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L_p-REGULARITY OF SOLUTIONS TO FIRST INITIAL-BOUNDARY VALUE PROBLEM FOR HYPERBOLIC EQUATIONS IN CUSP DOMAINS

NGUYEN MANH HUNG, VU TRONG LUONG

Abstract. In this article, we establish well-posedness and L_p-regularity of solutions to the first initial-boundary value problem for general higher order hyperbolic equations in cylinders whose base is a cusp domain.

1. Introduction

Initial boundary-value problems for hyperbolic and parabolic type equations in a cylinder with base containing conical point have been studied by many authors [8, 9, 10, 13, 14]. The main results are about the uniqueness and existence of the solutions, and asymptotic expansions of the solution near a neighborhood of conical point. Those results are mainly based on Galerkin's approximate method and L_2-theory.

Boundary-value problems for elliptic type equations and systems have also well studied. The main results, presented in [6, 15, 19, 20], established estimates in L_p for solutions of elliptic boundary value problems in domains with singular points on the boundary.

The question is whether similar results can be obtained based on these results for initial boundary-value problems for non-stationary equations. In this paper, we find the answer for this question.

Firstly, we show the existence of a sequence of smooth domains $\{\Omega^\epsilon\}_{\epsilon>0}$ such that $\Omega^\epsilon \subset \Omega$ and $\lim_{\epsilon\to 0} \Omega^\epsilon = \Omega$. Furthermore, we proved existence, uniqueness and smoothness, with respect to time variable, of the generalized solution by approximating boundary method, which can be applied for non-linear equations. Next, by modifying the arguments in [19], we take the term containing the derivative in time of the unknown function to the right-hand side of the equation, such that the problem can be considered as an elliptic problem. With the help of some auxiliary results, we apply the estimates in L_p for solution of the elliptic boundary value problem and our previous estimates to deal with the L_p-regularity with respect to both of time and spatial variables of the solution. Finally, in order to illustrate the

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results above we show an example for the Cauchy-Dirichlet problem for the beam equation in cylinder with base containing a cuspidal point.

2. Preliminaries

Let Ω be bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$. Let p, q be real numbers with $1 < p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We denote by $W_p^m(\Omega)$ the space of all $u = u(x), x \in \Omega$ that have generalized derivatives $D^\alpha u \in L_p(\Omega)$, $|\alpha| \leq m$. The norm in this space is defined as

$$\|u\|_{m;p} = \Big(\int_\Omega \sum_{|\alpha|=0}^{m} |D_x^\alpha u|^p \, dx \Big)^{1/p}.$$

In particular, $\mathring{W}_p^0(\Omega) \equiv L_p(\Omega)$. The space $\mathring{W}_p^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in norm of the space $W_p^m(\Omega)$.

Setting $Q_T = \Omega \times (0, T)$, $0 < T < +\infty$. We introduce the partial differential operator of order $2m$,

$$L = L(x, t; D_x) = \sum_{|\alpha|,|\beta|=0}^{m} D_x^\alpha \Big(a_{\alpha\beta}(x, t) D_x^\beta \Big), \tag{2.1}$$

where $D_x^\alpha = i^\alpha \partial_x^\alpha$, $a_{\alpha\beta}$ are $s \times s-$ matrices of functions with complex values, and $a_{\alpha\beta}$ are infinitely differentiable in $\overline{Q}_T$ and $a_{\alpha\beta} = {a^*}_{\alpha\beta}$, where ${a^*}_{\alpha\beta}$ denotes the transposed conjugate matrix of $a_{\alpha\beta}$. We have the following Green's formula

$$\int_\Omega Lu \, \overline{v} \, dx = B[u, v; t]$$

which is valid for all $u, v \in C_0^\infty(\Omega)$ and a.e. $t \in [0, T)$, where

$$B[u, v; t] = \sum_{|\alpha|,|\beta|=0}^{m} \int_\Omega a_{\alpha\beta}(., t) D_x^\beta u \, \overline{D_x^\alpha v} \, dx.$$

We also assume the Garding's inequality,

$$B[u, u\,; t] \geq \gamma_0 \|u\|^2_{W_2^m(\Omega)} \tag{2.2}$$

which is valid for all $u \in \mathring{W}_2^m(\Omega)$ and a.e. $t \in [0, T)$, where γ_0 is a positive constant independent of u and t.

Now we introduce spaces on Q_T. Let $W_p^{m,1}(Q_T)$ be the space consisting of functions $u = u(x, t), (x, t) \in Q_T$ having generalized derivatives $D^\alpha u \in L_p(Q_T), |\alpha| \leq m$, and $u_t \in L_p(Q_T)$, with norm

$$\|u\|_{m,1;p} = \Big(\int_{Q_T} \sum_{|\alpha|=0}^{m} |D_x^\alpha u|^p \, dx \, dt + \int_{Q_T} |u_t|^p \, dx \, dt \Big)^{1/p}.$$

The space $\mathring{W}_p^{m,1}(Q_T)$ is the closure in $W_p^{m,1}(Q_T)$ of the set consisting of all functions in $C^\infty(Q_T)$, which vanish near S_T denoting by $C_0^\infty(Q_T)$ for convenience.

We introduce the space $W_p^{-m,-1}(Q_T)$ of generalized functions on Q_T; it means that if $f \in W_p^{-m,-1}(Q_T)$, the f admits the representation

$$f = \sum_{|\alpha| \leq m} D_x^\alpha f^{(\alpha)} + f_t^{(t)} \tag{2.3}$$

where $f^{(\alpha)}, f^{(t)} \in L_p(Q_T)$, $p \in (1,+\infty)$. The norm in $W_p^{-m,-1}(Q_T)$ can also be defined by
$$\|f\|_{-m,-1;p} = \inf \sum_{|\alpha|\le m} \|f^{(\alpha)}\|_{L_p(Q_T)} + \|f_t^{(t)}\|_{L_p(Q_T)}.$$
Here the infimum is taken over the set of all representations (2.3). It is known that $W_p^{-m,-1}(Q_T)$ and $\mathring{W}_q^{m,1}(Q_T), q = \frac{p}{p-1}$, are dual to one another. We also define
$$\langle f,\eta\rangle = \int_{Q_T} f\overline{\eta}\,dx\,dt, \quad f \in W_p^{-m,-1}(Q_T),\ \eta \in \mathring{W}_q^{m,1}(Q_T).$$
It is clear that
$$\|f\|_{-m,-1;p} = \sup\{|\langle f,\eta\rangle| : \eta \in \mathring{W}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} = 1\};$$
$$\|\eta\|_{m,1;q} = \sup\{|\langle f,\eta\rangle| : f \in W_p^{-m,-1}(Q_T), \|f\|_{-m,-1;p} = 1\}.$$
In this paper, we consider the problem
$$Lu - u_{tt} = f \text{ in } Q_T, \tag{2.4}$$
$$u = 0, u_t = 0 \text{ on } \Omega, \tag{2.5}$$
$$\partial_\nu^j u = 0 \text{ on } S_T, j = 0,1,\dots,m-1, \tag{2.6}$$
where $f : Q_T \to \mathbb{C}$ is a given function and $\partial_\nu^j u$ are derivatives with respect to the outer unit normal of $S_T = \partial\Omega \times (0,T)$. Setting
$$B_1[u,\eta] = \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} D^\beta u\, \overline{D^\alpha \eta}\,dx\,dt + \int_{Q_T} u_t \overline{\eta_t}\,dx\,dt.$$
for all $u \in \mathring{W}_p^{m,1}(Q_T), \eta \in \mathring{W}_q^{m,1}(Q_T)$.

Definition 2.1. Let $f \in W_p^{-m,-1}(Q_T)$; a function u is called a generalized L_p-solution of problem (2.4)–(2.6) if and only if u belongs to $\mathring{W}_p^{m,1}(Q_T), u(x,0) = u_t(x,0) = 0$, and the equality
$$B_1[u,\eta] = \langle f,\eta\rangle \tag{2.7}$$
holds for all $\eta \in \mathring{W}_q^{m,1}(Q_T)$.

To prove uniqueness of the generalized L_p-solution of (2.4) -(2.6), we need to prove the following lemma.

