Electronic Journal of Differential Equations, Vol. 2009(2009), No. 149, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SYMMETRY IN REARRANGEMENT OPTIMIZATION PROBLEMS

BEHROUZ EMAMIZADEH, JYOTSHANA V. PRAJAPAT

Abstract. This article concerns two rearrangement optimization problems. The first problem is motivated by a physical experiment in which an elastic membrane is sought, built out of several materials, fixed at the boundary, such that its frequency is minimal. We capture some features of the optimal solutions, and prove a symmetry property. The second optimization problem is motivated by the physical situation in which an ideal fluid flows over a seamount, and this causes vortex formation above the seamount. In this problem we address existence and symmetry.

1. Introduction

In this article, we consider two rearrangement optimization problems which are physically relevant. A rearrangement optimization problem is referred to an optimization problem where the admissible set is a rearrangement class, see section 2 for precise definition. In both problems our focus will be on (radial) symmetry. More precisely, we will show that when the physical domain is a ball then the optimal solutions will be radial as well.

The first problem is concerned with the following non-linear eigenvalue problem:

$$\begin{gathered} -\Delta_p u + V(x)|u|^{p-2}u = \lambda g(x)|u|^{p-2}u, \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here V and g are given functions, and λ is an eigenvalue. There are some technical conditions on V and g, but we prefer to delay stating them until section 3. However, we mention that in case $p = 2$ and $g = 1$, (1.1) is the steady state case of an elastic membrane, fixed around the boundary, made out of various materials (this justifies placing V in the differential equation). The constant λ denotes the frequency of the membrane. There are infinitely many eigenvalues, but we are only interested in the first one, often referred to as the principal eigenvalue, and denote it by $\lambda(g, V)$ to emphasize its dependance on g and V. The following variational formulation for $\lambda(g, V)$ is well known:

$$\lambda(g, V) = \inf\Big\{ \int_\Omega (|\nabla u|^p + V(x)|u|^p)dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega g(x)|u|^p dx = 1\Big\}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 49K20, 35P15, 35J10, 74K15.
Key words and phrases. Minimization and maximization problems; rearrangements; principal eigenvalue; optimal solutions; symmetry.

Submitted October 28, 2009. Published November 25, 2009.

Given g_0 and V_0, we are interested in the following rearrangement optimization problem

$$\inf_{g\in\mathcal{R}(g_0),\ V\in\mathcal{R}(V_0)} \lambda(g,V), \tag{1.3}$$

where $\mathcal{R}(g_0)$ and $\mathcal{R}(V_0)$ are rearrangement classes generated by g_0 and V_0, respectively. This problem has already been considered in [7], where amongst other results the authors show (1.3) is solvable, see also [8, 9]. However, here we focus on some features of optimal solutions, and prove a symmetry result.

The second problem considered in this paper is motivated by fluids flowing over seamounts. More precisely, it is well accepted that a two dimensional ideal fluid flowing over a seamount (e.g. hills located at the bottom of an ocean) gives rise to vortex formation located above seamounts, see [11]. Existence of such flows turns out to be equivalent to existence of maximizers for certain functionals, representing some kind of energy, which can be formulated in terms of vorticity function and the height function. Mathematically, here is the problem we are interested in: Let us denote by $u_f \in W_0^{1,2}(\Omega)$ the unique solution of the Poisson differential equation

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Consider the energy functional:

$$J(f,h) = \frac{1}{2}\int_\Omega f u_f\,dx + \int_\Omega h u_f\,dx.$$

We are interested in the following rearrangement optimization problem:

$$\sup_{f\in\mathcal{R}(f_0),\ h\in\mathcal{R}(h_0)} J(f,h). \tag{1.4}$$

Here f represents the vorticity function and h the hight function (seamount). Details related to (1.4) are given in section 4, where we discuss existence of optimal solutions for (1.4), and address the question of symmetry. The reader can refer to [5, 6] for other examples of rearrangement optimization problems.

2. Preliminaries

In this section we review rearrangement theory with an eye on the optimization problems (1.3) and (1.4). So we only mention results that are going to assist us with the two optimization problems in question. The reader can refer to [1, 2] for an extensive account of rearrangement theory. Henceforth, we assume Ω is a smooth bounded domain in $\mathbb{R}^N$, unless stated otherwise.

