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EXISTENCE OF SOLUTIONS FOR FOURTH-ORDER PDES WITH VARIABLE EXPONENTS

ABDELRACHID EL AMROUSS, FOUZIA MORADI, MIMOUN MOUSSAOUI

Abstract. In this article, we study the following problem with Navier boundary conditions

$$\begin{gathered}\Delta^2_{p(x)} u = \lambda |u|^{p(x)-2}u + f(x,u) \quad \text{in } \Omega,\\ u = \Delta u = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator, $\lambda \leq 0$, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function. Using the Mountain Pass Theorem, we establish the existence of at least one solution of this problem. Especially, the existence of infinite many solutions is obtained.

1. Introduction

The study of differential and partial differential involving variable exponent conditions is a new and an interesting topic. The main references in this field can be found in an overview paper [13].

Fourth order elliptic equations arise in many applications such as: Micro Electro Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [16], [11]) and the references therein. There is also another important class of physical problems leading to higher order partial differential equations. An example of this is Kuramoto-Sivashinsky equation which models pattern formation in different physical contexts, such as chemical reaction -diffusion systems and a cellular gas flame in the presence of external stabilizing factors (see [20]).

This paper is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics, in particular, the electro-rheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which is not easy and which depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in elastic mechanics, fluid dynamics etc.. For more information, the reader can refer to [12, 17].

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These physical problems was facilitated by the development of Lebesgue and Sobolev spaces with variable exponent. The existence of solutions of p(x)-Laplacian problems has been studied by several authors (see [5, 6, 8, 9, 14]).

The purpose of the present article is to study the existence of weak solutions of a elliptic fourth order equation with variable exponent. This is a new topic.

Consider the following problem with Navier boundary conditions

$$\begin{gathered} \Delta^2_{p(x)} u = \lambda |u|^{p(x)-2} u + f(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \leq 0$, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

The operator $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, with $p(x) > 1$ is called the p(x)-biharmonic which is a natural generalization of the p-biharmonic (where $p > 1$ is a constant). When $p(x)$ is not constant, the $p(x)$-biharmonic possesses more complicated nonlinearity that the p-biharmonic, say, it is inhomogeneous.

In the constant case ($p(x) \equiv p$), there are many papers devoted the existence of solutions of the above problem; see for example [4, 18] and the references therein.

Recently, in [2], the authors interested to the spectrum of a fourth order elliptic equation with variable exponent. They proved the existence of infinitely many eigenvalue sequences and $\sup \Lambda = +\infty$, where Λ is the set of all eigenvalues. Moreover, they present some sufficient conditions for $\inf \Lambda = 0$.

In this paper, we start by proving the following results.

Theorem 1.1. *If $f(x,u) = f(x), f \in L^{\alpha(x)}(\Omega)$ with $\alpha \in C_+(\overline{\Omega})$ satisfies*

$$\frac{1}{\alpha(x)} + \frac{1}{p_2^*(x)} < 1, \quad \forall x \in \overline{\Omega},$$

then, for all $\lambda \leq 0$, problem (1.1) *has a unique weak solution.*

Theorem 1.2. *Suppose that f satisfies the condition*

$$|f(x,s)| \leq a(x) + b|s|^{\beta-1} \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

with $a(x) \geq 0$, $a(x) \in L^{\frac{\alpha(x)}{\alpha(x)-1}}(\Omega)$, $b \geq 0$, $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p_2^*(x)$ and $1 \leq \beta < p^-$. Then, for all $\lambda \leq 0$, problem (1.1) *admits at least one weak solution.*

The second purpose of this paper is to show the existence of at least one nontrivial solution of problem (1.1) via Mountain Pass Theorem and the following assumptions of the function f.