Lemma 2.2. *If $1 < p \le 2$, then there exists a constant $\gamma_2 = \gamma_2(p,n,m,|\Omega|,T) > 0$, such that*
$$\sup\{\big|B_1[u,\eta]\big| : \eta \in \mathring{W}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \le 1\} \ge \gamma_2 \|u\|_{m,1;p}, \tag{2.8}$$
for all $u \in \mathring{W}_p^{m,1}(Q_T)$.

Proof. We prove this result with $u \in C_0^\infty(Q_T)$. Suppose that there is no $\gamma_2 > 0$ such that (2.8) holds. Then there is a sequence $\{u_k\} \subset C_0^\infty(Q_T)$ with $\|u_k\|_{m,1;p} = 1$ and
$$\sup\{\big|B_1[u_k,\eta]\big| : \eta \in \mathring{W}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \le 1\} \le \frac{1}{k}, \quad \text{for every } k \ge 1. \tag{2.9}$$
Using Garding' inequality (2.2), we obtain
$$\big|B_1[u_k,u_k]\big| \ge \gamma_0 \|u_k\|_{m,0;2}^2 + \int_{Q_T} |u_{kt}|^2\,dx\,dt \ge c_1 \|u\|_{m,1;2}^2. \tag{2.10}$$

On the other hand, by using Hölder's inequality with $1 < p < 2$, $p^* = \frac{2}{p}$, $q^* = \frac{2}{2-p}$, we have

$$\|u_k\|_{m,1;p}^p = \sum_{|\alpha|=0}^{m} \int_{Q_T} |D_x^\alpha u|^p \,dx\,dt + \int_{Q_T} |u_t|^p \,dx\,dt \le C_2 \|u_k\|_{m,1;2}^p, \tag{2.11}$$

where $C_2 = C_2(p, |\Omega|, T) > 0$. Combining (2.10) and (2.11), we obtain

$$\big|B_1[u_k, u_k]\big| \ge C\|u_k\|_{m,1;p}^2,$$

where c is a constant independent of k. From the above inequality and (2.9), we have

$$\|u_k\|_{m,1;p}^2 \le \frac{1}{k\,C}, \quad \text{for } k = 1, 2, \dots .$$

which contradicts $\|u_k\|_{m,1;p} = 1$. Therefore, there is a constant $\gamma_2 > 0$ such that (2.8) holds. Since $u \in C_0^\infty(Q_T)$ which is dense in $\mathring{W}_p^{m,1}(Q_T)$, this completes the proof. □

Lemma 2.2 implies the uniqueness of generalized L_p-solution, according to the following theorem.

Theorem 2.3. *Assume that coefficients of operator* (2.1) *satisfy* (2.2) *and* $f \in W_p^{-m,-1}(Q_T)$. *Then* (2.4) -(2.6) *has at most one generalized* L_p*-solution.*

Proof. Firstly, we prove the theorem in the case $1 < p \le 2$. Suppose that (2.4)-(2.6) has two generalized L_p-solutions u_1, u_2. Put $u = u_1 - u_2$, then (2.7) implies that

$$B_1[u, \eta] = \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta}(x,t) D_x^\beta u \overline{D_x^\alpha \eta}\,dx\,dt + \int_{Q_T} u_t\,\overline{\eta}_t\,dx\,dt = 0$$

holds for all $\eta \in \mathring{W}_q^{m,1}(Q_T)$. Combining inequality (2.8) with the above equality, we obtain

$$\gamma_2 \|u\|_{m,1;p} \le \sup\{\big|B_1[u,\eta]\big| : \eta \in \mathring{W}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \le 1\} = 0.$$

Next, we prove the theorem in the case $p > 2$. Since $p > 2$, and Q_T is bounded, we have $\mathring{W}_p^{m,1}(Q_T) \hookrightarrow \mathring{W}_2^{m,1}(Q_T)$. Therefore, if u is a generalized L_p-solution, and then u is a generalized L_2-solution. We obtain the uniqueness of a generalized L_p-solution from the uniqueness of a generalized L_2-solution. Hence, $u \equiv 0$ in Q_T. This completes the proof of theorem. □

Next, we prove the approximate boundary lemma, which is the essential tool in solving (2.4) -(2.6).

Lemma 2.4 ([12]). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*; then there exists a sequence of smooth domains* $\{\Omega^\varepsilon\}$ *such that* $\Omega^\varepsilon \subset \Omega$ *and* $\lim_{\varepsilon\to 0} \Omega^\varepsilon = \Omega$.

Proof. For $\varepsilon > 0$ arbitrary, set $S^\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}, \Omega^\varepsilon = \Omega \setminus S^\varepsilon$ and $\partial\Omega^\varepsilon$ is the boundary of Ω^ε. Denote by $J(x)$ the characteristic function of Ω^ε and by $J_h(x)$ the mollification of $J(x)$; i.e.,

$$J_h(x) = \int_{\mathbb{R}^n} \theta_h(x-y) J(y) dy,$$

where θ_h is a mollifier. If $h < \varepsilon/2$, then $J_h(x)$ has following properties:

(1) $J_h(x) = 0$ if $x \notin \Omega^{\frac{\varepsilon}{2}}$;
(2) $0 \le J_h(x) \le 1, \forall x \in \Omega$;

(3) $J_h(x) = 1$ in $\Omega^{2\varepsilon}$;
(4) $J_h \in C_0^\infty(\mathbb{R}^n)$.

We now fix a constant $c \in (0,1)$, set $\Omega_c^\varepsilon = \{x \in \Omega : J_h(x) > c\}$. It is obvious that $\Omega^{\frac{\varepsilon}{2}} \supset \Omega_c^\varepsilon \supset \Omega^{2\varepsilon}$. Therefore, $\Omega_c^\varepsilon \subset \Omega$ and $\lim_{\varepsilon\to 0} \Omega_c^\varepsilon = \Omega$.

Assume that K is the critical set of J_h, i.e. K consisting of all point x, such that the gradient of J_h at x vanishes. A number $c \in \mathbb{R}$ such that $J_h^{-1}(c)$ contains at least one $x \in K$ is called a critical value. By Sard's theorem then the set of critical values of J_h is of measure zero (see[2, Theorem 1.30]), it implies that there exists a constant $c_0 \in (0,1)$ such that c_0 is not a critical value of J_h. Denote $\Omega_{c_0}^\varepsilon = \{x \in \Omega : J_h(x) > c_0\}$ and $F(x) = J_h(x) - c_0$. For all $x^0 \in \partial\Omega_{c_0}^\varepsilon$, then $F(x^0) = J_h(x^0) - c_0 = 0$ and vector $\mathrm{grad} J_h(x^0) \neq 0$. This implies that there exists a $\frac{\partial J_h}{\partial x_i}(x^0) \neq 0$, without loss of generality we can suppose that $\frac{\partial J_h}{\partial x_n}(x^0) \neq 0$. Using the implicit function theorem, we obtain that there exists a neighborhood $\mathcal{W}$ of $(x_1^0, \dots, x_{n-1}^0)$ in $\mathbb{R}^{n-1}$ a neighborhood $\mathcal{V}$ of x_n^0 in $\mathbb{R}$ and an infinitely differentiable function $z : \mathcal{W} \to \mathbb{R}$ such that $x \in \mathcal{U}_{x^0} \cap \partial\Omega_{c_0}^\varepsilon$, where $\partial\Omega_c^\varepsilon = \{x \in \Omega : J_h(x) = c\}$, $\mathcal{U}_{x^0} = \mathcal{W} \times \mathcal{V}$, if and only if $x = (x_1, \dots, x_n) \in U_{x^0}$, $x_n = z(x_1, \dots, x_{n-1})$. Hence, $\Omega_{c_0}^\varepsilon$ is smooth and $\lim_{\varepsilon\to 0} \Omega_{c_0}^\varepsilon = \Omega$. The lemma proved. □

Suppose that $\{\Omega^\epsilon\}$ is a smooth domain subsequence and $\lim_{\varepsilon\to 0} \Omega^\varepsilon = \Omega$. Set $Q_T^\epsilon = \Omega^\epsilon \times (0,T), S_T^\epsilon = \partial\Omega^\epsilon \times (0,T)$. It is known that the problem

$$\begin{gathered} Lu - u_{tt} = f \text{ in } Q_T^\epsilon, \\ u = 0, u_t = 0 \text{ on } \Omega^\epsilon, \\ \partial_\nu^j u = 0 \text{ on } S_T^\epsilon, j = 0, 1, \dots, m-1, \end{gathered}$$

has a unique function $u^\epsilon(x,t) \in C^\infty(\overline{Q_T^\epsilon})$; if $f \in C^\infty(\overline{Q_T^\epsilon}), f_{t^k}\big|_{t=0} = 0$, for $k = 0, 1, \dots$. Moreover, $u^\epsilon(.,t) \in \overset{\circ}{W}{}_2^m(\Omega^\epsilon)$, for all $t \in [0,T]$, (see[5, 18, 17]).