Definition. Two functions $f : (X, \Sigma_1, \mu_1) \to \mathbb{R}$, $g : (Y, \Sigma_2, \mu_2) \to \mathbb{R}$ are said to be rearrangements of each other if:

$$\mu_1(\{x \in X : f(x) \ge \alpha\}) = \mu_2(\{x \in Y : g(x) \ge \alpha\}), \quad \forall \alpha \in \mathbb{R}.$$

In case μ_i stands for the Lebesgue measure in $\mathbb{R}^N$, we replace it with $|\cdot|$. When f and g are rearrangements of each other we write $f \sim g$. For a fixed $f_0 : (X, \Sigma, \mu) \to \mathbb{R}$, the class of rearrangements generated by f_0, denoted $\mathcal{R}(f_0)$, is defined as follows:

$$\mathcal{R}(f_0) = \{f : f \sim f_0\}.$$

In case Ω is a ball in $\mathbb{R}^N$, say centered at the origin, and $f : \Omega \to \mathbb{R}$ is a Lebesgue measurable function then $f^* : \Omega \to \mathbb{R}$ and $f_* : \Omega \to \mathbb{R}$ denote the Schwarz decreasing and increasing rearrangements of f. That is, $f^* \sim f$, $f_* \sim f$, and f^* is a radial function which is decreasing as a function of $r := \|x\|$, whereas f_* is a radial function which is increasing as a function of r. We will use two well known rearrangement inequalities.

Lemma 2.1 ([10]). *Suppose Ω is a ball in $\mathbb{R}^N$. Then*

$$\int_\Omega f^* g_*\,dx \le \int_\Omega fg\,dx \le \int_\Omega f^* g^*\,dx, \tag{2.1}$$

where f and g are non-negative functions.

Lemma 2.2 ([4]). *Suppose Ω is a ball in $\mathbb{R}^N$. For $0 \le u \in W_0^{1,p}(\Omega)$, $u^* \in W_0^{1,p}(\Omega)$, and*

$$\int_\Omega |\nabla u|^p\,dx \ge \int_\Omega |\nabla u^*|^p\,dx. \tag{2.2}$$

If the equality holds in (2.2), and the set $\{x \in \Omega : \nabla u(x) = 0,\ 0 < u(x) < M\}$, $M := \operatorname{ess\,sup}_\Omega u(x)$, has zero measure, then $u = u^*$.

The next two results are fundamental tools in studying rearrangement optimization problems, see [1, 2].

Lemma 2.3. *Let $1 \le p < \infty$, and $q = \frac{p}{p-1}$. Let $f_0 \in L^p(\Omega)$ be a non-trivial function and $g \in L^q(\Omega)$. Then there exist $\hat{f}$ and $\overline{f}$ in $\mathcal{R}(f_0)$ such that*

$$\int_\Omega \overline{f} g\,dx \le \int_\Omega fg\,dx \le \int_\Omega \hat{f} g\,dx. \tag{2.3}$$

Lemma 2.4. *Let $1 \le p < \infty$, and let $\Psi : L^p(\Omega) \to \mathbb{R}$ be strictly convex, and weakly sequentially continuous. Then Ψ attains a maximum relative to $\mathcal{R}(f_0)$. Moreover, if $\hat{f}$ is a maximizer of Ψ, and $g \in \partial\Psi(\hat{f})$, the subdifferential of Ψ at $\hat{f}$, then*

$$\hat{f} = \phi(g), \tag{2.4}$$

almost everywhere in Ω, where ϕ is an increasing function unknown a priori.

We close this section with the following definition.

Definition. Given a measurable function $f : \Omega \to \mathbb{R}$, the distribution function of f is defined by:

$$\mu_f(\alpha) = |\{x \in \Omega : f(x) \ge \alpha\}|.$$

The function $f^\Delta : [0, |\Omega|] \to \mathbb{R}$ defined by

$$f^\Delta(s) = \inf\{\alpha : \mu_f(\alpha) \le s\}$$

is called the decreasing rearrangement of f. On the other hand, $f_\Delta : [0, |\Omega|] \to \mathbb{R}$, the increasing rearrangement of f, is defined as follows:

$$f_\Delta(s) = f^\Delta(|\Omega| - s).$$

3. Study of Problem (1.3)

This section is devoted to problem (1.3). We begin by introducing the function space:

$$\mathcal{S} = \{(g, V) \in L^\infty_+(\Omega) \times L^\infty_+(\Omega) : g(x) \ge A > 0,\ \|V\|_\infty < \frac{A}{C_p(A + \|g\|_\infty)}\},$$

where A is a positive constant, and C_p is the constant in the Poincarè inequality:

$$\int_\Omega |u|^p\,dx \le C_p \int_\Omega |\nabla u|^p\,dx, \quad u \in W_0^{1,p}(\Omega).$$

Next, we fix $(g_0, V_0) \in \mathcal{S}$, and let $\mathcal{R}(g_0)$ and $\mathcal{R}(V_0)$ to denote the rearrangement classes generated by g_0 and V_0, respectively. Note that from the definition of rearrangements, it readily follows that $\mathcal{R}(g_0) \times \mathcal{R}(V_0) \subset \mathcal{S}$.