- (H1) $|f(x,s)| \leq a(x) + b|s|^{\alpha(x)-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$, with $a(x) \geq 0$, $a(x) \in L^{\frac{\alpha(x)}{\alpha(x)-1}}(\Omega)$, $b \geq 0$, $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p_2^*(x)$.
- (H2) There exist $M > 0$, $\theta > p^+$ such that for all $|s| \geq M$ and $x \in \Omega$,
$$0 < F(x,s) \leq \frac{s}{\theta} f(x,s).$$
- (H3) $f(x,s) = o(|s|^{p^+-1})$ as $s \to 0$ and uniformly for $x \in \Omega$, with $\alpha^- > p^+$.

We can state the following result.

Theorem 1.3. *If f satisfies* (H1)–(H3)*, then, for all $\lambda \leq 0$, problem* (1.1) *has at least a nontrivial solution.*

Next, we obtain an infinite many pairs of solutions.

Theorem 1.4. *Suppose that f satisfies the conditions* (H1)–(H2) *and the following condition*

(H4) $f(x,-s) = -f(x,s)$, $x \in \Omega$, $s \in \mathbb{R}$.

Then, problem (1.1) *has infinite many weak solutions.*

Remark 1.5. (1) Condition (H1) indicates that the nonlinearity f is subcritical and (H2) indicates f is "superlinear". These two conditions enable us to use a variational approach for the study (1.1); they also provide the Palais-Smale compactness condition.

(2) Beginning with [1], many authors have obtained non trivial solutions of superlinear problems, $-\Delta_p u = f(x,u)$ in Ω; $u = 0$ on $\partial\Omega$, under various assumptions of the behavior of f near zero, in the semilinear case $p = 2$ and quasilinear $p \neq 2$.

Our work is motivated by [1, 3, 8, 2].

This paper is divided into four sections, organized as follows: In section 2, we introduce some basic properties of the Lebesgue and Sobolev spaces with variable exponent. In the third section, we present some important properties of the p(x)-biharmonic operator. In section 4, we proves our main results.

2. Preliminaries

To study $p(x)$-Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$-Laplacian, which we will use later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\big\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) := \big\{p \in C(\overline{\Omega}) : p(x) > 1 \quad \forall x \in \overline{\Omega}\big\}.$$

Denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$,

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the norm

$$|u|_{p(x)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\big\}.$$

The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.

Proposition 2.1 ([10])**.** *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\Big| \int_{\Omega} u(x)v(x)dx \Big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u$, (the derivation in distributions sense) with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [7, 10, 15, 19].

Proposition 2.2 ([10])**.** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

3. Properties of $p(x)$-Biharmonic operator

Note that the weak solutions of the problem (1.1) are considered in the generalized Sobolev space

$$X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

equipped with the norm

$$\|u\| = \inf \Big\{\alpha > 0 : \int_{\Omega} \Big(|\frac{\Delta u(x)}{\alpha}|^{p(x)} - \lambda |\frac{u(x)}{\alpha}|^{p(x)} \Big) dx \le 1 \Big\}.$$

Remark 3.1. (1) According to [21], the norm $\|.\|_{2,p(x)}$, cited in the preliminaries, is equivalent to the norm $|\Delta.|_{p(x)}$ in the space X. Consequently, the norms $\|.\|_{2,p(x)}, \|.\|$ and $|\Delta.|_{p(x)}$ are equivalent.

(2) By the above remark and proposition 2.2, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

We consider the functional

$$J(u) = \int_{\Omega} \big(|\Delta u|^{p(x)} - \lambda |u|^{p(x)} \big) dx,$$

and give the following fundamental proposition.

Proposition 3.2. *For $u \in X$ we have*

(1) $\|u\| < (=; > 1) \Leftrightarrow J(u) < (=; > 1)$,
(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le J(u) \le \|u\|^{p^-}$,

(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq J(u) \leq \|u\|^{p^+}$, *for all* $u_n \in X$ *we have*
(4) $\|u_n\| \to 0 \Leftrightarrow J(u_n) \to 0$,
(5) $\|u_n\| \to \infty \Leftrightarrow J(u_n) \to \infty$.