3. Main results

3.1. **Existence of generalized L_p-solutions.** In this subsection, we prove the existence of generalized L_p-solution. Firstly, we prove the needed following propositions:

Proposition 3.1. *Suppose that $1 < p \le 2$ and $f \in C^\infty(\overline{Q_T})$, and $f_{t^k}\big|_{t=0} = 0$, for $k = 0, 1, \dots$; then u^ϵ is a generalized L_p-solution of (2.4)-(2.5) in Q_T^ϵ satisfying*

$$\|u^\epsilon\|_{m,1;p} \le C\|f\|_{-m,-1;p}$$

where the constant C is independent of ϵ, u and f.

Proof. From u^ϵ satisfying system (2.4) in Q_T^ϵ; i. e.,

$$f = Lu^\epsilon - u_{tt}^\epsilon, \text{ in } Q_T^\epsilon,$$

we have

$$\langle f, \eta\rangle = \int_{Q_T^\epsilon} Lu^\epsilon\, \overline{\eta}\, dx\, dt - \int_{Q_T^\epsilon} u_{tt}^\epsilon\, \overline{\eta}\, dx\, dt$$

valid for all $\eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T^\epsilon)$.

By using Green's formula and integrating by parts with respect to t, we obtain from the equality above that

$$B_1[u^\epsilon, \eta] = \langle f, \eta\rangle \tag{3.1}$$

valid for all $\eta \in \mathring{W}_q^{m,1}(Q_T^\epsilon)$. This clearly shows that u^ϵ is a generalized L_p-solutions of problem (2.4) -(2.5) in Q_T^ϵ; otherwise, using inequality (2.8), we conclude from (3.1) that

$$\|u^\epsilon\|_{m,1;p} \le C\|f\|_{-m,-1;p}.$$

□

Now we prove the existence of the generalized L_p-solution of (2.4)-(2.6) in Q_T, when the assumptions of Proposition 3.1 are satisfied.

Proposition 3.2. *Let the following hypothesis be satisfied:*

- (i) $1 < p \le 2$,
- (ii) $f \in C^\infty(\overline{Q_T})$, *and* $f_{t^k}\big|_{t=0} = 0$, *for* $k = 0, 1, \dots$

Then (2.4)-(2.6) *in cylinder* Q_T *has a generalized* L_p*-solution* $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ *which satisfies*

$$\|u\|_{m,1;p} \le C\|f\|_{-m,-1;p} \tag{3.2}$$

where C *is a constant independent of* u *and* f.

Proof. By Proposition 3.1 we have

$$\|u^\epsilon\|_{m,1;p} \le C\|f\|_{-m,-1;p} \tag{3.3}$$

where the constant C does not depend on ϵ. Setting $\widetilde{u}^\epsilon = u^\epsilon$ in Q_T^ϵ, and vanishes outside Q_T^ϵ. From the inequality above we obtain

$$\|\widetilde{u^\epsilon}\|_{m,1;p} \le C\|f\|_{-m,-1;p} \tag{3.4}$$

where the constant C does not depend on ϵ.

It implies that the set $\{\widetilde{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in the space $\mathring{W}_p^{m,1}(Q_T)$. So we can take a subsequence, denoted also by $\widetilde{u}^\varepsilon$ for convenience, which converges weakly to a function $u \in \mathring{W}_p^{m,1}(Q_T)$. We will show that u is a generalized L_p-solution of (2.4)-(2.6) in cylinder Q_T. In fact for all $\eta \in \mathring{W}_q^{m,1}(Q_T)$, there exists $\eta_\delta \in C_0^\infty(Q_T)$ such that $\eta_\delta \equiv 0$ in $Q_T \setminus Q_T^\varepsilon$, and $\|\eta_\delta - \eta\|_{m,1;q} \to 0$ when $\delta \to 0$. Since $\widetilde{u}^\varepsilon$ is a generalized solution of (2.4)-(2.6) in the smooth cylinder Q_T^ε, we have

$$B_1[\widetilde{u}^\varepsilon, \eta_\delta] = \langle f, \eta_\delta\rangle$$

Passing to the limit when $\varepsilon \to 0, \delta \to 0$ for the weakly convergent sequence, we get

$$B_1[u, \eta] = \langle f, \eta\rangle$$

Since $\mathring{W}_p^{m,1}(Q_T)$ is imbedded continuously into $L_p(\Omega)$, the trace sequence $\{\widetilde{u}^\varepsilon(x,0)\}$ of $\{\widetilde{u}^\varepsilon(x,t)\}$ converges weakly to the trace $u(x,0)$ of $u(x,t)$ in $L_p(\Omega)$. On the other hand, $\widetilde{u}^\varepsilon(x,0) = 0$, so that $u(x,0) = 0$;by analogous arguments, we have $u_t(x,0) = 0$. Hence, $u(x,t)$ is a generalized L_p-solution of (2.4)-(2.6). Moreover, from (3.4) we have

$$\|u\|_{m,1;p} \le \varliminf_{\varepsilon \to 0} \|\widetilde{u}^\varepsilon\|_{m,1;p} \le C\|f\|_{-m,-1;p}.$$

□

Proposition 3.2 stated the existence of generalized L_p-solutions of (2.4)-(2.6) in $\mathring{W}_p^{m,1}(Q_T)$ when $f \in C^\infty(\overline{Q}_T)$ and $f_{t^k}\big|_{t=0} = 0$, for $k = 0, 1, \dots$. We now establish the problem when $f \in W_p^{-m,-1}(Q_T)$.

Theorem 3.3. *Suppose that* $f \in W_p^{-m,-1}(Q_T), p \in (1, +\infty)$*, then* (2.4)-(2.6) *has a generalized* L_p*-solution* $u \in \mathring{W}_p^{m,1}(Q_T)$*, and*

$$\|u\|_{m,1;p}^p \leq C\|f\|_{0,p}^p, \tag{3.5}$$

where C *is a constant independent of* u *and* f*.*

Proof. We start by studying the case $1 < p \leq 2$. Denote

$$f_h(x,t) = \begin{cases} 0, & \text{outside } Q_T^\epsilon \\ f(x,t), & t > h \\ 0, & t \leq h \end{cases}$$

for all $h > 0$. We denote by $g_{\frac{h}{2}}$ the mollification of f_h. Then $g_{\frac{h}{2}} \in C_0^\infty(Q_T), g_{\frac{h}{2}} \equiv 0, t < \frac{h}{2}$ and $g_{\frac{h}{2}} \to f$ in $W_p^{-m,-1}(Q_T)$. By Proposition 3.2, problem (2.4)-(2.6) has a generalized L_p-solution $u_h \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ with replacing f by $g_{\frac{h}{2}}$, and the following estimates holds

$$\|u_h\|_{m,1;p} \leq C\|g_{\frac{h}{2}}\|_{-m,-1;p} \tag{3.6}$$

where C is a constant independent of h, u and f. Since $\{g_{\frac{h}{2}}\}$ is a Cauchy sequence in $L_p(Q_T)$ and inequality (3.6), it follows that $\{u_h\}$ is a Cauchy sequence in $\overset{\circ}{W}{}_p^{m,1}(Q_T)$. Hence, $u_h \to u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$, then u is a generalized L_p-solutions of (2.4)-(2.6) and satisfies

$$\|u\|_{m,1;p} \leq C\|f\|_{-m,-1;p}\,.$$

Thus, the theorem is proved in the case $1 < p \leq 2$.