The question of existence of (optimal) solutions of (1.3) has already been addressed in [7, 8, 9]; where amongst other results the following result is proved.

Lemma 3.1. *(a) Problem* (1.3) *is solvable; that is, there exists* $(\hat{g}, \hat{V}) \in \mathcal{R}(g_0) \times \mathcal{R}(V_0)$ *such that*

$$\lambda(\hat{g}, \hat{V}) = \inf_{g \in \mathcal{R}(g_0),\, V \in \mathcal{R}(V_0)} \lambda(g, V).$$

(b) If $(\hat{g}, \hat{V})$ *is an optimal solution of* (1.3), *then*

$$\hat{g} = \phi(\hat{u}), \quad \text{a.e. in } \Omega, \tag{3.1}$$

$$\hat{V} = \psi(\hat{u}), \quad \text{a.e. in } \Omega, \tag{3.2}$$

where ϕ *and* ψ *are increasing and decreasing functions unknown a priori. Here* $\hat{u}$ *stands for the unique eigenfunction corresponding to* $\lambda(\hat{g}, \hat{V})$.

Let us point out some consequences of (3.1) and (3.2) before addressing the question of symmetry.

Lemma 3.2. *Suppose* $(\hat{g}, \hat{V})$ *is an optimal solution of* (1.3). *Then*

(a) *The function* $\hat{u}$ *attains its smallest values on the support of* $\hat{V} := \{x \in \Omega : \hat{V} > 0\}$. *In fact, for some* $t > 0$,

$$\{x \in \Omega : \hat{V} > 0\} = \{x \in \Omega : \hat{u} < t\}. \tag{3.3}$$

(b) *In case* Ω *is simply connected, the support of* $\hat{V}$ *is a connected tubular domain around* $\partial\Omega$.

(c) *The functions* ϕ *and* ψ *in* (3.1) *and* (3.2) *can be formulated as follows:*

$$\phi = \hat{g}^{\Delta}(\mu_{\hat{u}}), \quad \psi = \hat{g}_{\Delta}(\mu_{\hat{u}}).$$

Proof. (a) From (3.2), we obtain

$$\{x \in \Omega : \hat{V} > 0\} = \hat{V}^{-1}(0, \infty) = \hat{u}^{-1}(\psi^{-1}(0, \infty)).$$

Since ψ is decreasing, $\psi^{-1}(0, \infty)$ must be an interval of the form $(-\infty, t)$ or $(-\infty, t]$, for some $t \in \mathbb{R}$. Clearly the assertion is proved once we show that the set $E := \{x \in \Omega : \hat{u} = t\}$ has zero measure, since it is obvious that t can not be a non-positive constant. Let us assume the contrary and derive a contradiction. Specializing the differential equation (1.1) to the set E, we get:

$$\hat{V}(x)\hat{u}^{p-1} = \lambda(\hat{g}, \hat{V})\hat{g}(x)\hat{u}^{p-1}, \quad \text{in } E.$$

Since $\hat{u}$ is positive in Ω, in turn, we get $\hat{V}(x) = \hat{\lambda}\hat{g}(x)$, where we have replaced $\lambda(\hat{g}, \hat{V})$ by $\hat{\lambda}$, for simplicity. We show that this last equation can not hold. To see this observe that:

$$\hat{\lambda} = \int_{\Omega} (|\nabla \hat{u}|^p + \hat{V}(x)\hat{u}^p)\, dx \geq \int_{\Omega} |\nabla \hat{u}|^p\, dx. \tag{3.4}$$

On the other hand, since $\hat{u}$ is a normalized function, we have

$$1 = \int_{\Omega} \hat{g}\hat{u}^p\, dx \leq \|g_0\|_{\infty}, \quad \int_{\Omega} \hat{u}^p\, dx \leq C_p \|g_0\|_{\infty}, \quad \int_{\Omega} |\nabla \hat{u}|^p\, dx,$$

hence
$$\int_\Omega |\nabla \hat{u}|^p \, dx \geq \frac{1}{C_p \|g_0\|_\infty}. \tag{3.5}$$
Thus, from (3.4) and (3.5), we infer
$$\hat{\lambda} \geq \frac{1}{C_p \|g_0\|_\infty}.$$
This, in turn, implies $\hat{\lambda}\hat{g} - \hat{V} > 0$, thanks to the facts that $\hat{g} \geq A$, and $\|V\|_\infty < \frac{A}{C_p(A+\|g\|_\infty)}$. This completes the proof of part (a).