The proof of this proposition is similar to the proof in [10, Theorem 1.3].
It is clear that the energy functional associated to (1.1) is defined by

$$\Psi(u) = \int_\Omega \frac{1}{p(x)}\big(|\Delta u|^{p(x)} - \lambda|u|^{p(x)}\big)dx - \int_\Omega F(x,u)dx.$$

where $F(x,s) = \int_0^s f(x,t)dt$. Let us define the functionals

$$\gamma(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} - \lambda|u|^{p(x)})dx,$$
$$\Gamma(u) = \int_\Omega F(x,u)dx.$$

It is well known that γ is well defined, even and C^1 in X. For the operator Γ, if the function f satisfies condition (H1). Then we have the following result.

Proposition 3.3. (i) $\Gamma \in C^1(X,\mathbb{R})$ *and for* u, v *in* X, *we have*

$$\langle \Gamma'(u), v\rangle = \int_\Omega f(x,u)v dx.$$

(ii) *The operator* $\Gamma' : X \to X'$ *is completely continuous.*

Proof. (i) By condition (H1), we have

$$|F(x,s)| \leq a(x)|s| + \frac{b}{\alpha(x)}|s|^{\alpha(x)} \leq A(x) + B|s|^{\alpha(x)}$$

where $A(x) \geq 0$, $A \in L^1(\Omega)$, $B \geq 0$ and $\alpha < p_2^*$. Then the Nemytskii operator properties implies that Γ is a C^1 operator in $L^{\alpha(x)}(\Omega)$. Since there is a continuous embedding of X into $L^{\alpha(x)}(\Omega)$, the function Γ is also C^1 in X and

$$\langle \Gamma'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx.$$

(ii) Let $(u_n)_n \subset X$ be a sequence such that $u_n \rightharpoonup u$. Using the compact embedding of X into $L^{\alpha(x)}(\Omega)$, there exists a subsequence, noted also $(u_n)_n$, such that $u_n \to u$ in $L^{\alpha(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytskii operator

$$\begin{array}{cccc} N_f: & L^{\alpha(x)} & \to & L^{\frac{\alpha(x)}{\alpha(x)-1}} \\ & u & \longmapsto & f(.,u) \end{array}$$

is continuous. Hence, $N_f(u_n) \to N_f(u)$ in $L^{\frac{\alpha(x)}{\alpha(x)-1}}(\Omega)$. Also in view of the Holder's inequality and the continuous embedding of X into $L^{\alpha(x)}(\Omega)$, we obtain

$$\begin{aligned} |\langle \Gamma'(u_n) - \Gamma'(u), v\rangle| &= \Big|\int_\Omega (f(x,u_n) - f(x,u))v(x)dx\Big| \\ &\leq 2\|N_f(u_n) - N_f(u)\|_{\frac{\alpha(x)}{\alpha(x)-1}} \|v\|_{\alpha(x)} \\ &\leq C\|N_f(u_n) - N_f(u)\|_{\frac{\alpha(x)}{\alpha(x)-1}} \|v\|. \end{aligned}$$

Thus, $\Gamma'(u_n) \to \Gamma'(u)$ in X'. Which completes the proof. $\square$

Consequently, the weak solutions of (1.1) are the critical points of the functional

$$\Psi(u) = \int_\Omega \frac{1}{p(x)}\big(|\Delta u|^{p(x)} - \lambda |u|^{p(x)}\big)dx - \int_\Omega F(x,u)dx.$$

Moreover, the operator $L := \gamma' : X \to X'$ defined as

$$\langle L(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v - \lambda |u|^{p(x)-2}uv)dx \quad \forall u, v \in X$$

satisfies the assertions of the following theorem.