Now we study the case $p > 2$. It is clear that $q = \frac{p}{p-1} \in (1, 2)$; by the proof above, for any $g \in W_q^{-m,-1}(Q_T)$ there exists a solution $v \in \mathring{W}_q^{m,1}(Q_T)$ of the adjoint problem

$$B_1[v, u] = \langle g, u\rangle \tag{3.7}$$

for all $u \in \mathring{W}_p^{m,1}(Q_T)$, and

$$\|v\|_{m,1;q} \leq C\|g\|_{-m,-1;q}.$$

We suppose that $f \in C^\infty(\overline{Q_T})$, $f_{t^k}(x, 0) = 0$, $k = 0, 1, \dots$ and for $u = u^\epsilon$ in (3.7). Then, by (3.7), we have

$$\begin{aligned} |\langle g, u^\epsilon\rangle| &= |B_1[v, u^\epsilon]| = |\overline{B_1[u^\epsilon, v]}| = |\langle f, v\rangle| \\ &\leq \|f\|_{-m,-1;p}\|v\|_{m,1;q} \\ &\leq C\|f\|_{-m,-1;p}\|g\|_{-m,-1;q} \end{aligned}$$

for any $g \in W_q^{-m,-1}(Q_T)$. This implies

$$\|u^\epsilon\|_{m,1;p} = \sup\Big\{\frac{|\langle g, u^\epsilon\rangle|}{\|g\|_{-m,-1;q}} : 0 \neq g \in W_q^{-m,-1}(Q_T)\Big\} \leq C\|f\|_{-m,-1;p}.$$

From this inequality and arguments analogous to proofs above, we get the proof of the theorem in this case. The proof is complete. □

We should remark that by replacing the condition $f \in W_p^{-m,-1}(Q_T)$ by condition $f \in L_p(Q_T)$, and noting that

$$\|f\|_{W_p^{-m,-1}(Q_T)} \leq \|f\|_{L_p(Q_T)},$$

we obtain the following theorem.

Theorem 3.4. *If $f \in L_p(Q_T), p \in (1, +\infty)$, then (2.4)-(2.6), in the cylinder Q_T, has a generalized L_p-solution $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ which satisfies*

$$\|u\|_{m,1;p} \le C\|f\|_{L_p(Q_T)}$$

where C is a constant independent of u and f.

3.2. **Smoothness of the generalized L_p-solution with respect to time.** The following theorem shows that the generalized L_p-solution $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ of problem (2.4)-(2.6) is smooth with respect to time variable t if right hand-side f and coefficients of operator (2.1) are smooth enough with respect to t.

Theorem 3.5. *Let h be a positive integer, and assume that*

(1) $f_{t^k} \in L_p(Q_T)$, $k \le h$,
(2) $f_{t^k}\big|_{t=0} = 0, x \in \Omega$, $k \le h-1$,
(3) $\sup\left\{\left|\frac{\partial^k a_{\alpha\beta}}{\partial t^k}\right|, k < h+1 : (x,t) \in Q_T, 0 \le |\alpha|, |\beta| \le m\right\} \le \mu$.

Then the generalized solution $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ of (2.4)-(2.6) has generalized derivatives with respect to t up to order h in $\overset{\circ}{W}{}_p^{m,1}(Q_T)$ and satisfies the estimate

$$\|u_{t^h}\|_{m,1;p} \le c\sum_{k=0}^{h} \|f_{t^k}\|_{L_p(Q_T)}, \tag{3.8}$$

where c is a constant independent of u and f.

Proof. In the case $1 < p \le 2$, Clearly, we needed only to show that

$$\|u^\epsilon_{t^h}\|_{m,1;p} \le \sum_{k=0}^{h} \|f_{t^k}\|_{L_p(Q_T)} \tag{3.9}$$

where $f \in C^\infty(\overline{Q_T}), f_{t^k}(x,0) = 0, x \in \Omega$. It is proved by induction on h. According to Proposition 3.1, inequality (3.9) is valid for $h = 0$. Now let it be true for $h-1$; we will prove that this also holds for h.

From the fact that u^ϵ satisfies (2.4) in Q^ϵ_T, we have

$$f = Lu^\epsilon - u^\epsilon_{tt}. \tag{3.10}$$

Differentiating equality (3.10), h times with respect to t, it follows that

$$f_{t^h} = Lu^\epsilon_{t^h} + \sum_{k=0}^{h-1}\binom{h-1}{k} D^\alpha_x(a_{\alpha\beta t^{h-k}} D^\beta_x u^\epsilon_{t^k}) - u^\epsilon_{t^{h+2}}.$$

Therefore,

$$\begin{aligned}\langle f_{t^h}, v\rangle = &\int_{Q_T} Lu_{t^h} v\,dx\,dt + \sum_{k=0}^{h-1}\binom{h-1}{k}\int_{Q_T}\sum_{|\alpha|,|\beta|=0}^{m} D^\alpha_x(a_{\alpha\beta t^{h-k}} D^\beta_x u^\epsilon_{t^k})\overline{v}\,dx\,dt\\ &- \int_{Q_T} u^\epsilon_{t^{h+1}}\overline{v}\,dx\,dt\end{aligned}$$

for all $v \in \overset{\circ}{W}{}_q^{m,1}(Q_T)$.

By using Green's formula and integrating by parts,

$$B_1[u^\epsilon_{t^h}, v] = \langle f_{t^h}, v\rangle - \sum_{k=0}^{h-1}\binom{h-1}{k}\int_{Q_T}\sum_{|\alpha|,|\beta|=0}^{m} a_{\alpha\beta t^{h-k}} D^\beta_x u^\epsilon_{t^k}\overline{D^\alpha_x v}\,dx\,dt.$$

for all $v \in \mathring{W}_q^{m,1}(Q_T)$.

From the inequality above and Hölder's inequality, we have

$$|B_1[u^\epsilon_{t^h}, v]| \le C(\|f_{t^h}\|_{L_p(Q_T)} + \sum_{k=0}^{h-1} \|u^\epsilon_{t^k}\|_{m,1;p})\|v\|_{m,1;q} \tag{3.11}$$

for all $v \in \mathring{W}_q^{m,1}(Q_T)$. By using (2.8), (3.11) and the induction assumption, we obtain

$$\|u^\epsilon_{t^h}\|_{m,1;p} \le C \sum_{k=0}^{h} \|f_{t^k}\|_{L_p(Q_T)}$$

where C is a constant independent of ϵ, u. The proof is completed in this case.

In the case $p > 2$. It is easy to recognize that $q = \frac{p}{p-1} \in (1,2)$; by Theorem 3.3, for any $g \in W_q^{-m,-1}(Q_T)$ there exists a solution $v \in \mathring{W}_q^{m,1}(Q_T)$ of the adjoint problem

$$B_1[v, u] = \langle g, u\rangle \tag{3.12}$$

which for all $u \in \mathring{W}_p^{m,1}(Q_T)$, and

$$\|v\|_{m,1;q} \le C\|g\|_{-m,-1;q}.$$

We assume that $f \in C^\infty(\overline{Q_T})$, $f_{t^k}(x,0) = 0$, $k = 0, 1, \dots$ and for $u = u^\epsilon_{t^h}$ in (3.12). Then, by (3.12) and (3.11), we have

$$\begin{aligned}
|\langle g, u^\epsilon_{t^h}\rangle| &= |B_1[v, u^\epsilon_{t^h}]| = |\overline{B_1[u^\epsilon_{t^h}, v]}| \\
&\le C(\|f_{t^h}\|_{L_p(Q_T)} + \sum_{k=0}^{h-1} \|u^\epsilon_{t^k}\|_{m,1;p})\|v\|_{m,1;q} \\
&\le C(\|f_{t^h}\|_{L_p(Q_T)} + \sum_{k=0}^{h-1} \|u^\epsilon_{t^k}\|_{m,1;p})\|g\|_{-m,-1;q}
\end{aligned}$$

for any $g \in W_q^{-m,-1}(Q_T)$. Hence,

$$\begin{aligned}
\|u^\epsilon_{t^h}\|_{m,1;p} &= \sup\big\{\frac{|\langle g, u^\epsilon_{t^h}\rangle|}{\|g\|_{-m,-1;q}} : 0 \ne g \in W_q^{-m,-1}(Q_T)\big\} \\
&\le C(\|f_{t^h}\|_{L_p(Q_T)} + \sum_{k=0}^{h-1} \|u^\epsilon_{t^k}\|_{m,1;p}).
\end{aligned}$$