(b) Since $\hat{u} = 0$ on $\partial\Omega$, and $\hat{u} \in C(\overline{\Omega})$, it follows from part (a) that $\{x \in \Omega : \hat{V}(x) > 0\}$ contains a tubular domain around $\partial\Omega$. To show that $\{x \in \Omega : \hat{V}(x) > 0\}$ is connected, it suffices to show that the boundary of every component of the support of $\hat{V}$ must intersect $\partial\Omega$. To derive a contradiction, we assume the contrary. Let us assume E is a component of the support of $\hat{V}$ such that $\partial E \cap \partial\Omega$ is empty. From part (a), we observe that $\partial E \subseteq \{x \in \Omega : \hat{u} = t\}$, for some positive t. We set $w(x) := \hat{u}(x) - t$, so $-\Delta_p w = -\Delta_p \hat{u} = (\hat{\lambda}\hat{g} - \hat{V})\hat{u}^{p-1} > 0$, in E. In addition, $w(x) = 0$, for $x \in \partial E$. Therefore, by the strong maximum principle, we derive $w(x) > 0$, in E. Hence, $\hat{u} > t$ in E, which is a contradiction to the assertion in part (a).

(c) We only show $\phi = \hat{g}^\Delta(\mu_{\hat{u}})$, since $\psi = \hat{g}_\Delta(\mu_{\hat{u}})$ is proved similarly. As in the proof of part (a), one can show that the graph of $\hat{u}$ has no flat sections; that is, sets of the form $\{x \in \Omega : \hat{u} = \beta\}$ have zero measure. Thus, $\hat{u}^\Delta$ is strictly decreasing; hence, the inverse of $\hat{u}^\Delta$ exists and coincides with $\mu_{\hat{u}}$. On the other hand, from (3.1), we infer $\hat{g}^\Delta = \phi(\hat{u})^\Delta$. But, $\phi(\hat{u})^\Delta = \phi(\hat{u}^\Delta)$, since $\phi(\hat{u}) \sim \phi(\hat{u}^\Delta)$. Thus, we have $\hat{g}^\Delta = \phi(\hat{u}^\Delta)$, hence $\hat{g}^\Delta(\mu_{\hat{u}}) = \phi(\hat{u}^\Delta \circ \mu_{\hat{u}}) = \phi$. □

Now we state the main result of this section.

Theorem 3.3. *Suppose Ω is a ball centered at the origin. Suppose that $(\hat{g}, \hat{V})$ is an optimal solution of* (1.3)*. Then*
$$\hat{g} = {g_0}^*, \quad \hat{V} = (V_0)_*, \tag{3.6}$$
modulo sets of measure zero.

Proof. Again we write $\hat{\lambda}$ in place of $\lambda(\hat{g}, \hat{V})$. Recall $\int_\Omega \hat{g}\hat{u}^p \, dx = 1$, hence $1 \leq \int_\Omega \hat{g}^*(\hat{u}^*)^p \, dx =: \gamma$. Let $v = \gamma^{-1/p}\hat{u}^*$, so $\int_\Omega \hat{g}^* v^p \, dx = 1$. We also have
$$\begin{aligned}
\hat{\lambda} &= \int_\Omega |\nabla \hat{u}|^p \, dx + \int_\Omega \hat{V}(x)\hat{u}^p \, dx \\
&\geq \int_\Omega |\nabla \hat{u}^*|^p \, dx + \int_\Omega \hat{V}(x)\hat{u}^p \, dx \\
&\geq \int_\Omega |\nabla \hat{u}^*|^p \, dx + \int_\Omega \hat{V}_*(x)(\hat{u}^*)^p \, dx \\
&= \gamma\Big(\int_\Omega |\nabla v|^p \, dx + \int_\Omega \hat{V}_* v^p \, dx\Big) \\
&\geq \int_\Omega |\nabla v|^p \, dx + \int_\Omega \hat{V}_* v^p \, dx \\
&\geq \lambda(\hat{g}^*, \hat{V}_*) \geq \hat{\lambda},
\end{aligned} \tag{3.7}$$

where in the first inequality we have used Lemma 2.2, and in the second one we have used Lemma 2.1. Therefore all inequalities in (3.7) are in fact equalities. In particular, we infer

$$\begin{gathered}\int_\Omega |\nabla \hat{u}|^p \, dx = \int_\Omega |\nabla \hat{u}^*|^p \, dx, \\ \int_\Omega \hat{V} \hat{u}^p \, dx = \int_\Omega \hat{V}_* (\hat{u}^*)^p \, dx, \\ \int_\Omega \hat{g} \hat{u}^p \, dx = \int_\Omega \hat{g}^* (\hat{u}^*)^p \, dx.\end{gathered}$$

