde Q\cap Q_\delta\setminus Q}\chi_{\{v_{p,q}\ge k\}}(a(v_{p,q},\nabla g(v_{p,q}))-a(v_{p,q},0))\cdot\nabla\xi\\&\quad-\int_{\tilde Q\cap Q_\delta\setminus Q}\chi_{\{v_{p,q}>k\}}\beta_{p,q}(v_{p,q})\xi.\end{aligned}\tag{4.28}$$

As $-\int_{\tilde{Q}\cap Q_\delta\setminus Q}\chi_{\{v_{p,q}>k\}}\beta_{p,q}(v_{p,q})\xi \le 0$, we can neglect the last term in the above inequality. Using also (4.26), it is clear that

$$\limsup_{p,q\to\infty}\int_{\tilde{Q}\cap Q_\delta\setminus Q}\chi_{\{v_{p,q}\ge k\}}(a(v_{p,q},\nabla g(v_{p,q}))-a(v_{p,q},0))\cdot\nabla\xi+(b(v_{p,q})-b(k)))^+\xi_t = 0.$$

Hence, passing to the limit with $p,q\to\infty$ in (4.28), we find

$$\begin{aligned}\int_Q(\psi(v)-\psi(k))^+\xi &\le \int_\Omega(b(v_0)-b(k))^+\xi(0,x)+\int_Q(b(v)-b(k)))^+\xi_t\\ &\quad+\int_Q\chi_{\{v_{p,q}\ge k\}}(f\xi+(a(v_{p,q},\nabla g(v_{p,q}))-a(v_{p,q},0))\cdot\nabla\xi)\end{aligned}\tag{4.29}$$

for any $k\ge\max_{\Sigma\cap B((t,x),r)}u+\epsilon$. As ϵ is arbitrary, the above inequality holds for any $k\ge\max_{\Sigma\cap B((t,x),r)}u$. Thanks to Remark 3.6, we can apply the comparison principle for v as a function satisfying the "local property" (4.29) and any $k\in\mathbb{R}$ with $(-g(k))^+\xi=0$ as classical solution of $P_{b,g}(k,k,\psi(k),\psi)$ in Q to get for all $\xi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N$ with $(-g(k))^+\xi=0$ and for some $\kappa\in\operatorname{sign}^+(v-k)$,

$$\begin{aligned}&-\int_\Sigma\omega^+((t,x),u,k)\xi+\int_Q(\psi(v)-\psi(k))^+\xi\\ &\le\int_Q(b(v)-b(k)))^+\xi_t+\int_\Omega(b(v_0)-b(k))^+\xi(0,x)\\ &\quad+\int_Q\chi_{\{v\ge k\}}(a(v,\nabla g(v))-a(k,0))\cdot\nabla\xi+\int_Q\kappa f\xi.\end{aligned}$$

Choosing $(k_n)_n\subset\mathbb{R}$ with $k_n\downarrow k$ as $n\to\infty$, passing to the limit in the above inequality written with k replaced by k_n, using the fact that, for any $\kappa_n\in\operatorname{sign}^+(v-k_n)$ a.e. in Q, $\lim_{n\to\infty}\kappa_n\in\operatorname{sign}_0^+(v-k)$, we obtain (4.22). In the same way, we prove (4.23).

Hence, the limit function v satisfies the first entropy inequality for any $k\in\mathbb{R}$ and any $\xi\in\mathcal{D}([0,T)\times\mathbb{R}^N)$ such that $(-g(k))^+\xi=0$ and with similar arguments we prove that v verifies the second entropy inequality. The rest of the proof follows the same lines as in [2].

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