From this inequality and induction assumption, we have the proof of this case, and complete the proof. □

3.3. **Regularity of the generalized L_p-solution.** In this section, we consider problem (2.4)-(2.6) in cylinders $Q_T = \Omega \times (0, T)$, where its base Ω is described as follows:

Let φ be an infinitely differentiable positive function on the interval $(0, 1]$ satisfying the conditions

(i) $\lim_{\tau\to 0} \varphi(\tau)^{k-1}\varphi(\tau)^{(k)} < \infty$ for $k = 1, 2, \dots$;

(ii) $\int_0^1 \frac{d\tau}{\varphi(\tau)} = +\infty$

These conditions are satisfied, for example, by the function $\varphi(\tau) = \tau^\alpha$ if $\alpha \geq 1$. Obviously, conditions (i) and (ii) imply $\varphi(0) = 0$. Suppose that Ω is a bounded domain in $\mathbb{R}^n (n \geq 2), \partial\Omega \backslash \{\mathcal{O}\}$ is smooth, and

$$\{x \in \Omega :\ 0 < x_n < 1\} = \{x \in \mathbb{R}^n : x_n < 1,\ x' \in \varphi(x_n)\omega\},$$

where $x' = (x_1, \dots, x_{n-1}), \omega$ is a smooth domain in $\mathbb{R}^{n-1}$. Then the mapping

$$y_j = \frac{x_j}{\varphi(x_n)},\ \text{if } j = 1, \dots, n-1, \quad \text{and} \quad y_n = \int_{x_n}^1 \frac{d\tau}{\varphi(\tau)} \tag{3.13}$$

takes the set $\{x \in \Omega :\ 0 < x_n < 1\}$ onto the half-cylinder $\mathcal{C}_+ = \{y \in \mathbb{R}^n : y' \in \omega,\ y_n > 0\} = \omega \times (0, +\infty)$. Moreover, it follows that

$$\det\Big(\frac{\partial y_j}{\partial x_k}\Big)_{j,k=1,\dots,n} = \varphi(x_n)^{-n}.$$

It is known that the function φ can be extended to an infinitely differentiable positive function on the interval $(0, +\infty)$. To consider the problem, we need to introduce some weighted Sobolev spaces. The space $W^l_{p,\beta,\gamma}(\Omega)$ can be defined as the closure of the set $C_0^\infty(\overline{\Omega}\backslash\{\mathcal{O}\})$ with respect to the norm

$$\|u\|_{W^l_{p,\beta,\gamma}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \leq l} e^{p\beta y_n(x_n)} \varphi(x_n)^{p(\gamma - l + |\alpha|)} |D^\alpha u|^p\, dx\Big)^{1/p}.$$

Let X, Y be Banach spaces, we denote by $L_p(0,T;X)$ the spaces consisting of all measurable functions $u : (0,T) \to X$ with norm

$$\|u\|_{L_p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p\, dt\Big)^{1/p},$$

and by $W^k_p(0,T;X,Y), k = 1,2$, the spaces consisting of functions $u \in L_p(0,T;X)$ such that generalized derivatives $u_{t^k} = u^{(k)}$ exist and belong to $L_p(0,T;Y)$, (see [4]), with norm

$$\|u\|_{W^k_p(0,T;X,Y)} = \Big(\|u\|^2_{L_p(0,T;X)} + \sum_{j=1}^k \|u_{t^j}\|^p_{L_p(0,T;Y)}\Big)^{1/p}.$$

For short notation, we set

$$V^l_p(\Omega) = W^l_{p,0,0}(\Omega), \quad V^{l,k}_p(Q_T) = W^k_p(0,T; V^l_p(\Omega), L_p(\Omega)),$$
$$W^{l,k}_{p,\beta,\gamma}(Q_T) = W^k_p(0,T; W^l_{p,\beta,\gamma}(\Omega), L_p(\Omega)).$$

Finally, we define the weighted Sobolev space $W^l_{p,\beta,\gamma}(Q_T)$ as the set of functions defined in Q_T such that

$$\|u\|_{W^l_{p,\beta,\gamma}(Q_T)} = \Big(\int_{Q_T} \sum_{|\alpha|+k \leq l} e^{2\beta y(x_n)} \varphi(x_n)^{p(|\gamma - l + \alpha| + k)} |D^\alpha u_{t^k}|^p\, dx\, dt\Big)^{1/p} < +\infty.$$

To simplify notation, we continue to write $V^l_p(Q_T)$ instead of $W^l_{p,0,0}(Q_T)$.

Moreover, we assume that the functions

$$\widehat{a}_{\alpha\beta}(y, .) = \varphi(x(y))^{2m - |\alpha| - |\beta|} a_{\alpha\beta}(x(y), .) \tag{3.14}$$

satisfy the condition of stabilization for $y_n \to +\infty$ for a.e. t in $(0,T)$(see[19, Sec.9]). Then the coefficients of the operators $\widehat{L}(y,t;\ D_y)$, which arises from the operators $\varphi(x_n)^{2m} L(x,t; D_x)$ via the coordinate change $x \to y$, stabilize for $y_n \to +\infty$. If we replace the coefficients of the differential operator $\widehat{L}(y,t;\, D_y)$ by their limits for $y_n \to +\infty$, we get differential operator (denote also by $\widehat{L}(y',t;\, D_{y'}, D_{y_n})$ for convenience) which has coefficients depending only on y' and t.

By the following proposition, we can apply the results of the Dirichlet problem to elliptic equations in domains with cuspidal points on boundary.

Proposition 3.6. *Suppose that $u = u(x,t)$ is a generalized solution of problem (2.4) -(2.6) and $u_{tt} \in L_p(Q_T)$. Then for a.e. $t \in (0,T)$, $u(t) = u(.,t)$ is a generalized solution in $\mathring{W}_p^m(\Omega)$ of the Dirichlet problem for elliptic equation*

$$L(.,t;D_x)u = f_1(.,t) \tag{3.15}$$

where $f_1 = u_{tt} + f$.

Proof. For any $\psi \in \mathring{W}_q^m(\Omega)$, $\theta \in C_0^\infty(0,T)$ and setting $v(x,t) = \psi(x)\theta(t)$, we substitute the function $v(x,t)$ into (2.7), we conclude that

$$\int_{Q_T} \Big[\sum_{|\alpha|,|\beta|=0}^{m} a_{\alpha\beta} D_x^\beta\, u\overline{D_x^\alpha \psi} - (u_{tt}\, \overline{\psi} + f\overline{\psi})\Big]\overline{\theta(t)}\, dx\, dt = 0. \tag{3.16}$$

We will denote by

$$\xi(t) = \int_\Omega \Big[\sum_{|\alpha|,|\beta|=0}^{m} a_{\alpha\beta} D_x^\beta\, u\overline{D_x^\alpha \psi} - (u_{tt}\, \overline{\psi} + f\overline{\psi})\Big]\, dx,$$

then $\xi(t) \in L_p(0,T)$. Noting that $\theta \in C_0^\infty(0,T)$, which dense in $L_q(0,T)$ and using Fubini's theorem, we obtain from (3.16) that

$$\int_0^T \xi(t)\overline{\theta}(t)\, dt = 0, \quad \text{for any } \theta \in L_q(0,T), (1/p + 1/q = 1). \tag{3.17}$$

Therefore,

$$\|\xi\|_{L_p(0,T)} = \sup\big\{ \int_0^T \xi(t)\overline{\theta}(t)\, dt : \theta \in L_q(0,T), \|\theta\|_{L_q(0,T)} = 1\big\} = 0.$$