To complete the proof, by [2, Lemma 2.9 and Lemma 2.4 (ii)], it suffices to show that $\hat{u} = \hat{u}^*$. From $\int_\Omega |\nabla \hat{u}|^p \, dx = \int_\Omega |\nabla \hat{u}^*|^p \, dx$, in conjunction with Lemma 2.2, we need to show the set $Q = \{x \in \Omega : \nabla \hat{u} = 0, \ 0 < \hat{u}(x) < M\}$ has zero measure. We will achieve this by showing that in fact Q is empty. To this end, fix $x_0 \in \Omega$, such that $0 < \hat{u}(x_0) < M$. Next we consider the set $N := \{x \in \Omega : \hat{u}(x) \geq \hat{u}(x_0)\}$. Since N is a translation of $\{x \in \Omega : \hat{u}^*(x) \geq \hat{u}(x_0)\}$, see [4], N is a ball. Moreover, because of continuity of $\hat{u}$, $x_0 \in \partial N$. Now let $w(x) := \hat{u}(x) - \hat{u}(x_0)$, hence $-\Delta_p w(x) = -\Delta_p \hat{u}(x) > 0$, in N, thanks to the fact that $(\hat{g}, \hat{V}) \in \mathcal{S}$. Also, $\partial N \subset \{x \in \Omega : \hat{u}(x) = \hat{u}(x_0)\}$. So by the strong maximum principle we see that $w > 0$, in the interior of N. Since $w(x_0) = 0$, we can apply the Hopf boundary point lemma to conclude that $\frac{\partial w}{\partial \nu}(x_0) < 0$, where ν stands for the outward unit normal to ∂N at x_0. This, in turn, implies that $\frac{\partial \hat{u}}{\partial \nu}(x_0) < 0$, hence $\nabla \hat{u}(x_0) \neq 0$. So Q is empty, as desired. $\square$

4. Problem (1.4)

In this section we study problem (1.4), where we address both questions of existence and symmetry. Henceforth we assume $1 \leq p < \infty$, and $p > \frac{2N}{N+2}$, where $N \geq 2$ is the space dimension. We fix two non-negative functions f_0 and h_0 in $L^p(\Omega)$, and as before let $\mathcal{R}(f_0)$ and $\mathcal{R}(h_0)$ denote rearrangement classes generated by f_0 and h_0, respectively. The energy functional $J : L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}$ is defined by:

$$J(f,h) = \frac{1}{2} \int_\Omega f u_f \, dx + \int_\Omega h u_f \, dx.$$

Note that J is finite. Indeed, for $f, g \in L^p(\Omega)$,

$$|J(f,h)| \leq \frac{1}{2} \|f\|_p \|u_f\|_q + \|h\|_p \|u_f\|_q,$$

where q is the conjugate of p; that is, $q = \frac{p}{p-1}$. An application of the embedding $W_0^{1,2}(\Omega) \to L^q(\Omega)$, since $p > \frac{2N}{N+2}$, implies

$$|J(f,h)| \leq \frac{1}{2} C \|f\|_p^2 + C \|h\|_p \|f\|_p, \tag{4.1}$$

where C is a positive constant, thus J is finite. Note that from (4.1), we readily infer that J is bounded on $\mathcal{R}(f_0) \times \mathcal{R}(h_0)$. Now we prove problem (1.4) is solvable.