Theorem 3.4. (1) *L is continuous, bounded and strictly monotone.*
(2) *L is of (S_+) type.*
(3) *L is a homeomorphism.*

Proof. (1) Since L is the Fréchet derivative of γ, it follows that L is continuous and bounded. Let us define the sets

$$U_p = \{x \in \Omega : p(x) \ge 2\}, \quad V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

Using the elementary inequalities

$$|x-y|^\gamma \le 2^\gamma (|x|^{\gamma-2}x - |y|^{\gamma-2}y).(x-y) \quad \text{if } \gamma \ge 2,$$
$$|x-y|^2 \le \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y).(x-y) \quad \text{if } 1 < \gamma < 2,$$

for all $(x,y) \in (\mathbb{R}^N)^2$, where $x.y$ denotes the usual inner product in $\mathbb{R}^N$, we obtain for all $u, v \in X$ such that $u \ne v$

$$\langle L(u) - L(v), u - v\rangle > 0,$$

which means that L is strictly monotone.

(2) Let $(u_n)_n$ be a sequence of X such that

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \limsup_{n\to+\infty}\langle L(u_n), u_n - u\rangle \le 0.$$

From proposition 3.2, it suffices to shows that

$$\int_\Omega (|\Delta u_n - \Delta u|^{p(x)} - \lambda |u_n - u|^{p(x)})dx \to 0. \tag{3.1}$$

In view of the monotonicity of L, we have

$$\langle L(u_n) - L(u), u_n - u\rangle \ge 0,$$

and since $u_n \rightharpoonup u$ in X, it follows that

$$\limsup_{n\to+\infty}\langle L(u_n) - L(u), u_n - u\rangle = 0. \tag{3.2}$$

Put

$$\varphi_n(x) = (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u).(\Delta u_n - \Delta u),$$
$$\xi_n(x) = (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u).(u_n - u).$$

By the compact embedding of X into $L^{p(x)}(\Omega)$, it follows that

$$u_n \to u \quad \text{in} \quad L^{p(x)}(\Omega),$$
$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \quad \text{in} \quad L^{q(x)}(\Omega)$$

where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ for all $x \in \Omega$. It results that

$$\int_\Omega \xi_n(x)dx \to 0. \tag{3.3}$$

It follows by (3.2) and (3.3) that

$$\limsup_{n\to+\infty} \int_\Omega \varphi_n(x)dx = 0. \tag{3.4}$$

Thanks to the above inequalities,

$$\int_{U_p} |\Delta u_n - \Delta u_k|^{p(x)}dx \le 2^{p^+} \int_{U_p} \varphi_n(x)dx,$$
$$\int_{U_p} |u_n - u_k|^{p(x)}dx \le 2^{p^+} \int_{U_p} \xi_n(x)dx.$$

Then

$$\int_{U_p} (|\Delta u_n - \Delta u|^{p(x)} - \lambda |u_n - u|^{p(x)})dx \to 0 \quad \text{as } n \to +\infty. \tag{3.5}$$

On the other hand, in V_p, setting $\delta_n = |\Delta u_n| + |\Delta u|$, we have

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}dx \le \frac{1}{p^- - 1} \int_{V_p} (\varphi_n)^{\frac{p(x)}{2}} (\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx\,.$$

By Young's inequality,

$$\begin{aligned} d\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}dx &\le \int_{V_p} [d(\varphi_n)^{\frac{p(x)}{2}}](\delta_n)^{\frac{p(x)}{2}(2-p(x))}dx, \\ &\le \int_{V_p} \varphi_n(d)^{\frac{2}{p(x)}}dx + \int_{V_p} (\delta_n)^{p(x)}dx. \end{aligned} \tag{3.6}$$

From (3.4) and since $\varphi_n \ge 0$, one can consider that

$$0 \le \int_{V_p} \varphi_n dx < 1.$$

If $\int_{V_p} \varphi_n dx = 0$ then $\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}dx = 0$. If not, we take

$$d = (\int_{V_p} \varphi_n(x)dx)^{-1/2} > 1,$$

and the fact that $\frac{2}{p(x)} < 2$, inequality (3.6) becomes

$$\begin{aligned} \int_{V_p} |\Delta u_n - \Delta u|^{p(x)}dx &\le \frac{1}{d}\Big(\int_{V_p} \varphi_n d^2 dx + \int_\Omega \delta_n^{p(x)}dx\Big), \\ &\le \Big(\int_{V_p} \varphi_n dx\Big)^{1/2}\Big(1 + \int_\Omega \delta_n^{p(x)}dx\Big). \end{aligned}$$