This implies $\xi = 0$ for a.e. $t \in (0,T)$. Hence,

$$\int_\Omega \sum_{|\alpha|,|\beta|=0}^{m} a_{\alpha\beta} D_x^\beta\, u\overline{D_x^\alpha \psi}\, dx = \int_\Omega (u_{tt} + f)\overline{\psi}\, dx$$

for all $\psi \in \mathring{W}_q^m(\Omega)$, for a.e. $t \in (0,T)$. It follows that $u(t)$ is a generalized solution in $\mathring{W}_p^m(\Omega)$ of the Dirichlet problem for elliptic equation (3.15), for a.e. $t \in (0,T)$. $\square$

In this section, we present the main results which is based on our previous subsection and the results for elliptic equations in cusp domains (cf. [19]). For the start of this section, we denote by $\mathcal{U}(\lambda,t)(\lambda \in \mathbb{C}, t \in (0,T))$ the operator corresponding to the parameter-depending boundary-value problem

$$\begin{gathered} \widehat{L}(y',t;\, D_{y'},\lambda)v = 0 \quad \text{in } \omega; \\ \partial_\nu^j v = 0 \quad \text{on } \partial\omega,\ j = 1,\dots,m-1. \end{gathered} \tag{3.18}$$

Where $\widehat{L}(y',t;\, D_{y'},\, \lambda)$ is the Fourier transformation $y_n \to \lambda$ of $\widehat{L}(y',t;\, D_{y'},\, D_{y_n})$.

For each $t \in (0,T)$, the operator pencil $\mathcal{U}(\lambda,t)$ is Fredholm, and its spectrum consists of a countable numbers of isolated eigenvalues. The similarly, to Theorem 9.1 in [19], we have the following lemma.

Lemma 3.7. *Assume that $f_1 \in W^k_{p,\beta,\gamma}(\Omega)$, where β, γ are real numbers. Suppose further that no eigenvalues of $\mathcal{U}(\lambda, t), t \in (0,T))$ line in strip*

$$\operatorname{Im}\lambda_- \leq \operatorname{Im}\lambda \leq \operatorname{Im}\lambda_+; \quad \operatorname{Im}\lambda_- < \beta < \ Im\lambda_+$$

where λ_+, λ_- are eigenvalues of $\mathcal{U}(\lambda,t)$, and $\operatorname{Im}\lambda_- < 0 < \operatorname{Im}\lambda_+$. Then the generalized solution u of the Dirichlet problem for the elliptic equation (3.15), *$u \equiv 0$ if $x_n > 1$, belongs to the space $W^{2m+k}_{p,\beta,\gamma}(\Omega)$ and satisfies the inequality*

$$\|u\|^2_{W^{2m+k}_{p,\beta,\gamma}(\Omega)} \leq C\|f_1\|^2_{W^k_{p,\beta,\gamma}(\Omega)} \tag{3.19}$$

where the constant C is independent of f_1.

Proof. Setting $\omega_\tau = \varphi(\tau)\omega$ by the Friederichs inequality, we have

$$\int_{\omega_\tau} |u|^p dx' \leq C\varphi(\tau)^{pk} \sum_{|\gamma|=k} \int_{\omega_\tau} |D^\gamma_{x'} u|^p dx' ;$$

therefore,

$$\varphi(x_n)^{p(|\gamma|-m)} \int_{\omega_{x_n}} |D^\gamma_{x'} u|^p dx' \leq C \sum_{|\alpha|=m} \int_{\omega_{x_n}} |D^\alpha_{x'} u|^p dx'$$

for all $|\gamma| \leq m$. Hence,

$$\sum_{|\gamma|\leq m} \int_\Omega \varphi(x_n)^{p(|\gamma|-m)} |D^\gamma_x u|^p dx \leq C \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha_x u|^p dx \tag{3.20}$$

Let $v = v(y)$ be the function that arises from $\varphi(x_n)^{m-\frac{n}{p}} u(x)$ via the coordinate change $x \to y$. We set $\overline{\varphi}(y_n) = \varphi(x_n)$, from the properties of the mapping (3.13) and from inequality (3.20), it follows that $(\overline{\varphi})^{-m+\frac{n}{p}} v \in \mathring{W}^m_p(\mathcal{C}_+)$. Since $(\overline{\varphi})^{-m+\frac{n}{p}} v$ is the solution of an elliptic equation in $\mathcal{C}_+$ with coefficients which stabilize for $y_n \to +\infty$, i.e.

$$\widehat{L}(\overline{\varphi})^{-m+\frac{n}{p}} v = \widehat{f_1}$$

where $\widehat{f_1} = (\overline{\varphi})^{2m} f_1$, we obtain $(\overline{\varphi})^{-m+\frac{n}{p}} v \in \mathring{W}^{2m+k}_p(\mathcal{C}_+)$ (cf. [19, Theorem 8.1, 8.2]). This implies $u \in W^{2m+k}_{p,0,m+k}(\Omega)$. Using the fact that

$$\varphi(x_n)^{\gamma-m+k} e^{-\epsilon y_n(x_n)} \to 0$$

as $x_n \to 0$, if $0 < \epsilon < \beta$, we conclude that $u \in W^{2m+k}_{p,-\epsilon,\gamma}(\Omega)$. In a similar manner, Theorem 8.2 in [19] it follows that $u \in W^{2m+k}_{p,\beta,\gamma}(\Omega)$. Furthermore, (3.19) is valid. □

Lemma 3.8. *Suppose that $f, f_t \in L_p(Q_T), f(x,0) = 0$, and the strip $\operatorname{Im}\lambda_- \leq \operatorname{Im}\lambda \leq \operatorname{Im}\lambda_+$ does not contain eigenvalues of $\mathcal{U}(\lambda,t), t \in (0,T))$. Then the generalized solution u of problem* (2.4)-(2.6), *$u \equiv 0$ if $x_n > 1$, belongs to the $V^{2m,2}_p(Q_T)$ and satisfies the inequality*

$$\|u\|_{V^{2m,2}_p(Q_T)} \leq C[\|f\|_{L_p(Q_T)} + \|f_t\|_{L_p(Q_T)}], \tag{3.21}$$

where the constant C is independent of f.

Proof. Using the smoothness of the generalized solution of (2.4)-(2.6) with respect to t in Theorem 3.5 and Proposition 3.6, we can see that for a.e. $t \in (0,T), u \in \mathring{W}^m_p(\Omega)$ is the generalized solution of Dirichlet problem for equation (3.15) with compact support, where $f_1 = u_{tt} + f \in L_p(\Omega) = W^0_{p,0,0}(\Omega) = V^0_p(\Omega)$. From Lemma 3.7, it implies that $u \in V^{2m}_p(\Omega)$ for a.e. $t \in (0,T)$ and satisfies the inequality

$$\|u\|_{V^{2m}_p(\Omega)} \le C_1\|f_1\|_{L_p(\Omega)} \le C\left(\|f\|_{L_p(\Omega)} + \|u_{tt}\|_{L_p(\Omega)}\right).$$

By integrating the inequality above with respect to t from 0 to T, and using the estimates for derivatives of u with respect to t again, we obtain $u \in V^{2m,2}_p(Q_T)$, which satisfies (3.21). $\square$

Theorem 3.9. *Let the assumptions of Lemma 3.8 be satisfied, and $f_{t^k} \in L_p(Q_T)$, $k \le 2m$, $f_{t^k}(x,0) = 0$, for $k = 0,1,\dots,2m-1$. Then the generalized solution u of problem (2.4)-(2.6), $u \equiv 0$ if $x_n > 1$, belongs to the $V^{2m}_p(Q_T)$ and satisfies the inequality*

$$\|u\|_{V^{2m}_p(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{L_p(Q_T)} \tag{3.22}$$

where the constant C is independent of f.