Theorem 4.1. *Problem* (1.4) *is solvable; that is, there exists* $(\overline{f}, \overline{h}) \in \mathcal{R}(f_0) \times \mathcal{R}(h_0)$ *such that*

$$J(\overline{f}, \overline{h}) = \sup_{f \in \mathcal{R}(f_0),\, h \in \mathcal{R}(h_0)} J(f,h) =: I. \tag{4.2}$$

Proof. Let $(f_n, h_n) \in \mathcal{R}(f_0) \times \mathcal{R}(h_0)$ be a maximizing sequence. Since $\|f_n\|_p = \|f_0\|_p$ and $\|h_n\|_p = \|h_0\|_p$, we infer existence of $\hat{f} \in \overline{\mathcal{R}}(f_0)$ and $\hat{h} \in \overline{\mathcal{R}}(h_0)$ such that

$$f_n \rightharpoonup \hat{f}, \quad h_n \rightharpoonup \hat{h},$$

where "$\rightharpoonup$" stands for weak convergence in $L^p(\Omega)$. Hence we obtain $I = J(\hat{f}, \hat{h})$, since J is weakly sequentially continuous in $L^p(\Omega) \times L^p(\Omega)$. This, in particular, implies that $J(\hat{f}, \hat{h}) \geq J(\hat{f}, h)$, for every $h \in \mathcal{R}(h_0)$. Thus,

$$\frac{1}{2}\int_\Omega \hat{f} u_{\hat{f}}\, dx + \int_\Omega \hat{h} u_{\hat{f}}\, dx \geq \int_\Omega \hat{f} u_{\hat{f}}\, dx + \int_\Omega h u_{\hat{f}}\, dx, \quad \forall h \in \mathcal{R}(h_0).$$

So, $\int_\Omega \hat{h} u_{\hat{f}}\, dx \geq \int_\Omega h u_{\hat{f}}\, dx$, for every $h \in \mathcal{R}(h_0)$. That is to say, $\hat{h}$ maximizes the linear functional $l(h) := \int_\Omega h u_{\hat{f}}\, dx$, relative to $h \in \mathcal{R}(h_0)$. Hence an application of Lemma 2.3 implies existence of $\overline{h} \in \mathcal{R}(h_0)$ such that $l(\overline{h}) \geq l(h)$, relative to $h \in \mathcal{R}(h_0)$. By continuity of $l : L^p(\Omega) \to \mathbb{R}$, we obtain $l(\overline{h}) \geq l(h)$, for every $h \in \overline{\mathcal{R}}(h_0)$, the weak closure of $\mathcal{R}(h_0)$ in $L^p(\Omega)$. In particular, we deduce $l(\overline{h}) \geq l(\hat{h})$, hence we must have $l(\overline{h}) = l(\hat{h})$. Whence

$$J(\hat{f}, \hat{h}) = \frac{1}{2}\int_\Omega \hat{f} u_{\hat{f}}\, dx + \int_\Omega \hat{h} u_{\hat{f}}\, dx = \int_\Omega \hat{f} u_{\hat{f}}\, dx + \int_\Omega \overline{h} u_{\hat{f}}\, dx.$$

Next, we introduce $\Phi : L^p(\Omega) \to \mathbb{R}$ by

$$\Phi(f) = \frac{1}{2}\int_\Omega f u_f\, dx + \int_\Omega \overline{h} u_f\, dx.$$

It is easy to check that Φ is strictly convex, weakly sequentially continuous in $L^p(\Omega)$. Hence, by Lemma 2.4, Φ has a maximizer, say $\overline{f}$, relative to $\mathcal{R}(f_0)$. By weak continuity of Φ we deduce $\Phi(\overline{f}) \geq \Phi(\hat{f})$. Thus,

$$J(\overline{f}, \overline{h}) = \Phi(\overline{f}) \geq \Phi(\hat{f}) = \frac{1}{2}\int_\Omega \hat{f} u_{\hat{f}}\, dx + \int_\Omega \overline{h} u_{\hat{f}}\, dx = J(\hat{f}, \hat{h}) \geq J(\overline{f}, \overline{h}).$$

Therefore, $J(\overline{f}, \overline{h}) = I$. Hence $(\overline{f}, \overline{h}) \in \mathcal{R}(f_0) \times \mathcal{R}(h_0)$ is an optimal solution of the problem (1.4), as desired. $\square$

Corollary 4.2. *Let $(\overline{f}, \overline{h})$ be an optimal solution of* (1.4), *and assume* $|\{x \in: f_0(x) > 0\}| < |\Omega|$. *Then $\partial\{x \in \Omega : \overline{f}(x) > 0\}$, boundary of the support of $\overline{f}$, does not intersect $\partial\Omega$.*