Proof. Let us first prove that u_{t^s} belongs to $V^{2m,0}_p(Q_T)$ for $s = 0,\dots,2m-1$ and satisfy

$$\|u_{t^s}\|_{V^{2m,0}_p(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{L_p(Q_T)}. \tag{3.23}$$

The proof is by done induction on s. According to Lemma 3.8, it is valid for $s = 0$. Now let this assertion be true for $s-1$, we will prove that this also holds for s. Due to Lemma 3.8 then u satisfies (2.4), by differentiating both sides of (2.4) with respect to t, s times, we obtain

$$Lu_{t^s} = f_{t^s} + u_{t^{s+2}} - \sum_{k=1}^{s}\binom{s}{k}L_{t^k}u_{t^{s-k}} \tag{3.24}$$

where

$$L_{t^k} = L_{t^k}(x,t;D_x) = \sum_{\alpha,\beta=0}^{m} D^\alpha_x\Big(\frac{\partial^k a_{\alpha\beta}(x,t)}{\partial t^k}\,D^\beta_x\Big).$$

By the supposition of the theorem and the inductive assumption, the right-hand side of (3.24) belongs to $L_p(Q_T)$. By the arguments analogous to the proof of Lemma 3.8, we get $u_{t^s} \in V^{2m,0}_p(Q_T)$ and

$$\|u_{t^s}\|_{V^{2m,0}_p(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{L_p(Q_T)} \tag{3.25}$$

where C is a constant independent of u, f, and $s \le m-1$.

Using (3.25) and estimates for derivatives of u with respect to t in Theorem 3.4, we have

$$\|u\|_{V^{2m}_p(Q_T)} \le \sum_{k=0}^{2m-1}\|u_{t^k}\|_{V^{2m,0}_p(Q_T)} + \|u_{t^{2m}}\|_{L_p(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{L_p(Q_T)}.$$

$\square$

Remark. Let β be a sufficiently small positive number. Suppose that

$$e^{\beta y_n(x_n)}f \in L_p(Q_T), \quad \operatorname{Im}\lambda_- < \beta < \operatorname{Im}\lambda_+$$

and the strip

$$\operatorname{Im}\lambda_- \le \operatorname{Im}\lambda \le \operatorname{Im}\lambda_+$$

contains no eigenvalues of $\mathcal{U}(\lambda,t), t \in (0,T)$); then the generalized solution u of (2.4) -(2.6), $u \equiv 0$ if $x_n > 1$, belongs to the $W^{2m}_{p,\beta,0}(Q_T)$. In fact that, setting $u = e^{-\beta y_n(x_n)}U$, we obtain the first initial boundary value problem which differs little from (2.4)-(2.6). Therefore, $U \in V^{2m}_p(Q_T)$, and then $u \in W^{2m}_{p,\beta,0}(Q_T)$. Using the remark above and Lemma 3.7, we obtain the following theorem.

Theorem 3.10. *Let the assumptions of Lemma 3.7 be satisfied. Furthermore, we assume that $f_{t^k} \in W^0_{p,\beta,\gamma}(Q_T), k \le 2m$ and $f_{t^k}(x,0) = 0$, for $k = 0,1,\dots,2m-1$. Then the generalized solution u of (2.4)-(2.6), such that $u \equiv 0$ if $x_n > 1$, belongs to the $W^{2m}_{p,\beta,\gamma}(Q_T)$ and satisfies the inequality*

$$\|u\|_{W^{2m}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^0_{p,\beta,\gamma}(Q_T)} \tag{3.26}$$

where the constant C is independent of f.

This theorem is proved by arguments analogous to those proofs of Lemma 3.8 and Theorem 3.5. Next, we will prove the regularity of the generalized solution of problem (2.4)-(2.6).

Theorem 3.11. *Let the assumptions of Lemma 3.7 be satisfied. Furthermore, we assume that $f_{t^k} \in W^h_{p,\beta,\gamma}(Q_T), k \le 2m+h$ and $f_{t^k}(x,0) = 0$, for $k = 0,1,\dots,2m+h-1$, $h \in \mathbb{N}$. Then the generalized solution u of (2.4)-(2.6), such that $u \equiv 0$ if $x_n > 1$, belongs to $W^{2m+h}_{p,\beta,\gamma}(Q_T)$ and satisfies the inequality*

$$\|u\|_{W^{2m+h}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)} \tag{3.27}$$

where the constant C is independent of u and f.

Proof. The theorem is proved by induction on h. Thanks to Theorem 3.9, this theorem is obviously valid for $h = 0$. Assume that the theorem is true for $h-1$, we will prove that it also holds for h. It is only needed to show that

$$\begin{gathered} u_{t^s} \in W^{2m+h-s,0}_{p,\beta,\gamma}(Q_T) \quad \text{for } s = h, h-1\dots,0; \\ \|u_{t^s}\|_{W^{2m+h-s}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}. \end{gathered} \tag{3.28}$$

Differentiating both sides of (2.4) again with respect to t, h times, we obtain

$$Lu_{t^h} = f_{t^h} + u_{t^{h+2}} - \sum_{k=1}^{h}\binom{h}{k}L_{t^k}u_{t^{h-k}} \tag{3.29}$$

By the supposition of the theorem and the inductive assumption, the right-hand side of (3.29) belongs to $W^0_{p,\beta,\gamma}(\Omega)$ for a.e. $t \in (0,T)$. Using Lemma 3.7, we conclude that $u_{t^h} \in W^{2m,0}_{p,\beta,\gamma}(Q_T)$. It implies that (3.28) holds for $s = h$. Suppose that (3.28) is true for $s = h, h-1,\dots,j+1$ and set $v = u_{t^j}$, we obtain

$$Lv = F_j, \tag{3.30}$$

where $F_j = f_{t^j} + v_{tt} - \sum_{k=1}^{j}\binom{j}{k} L_{t^k} u_{t^{j-k}}$. By the inductive assumption with respect to s, v_{tt} belongs to $W^{h-j}_{p,\beta,\gamma}(\Omega)$ for a.e. $t \in (0,T)$. Thus, the right-hand side of (3.30) belongs to $W^{h-j}_{p,\beta,\gamma}(\Omega)$. Applying Lemma 3.7 again for $k = h-j$, we get that $v = u_{t^j} \in W^{2m+h-j}_{p,\beta,\gamma}(\Omega)$ for a.e. $t \in (0,T)$. It means that $v = u_{t^j}$ belongs to $W^{2m+h-j,0}_{p,\beta,\gamma}(Q_T)$. Furthermore, we have

$$\|v\|_{W^{2m+h-j,0}_{p,\beta,\gamma}(Q_T)} \le C\|F_j\|_{W^{h-j,0}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}. \tag{3.31}$$

Therefore,

$$\begin{aligned}\|u_{t^j}\|_{W^{2m+h-j}_{p,\beta,\gamma}(Q_T)} &\le \|u_{t^{j+1}}\|_{W^{2m+h-j-1}_{p,\beta,\gamma}(Q_T)} + \|u_{t^j}\|_{W^{2m+h-j,0}_{p,\beta,\gamma}(Q_T)}\\ &\le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}.\end{aligned}$$

It implies that (3.28) holds for $s = j$. The proof is complete. □

Now we prove the global regularity of the solution.

Theorem 3.12. *Let the hypotheses of Lemma 3.7 be satisfied. Furthermore, assume that $f_{t^k} \in W^h_{p,\beta,\gamma}(Q_T)$, $k \le 2m+h$ and $f_{t^k}(x,0) = 0$, for $k = 0,1,\dots,2m+h-1$, $h \in \mathbb{N}$. Then the generalized solution u of (2.4)-(2.6) belongs to $W^{2m+h}_{p,\beta,\gamma}(Q_T)$ and satisfies the inequality*

$$\|u\|_{W^{2m+h}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)} \tag{3.32}$$

where the constant C is independent of u and f.