Proof. Since $(\overline{f}, \overline{h})$ is a solution of (1.4), we deduce, in particular, that $J(\overline{f}, \overline{h}) \geq J(f, \overline{h})$, for every $f \in \mathcal{R}(f_0)$. That is, $\overline{f}$ maximizes $J(f, \overline{h})$ relative to $f \in \mathcal{R}(f_0)$. The functional $J(\cdot, \overline{h})$ is strictly convex, and weakly sequentially continuous in $L^p(\Omega)$. For fixed $g \in L^p(\Omega)$, and $t > 0$, it is straightforward to obtain

$$\begin{aligned} J(\overline{f} + tg, \overline{h}) &= J(\overline{f}, \overline{h}) + t\int_\Omega \overline{f} u_g\, dx + \frac{1}{2}t^2 \int_\Omega g u_g\, dx + t\int_\Omega \overline{h} u_g\, dx \\ &= J(\overline{f}, \overline{h}) + t\int_\Omega (\overline{f} + \overline{h}) u_g\, dx + \frac{1}{2}t^2 \int_\Omega g u_g\, dx. \end{aligned}$$

This, in turn, implies, $\partial_1 J(\overline{f}, \overline{h})$, the subdifferential of J at $\hat{f}$, for fixed $\overline{h}$, can be identified with $u_{\overline{f}+\overline{h}}$; note that this is a consequence of the following symmetry:

$$\int_\Omega (\overline{f} + \overline{h}) u_g\, dx = \int_\Omega g u_{\overline{f}+\overline{h}}\, dx.$$

Now we can apply Lemma 2.4 to deduce that $\overline{f} = \phi(u_{\overline{f}+\overline{h}})$, almost everywhere in Ω, for some increasing function ϕ. Hence, the largest values of $u_{\overline{f}+\overline{h}}$ are obtained on $\{x \in \Omega : \overline{f}(x) > 0\}$. On the other hand, we know that $u_{\overline{f}+\overline{h}}$ vanishes on $\partial\Omega$, hence $\partial\{x \in \Omega : \overline{f}(x) > 0\}$ must avoid $\partial\Omega$, as desired. $\square$

We need the following lemma before stating our symmetry result.

Lemma 4.3. *Let Ω be a ball centered at the origin, and $f \in L^p(\Omega)$. Suppose $u_f^*(0) = u_{f^*}(0)$. Then $u_f^*(x) = u_{f^*}(x)$ in Ω.*

Proof. Let us denote the distribution function of u_f by $\mu(t)$; that is,

$$\mu(t) = |\{x \in \Omega : u_f(x) \geq t\}|, \quad 0 \leq t \leq M := \sup_{\overline{\Omega}} u_f.$$

It is well known that the function

$$\xi(t) := \frac{1}{N^2 C_N^{2/N}} (-\mu'(t))\mu(t)^{-2+2/N} \int_0^{\mu(t)} f^{\Delta}(s)\, ds,$$

where C_N is the measure of the the N-dimensional unit ball, satisfies $\xi(t) \geq 1$ and

$$\int_0^{u_f^*(x)} \xi(t)\, dt = u_{f^*}(x), \tag{4.3}$$

see [12]. We claim $\xi(t) \equiv 1$. To prove the claim we assume the contrary and derive a contradiction. To this end, we assume $\xi(t) > 1$ on a set of positive measure. Then, from (4.3), we obtain

$$u_{f^*}(0) = \int_0^{u_f^*(0)} \xi(t)\, dt > u_f^*(0).$$

This is a contradiction to our hypothesis that $u_{f^*}(0) = u_f^*(0)$. Thus we must have $\xi(t) \equiv 1$. This, in conjunction with (4.3), implies

$$u_{f^*}(x) = \int_0^{u_f^*(x)} \xi(t)\, dt = \int_0^{u_f^*(x)} dt = u_f^*(x).$$

This completes the proof of the lemma. $\square$

Our symmetry result is the following, compare with [3].

Theorem 4.4. *Let Ω be a ball centered at the origin. Let (f, h) be an optimal solution of (1.4). Then*

$$f = f_0^*, \quad h = h_0^*. \tag{4.4}$$

In particular, we deduce that (1.4) has a unique solution, (f_0^*, h_0^*).

Proof. From [12], we know $u_f^* \le u_{f^*}$. Hence, we have

$$
\begin{aligned}
J(f,h) &= \int_\Omega f u_f \, dx + \int_\Omega h u_f \, dx \\
&\le \int_\Omega f^* u_f^* \, dx + \int_\Omega h u_f \, dx \\
&\le \int_\Omega f^* u_{f^*} \, dx + \int_\Omega h u_f \, dx \\
&\le \int_\Omega f^* u_{f^*} \, dx + \int_\Omega h^* u_f^* \\
&\le \int_\Omega f^* u_{f^*} \, dx + \int_\Omega h^* u_{f^*} \, dx \\
&= J(f^*, h^*) \le J(f,h)
\end{aligned}
\tag{4.5}
$$

where in the first and third inequalities we have used Lemma 2.1, whereas the last inequality follows from the optimality of (f,h). Hence, all inequalities in (4.5) are in fact equalities. This, in turn, implies

$$
\int_\Omega f^* u_f^* \, dx = \int_\Omega f^* u_{f^*} \, dx, \tag{4.6}
$$

$$
\int_\Omega h^* u_f^* \, dx = \int_\Omega h^* u_{f^*}. \tag{4.7}
$$

From (4.6), we derive $\int_\Omega f^*(u_{f^*} - u_f^*) \, dx = 0$. So, we must have $u_{f^*}(x) = u_f^*(x)$, for every x in the support of f^*, thanks to the fact that $u_f^* \le u_{f^*}$. Hence, in particular, $u_{f^*}(0) = u_f^*(0)$. Now we can apply Lemma 4.3 to deduce $u_f^* = u_{f^*}$, in Ω. Note that from (4.5), we get $\int_\Omega f u_f \, dx = \int_\Omega f^* u_f^* \, dx$. This coupled with $u_f^* = u_{f^*}$ yield $\int_\Omega f u_f \, dx = \int_\Omega f^* u_{f^*} \, dx$. This clearly implies $\int_\Omega |\nabla u_f|^2 \, dx = \int_\Omega |\nabla u_{f^*}|^2 \, dx = \int_\Omega |\nabla u_f^*|^2 \, dx$. Similarly to the proof of Theorem 3.3, one can show that the set $\{x \in \Omega : \nabla u_f(x) = 0,\ 0 < u_f(x) < M\}$ is empty, hence its measure is zero. Therefore by Lemma 2.2, we infer $u_f = u_f^*$. Whence, $u_f = u_{f^*}$, which implies that $f = f^* = f_0^*$. Along the same lines as above one can show that (b) implies $h = h^* = h_0^*$, as desired. □

References

[1] Burton, G. R.; *Rearrangements of functions, maximization of convex functionals, and vortex rings*. Math. Ann. 276 (1987), no. 2, 225–253.

[2] Burton, G. R.; *Variational problems on classes of rearrangements and multiple configurations for steady vortices*. Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 4, 295–319.

[3] Burton, G. R.; McLeod, J. B.; *Maximisation and minimisation on classes of rearrangements*. Proc. Roy. Soc. Edinburgh Sect. A 119 (1991), no. 3-4, 287–300.

[4] Brothers, John E.; Ziemer, William P.; *Minimal rearrangements of Sobolev functions*. J. Reine Angew. Math. 384 (1988), 153–179.

[5] Cuccu, Fabrizio; Emamizadeh, Behrouz; Porru, Giovanni; *Optimization of the first eigenvalue in problems involving the p-Laplacian*. Proc. Amer. Math. Soc. 137 (2009), no. 5, 1677–1687.

[6] Cuccu, Fabrizio; Emamizadeh, Behrouz; Porru, Giovanni; *Nonlinear elastic membranes involving the p-Laplacian operator*. Electron. J. Differential Equations 2006, No. 49, 1-10.

[7] L. Del Pezzo and J. Fernandez Bonder; *An optimization problem for the first weighted eigenvalue problem plus a potential*, preprint.

[8] L. Del Pezzo and J. Fernandez Bonder; *Remarks on an optimization problem for the p-Laplacian*. To appear in Applied Mathematical Letters.

[9] L. Del Pezzo and J. Fernandez Bonder; *Some optimization problems for p-Laplacian type equations*. Appl. Math. Optim., 59 (2009), 365–381.

[10] Hardy, G. H.; Littlewood, J. E.; Pólya, G.; *Inequalities*. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. xii+324 pp.

[11] Nycander, Jonas; Emamizadeh, Behrouz; *Variational problem for vortices attached to seamounts*. Nonlinear Anal. 55 (2003), no. 1-2, 15–24.

[12] Talenti, Giorgio; *Elliptic equations and rearrangements*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3 (1976), no. 4, 697–718.

Behrouz Emamizadeh
Department of Mathematics, The Petroleum Institute, Abu Dhabi, UAE
E-mail address: bemamizadeh@pi.ac.ae

Jyotshana V. Prajapat
Department of Mathematics, The Petroleum Institute, Abu Dhabi, UAE
E-mail address: jprajapat@pi.ac.ae