Proof. We denote by B the unit ball and suppose that $\zeta \in C_0^\infty(B)$, and $\zeta \equiv 1$ in the neighborhood of the origin $\mathcal{O}$. We have

$$L(\zeta u) - (\zeta u)_{tt} = \zeta f + L_1 u$$

where L_1 is a differential operator, whose coefficients have compact support in a neighborhood of the origin. By Theorem 3.10, we obtain

$$\|\zeta u\|_{W^{2m+h}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}.$$

Setting $\zeta_1 u = (1-\zeta)u$, then $\zeta_1 u \equiv 0$ in a neighborhood of the origin and $u = \zeta u + (1-\zeta)u$, and using the smoothness of the solution of this problem in domain with smooth boundary, we get

$$\|\zeta_1 u\|_{W^{2m+h}_p(Q_T)} \sim \|\zeta_1 u\|_{W^{2m+h}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2m}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}.$$

The proof is complete . □

4. An example

In this section, we apply the results of the previous section to the Cauchy-Dirichlet problem for the beam equation. Suppose that Ω is a bounded domain in $\mathbb{R}^n, \partial\Omega\backslash\{\mathcal{O}\}$ is smooth, and

$$\{x \in \Omega : 0 < x_n < 1\} \equiv \{x \in \mathbb{R}^n : \ 0 < x_n < 1, |x'| < \varphi(x_n)\},$$

where $x' = (x_1, \dots, x_{n-1}), \varphi \in C^\infty[0,1), \varphi'(x_n) \to 0, \varphi(x_n)\varphi''(x_n) \to 0$ as $x_n \to 0$ and $\varphi(0) = 0$. Set $Q_T = \Omega \times (0,T), S_T = \partial\Omega \setminus \{\mathcal{O}\} \times (0,T)$.

We consider the Cauchy-Dirichlet problem for the beam equation in Q_T:

$$\Delta^2 u - \Delta u - u_{tt} = -f \quad \text{in } Q_T, \tag{4.1}$$

$$u = 0, u_t = 0 \quad \text{on } \Omega, \tag{4.2}$$

$$u = 0, \partial_\nu u = 0 \quad \text{on } S_T \tag{4.3}$$

where $f : Q_T \to \mathbb{C}$ is given and $Lu = \Delta_x^2 u - \Delta_x u$. By using Green's formula, we get

$$B[u,u;t] = \int_\Omega \left(|D_x^2 u|^2 + |D_x u|^2\right) dx$$

for all $u \in \mathring{W}_2^2(\Omega)$. On other hand, by the Friedrich inequality

$$\int_\Omega |u|^2\, dx \le C \int_\Omega |D_x u|^2\, dx,$$

it implies that there exists a constant $\gamma_0 > 0$ such that

$$B[u,u;t] = \int_\Omega \left(|D_x^2 u|^2 + |D_x u|^2\right) dx \ge \gamma_0 |u|^2_{W_2^2(\Omega)}.$$

Hence, (2.2) is satisfied for all $u \in \mathring{W}_2^2(\Omega)$, for all $t \in (0,T)$.

For simplicity, we consider (4.1)-(4.3) in the two-dimensional case ($n = 2$), and let $\varphi(\tau) = \tau^2$; Then Ω is a bounded domain in $\mathbb{R}^2$, $\partial\Omega\backslash\{\mathcal{O}\}$ is smooth, and

$$\{(x,y) \in \Omega : 0 < x < 1\} \equiv \{(x,y) \in \mathbb{R}^2 : \ 0 < x < 1, |y| < x^2\},$$

on the change of variables

$$\xi = \int_x^1 \frac{d\tau}{\tau^2} = x^{-1} - 1, \quad \eta = yx^{-2}, \tag{4.4}$$

which transforms $\{(x,y) \in \Omega : 0 < x < 1\}$ onto

$$\mathcal{C}_+ := \{(\xi,\eta) : \xi > 0,\ \eta \in (-1,1)\}.$$

With the notation $v(\xi,\eta,t) = u(x,y,t)$, we have

$$u(x,y,t) = v(x^{-1} - 1, yx^{-2}, t)$$

and

$$\begin{aligned}
\partial_y u &= x^{-2}\partial_\eta v, \\
\partial_x u &= -x^{-2}\partial_\xi v - 2yx^{-3}\partial_\eta v, \\
\partial_{yy}^2 u &= x^{-4}\partial_{\eta\eta}^2 v \\
\partial_{xx}^2 u &= x^{-3}\partial_\xi v + 6yx^{-4}\partial_\eta v + x^{-4}\partial_{\xi\xi}^2 v + 4yx^{-5}\partial_{\xi\eta}^2 v + 4y^2x^{-6}\partial_{\eta\eta}^2 v \\
&= x^{-4}\left[x\partial_\xi v + 6y\partial_\eta v + \partial_{\xi\xi}^2 v + 4yx^{-1}\partial_{\xi\eta}^2 v + 4y^2x^{-3}\partial_{\eta\eta}^2 v\right]
\end{aligned}$$

$$
\begin{aligned}
&= x^{-4}\big[\partial^2_{\xi\xi}v + 4\eta(\xi+1)^{-1}\partial^2_{\xi\eta}v + 4\eta^2(\xi+1)^{-2}\partial^2_{\eta\eta}v \\
&\quad + (\xi+1)^{-1}\partial_\xi v + 6\eta(\xi+1)^{-2}\partial_\eta v\big].
\end{aligned}
$$

Hence, the differential operator $\widehat{\Delta}$, which arises from the differential operator $x^8\Delta u, (\varphi(x) = x^2, 2m = 4)$ via the coordinate change $(x, y) \to (\xi, \eta)$, turns out to be

$$
\begin{aligned}
\widehat{\Delta}v &= (\xi+1)^{-4}\left(\partial^2_{\xi\xi}v + \partial^2_{\eta\eta}v\right) + 4\eta(\xi+1)^{-5}\partial^2_{\xi\eta}v + 4\eta^2(\xi+1)^{-6}\partial^2_{\eta\eta}v \\
&\quad + (\xi+1)^{-5}\partial_\xi v + 6\eta(\xi+1)^{-6}\partial_\eta v;
\end{aligned}
$$

the similar calculation for $\widehat{\Delta^2}$. Clearly, coefficients of differential operator $\widehat{L} = \widehat{\Delta^2} - \widehat{\Delta}$ stabilize for $\xi \to +\infty$ and the limit differential operator of $\widehat{L}$ (denote by $\widehat{L}$ for convenience) is

$$
\widehat{L} = \widehat{\Delta^2}v = \partial^4_{\xi^4}v + 2\partial^4_{\eta^2\xi^2}v + \partial^4_{\eta^4}v.
$$

We denote also by $\mathcal{U}(\lambda)(\lambda \in \mathbb{C})$ the operator corresponding to the parameter-depending boundary value problem

$$
\begin{gathered}
\frac{d^4v}{d\eta^4} - 2\lambda^2\frac{d^2v}{d\eta^2} + \lambda^4 v = 0, \\
v(-1) = v(1) = 0, \\
v'(-1) = v'(1) = 0.
\end{gathered}
$$

It is easy to see that $\mathcal{U}(\lambda)$ is invertible for all $\lambda \in \mathbb{C}$. From arguments above in combination with Theorem 3.10 and Theorem 3.12, we obtain the following results.

Theorem 4.1. *Suppose that* $e^{\beta(\frac{1}{x}-1)}x^{2\gamma}f_{t^k} \in L_p(Q_T)$, $k \le 2, \beta, \gamma$ *are real numbers and* $f_{t^k}(x, 0) = 0$, *for* $k = 0, 1$. *Then* (4.1)-(4.3) *has a unique solution* u *in* $W^2_{p,\beta,\gamma}(Q_T)$ *and*

$$
\|u\|_{W^2_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2}\|e^{\beta(\frac{1}{x}-1)}x^{2\gamma}f_{t^k}\|_{L_p(Q_T)}.
$$

Moreover, if $f_{t^k} \in W^h_{p,\beta,\gamma}(Q_T)$, $k \le 2+h$, *and* $f_{t^k}(x, 0) = 0$ *for* $k = 0, 1, \dots, 1+h$, *then* $u \in W^{2+h}_{p,\beta,\gamma}(Q_T)$ *and satisfies*

$$
\|u\|_{W^{2+h}_{p,\beta,\gamma}(Q_T)} \le C\sum_{k=0}^{2}\|f_{t^k}\|_{W^h_{p,\beta,\gamma}(Q_T)}.
$$

In case boundary when Ω has cuspidal points, then by arguments analogous to Section 3, we obtain the similar results.

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Nguyen Manh Hung

Faculty of Mathematics, Hanoi National University of Education, Vietnam

E-mail address: hungnmmath@hnue.edu.vn

Vu Trong Luong

Department of Mathematics, Taybac University, Sonla City, Vietnam

E-mail address: vutrongluong@gmail.com