Note that, $\int_\Omega \delta_n^{p(x)}dx$ is bounded, which implies

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}dx \to 0 \quad \text{as } n \to +\infty.$$

A similar method gives

$$\int_{V_p} |u_n - u|^{p(x)}dx \to 0 \quad \text{as } n \to +\infty.$$

Hence, it result that

$$\int_{V_p} (|\Delta u_n - \Delta u|^{p(x)} - \lambda |u_n - u|^{p(x)}) dx \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.7}$$

Finally, (3.1) is given by combining (3.5) and (3.7).

(3) Note that the strict monotonicity of L implies this injectivity. Moreover, L is a coercive operator. Indeed, since $p^- - 1 > 0$, for each $u \in X$ such that $\|u\| \geq 1$ we have

$$\frac{\langle L(u), u\rangle}{\|u\|} = \frac{J(u)}{\|u\|} \geq \|u\|^{p^- - 1} \to \infty \quad \text{as } \|u\| \to \infty.$$

Consequently, thanks to a Minty-Browder theorem [22], the operator L is an surjection and admits an inverse mapping. It suffices then to show the continuity of L^{-1}. Let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X'. Let u_n and u in X such that

$$L^{-1}(f_n) = u_n \quad \text{and} \quad L^{-1}(f) = u.$$

By the coercivity of L, one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, which implies

$$\lim_{n\to+\infty} \langle L(u_n) - L(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty} \langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the second assertion and the continuity of L that

$$u_n \to \widehat{u} \quad \text{in } X \quad \text{and} \quad L(u_n) \to L(\widehat{u}) = L(u) \quad \text{in} \quad X'.$$

Moreover, since L is an injection, we conclude that $u = \widehat{u}$. This completes the proof. □

4. Proof of main results

Proof of theorem 1.1. Let A be the linear function

$$\begin{array}{cccc} A: & X & \to & \mathbb{R} \\ & v & \longmapsto & \int_\Omega f(x) v dx. \end{array}$$

A is a continuous function , indeed, let $\beta \in C_+(\overline{\Omega})$ such that

$$\frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = 1, \forall x \in \overline{\Omega},$$

thus, we have $\beta(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$. Using the second assertion of remark 3.1, there is a continuous embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$ which implies that there exists $C > 0$ such that

$$|v|_{\beta(x)} \leq C\|v\| \quad \text{for all } v \in X.$$

By proposition 2.1, we conclude that

$$\begin{aligned} |A(v)| &\leq \Big(\frac{1}{\alpha^-} + \frac{1}{\beta^-}\Big)|f|_{\alpha(x)} |v|_{\beta(x)} \\ &\leq C\Big(\frac{1}{\alpha^-} + \frac{1}{\beta^-}\Big)|f|_{\alpha(x)} \|v\|. \end{aligned}$$

Therefore, A is continuous. Since the operator L, in theorem 3.4, is an homeomorphism, there exists a unique $u \in X$ verifies $L(u) = A$. The proof is complete. □

Proof of theorem 1.2. From the condition of theorem 1.2, we have for all $(x,s) \in \Omega \times \mathbb{R}$,

$$|F(x,s)| \le a(x)|s| + \frac{b}{\beta}|s|^\beta \le A(x) + B|s|^\beta$$

where $A(x) \ge 0$, $A(x) \in L^1(\Omega)$ and $B \ge 0$. It follows that

$$\begin{aligned} \Psi(u) &= \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} - \lambda |u|^{p(x)})dx - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{p^+}J(u) - B\int_\Omega |u|^\beta dx - \|A\|_{L^1}. \end{aligned}$$

Note that for $\|u\|$ large enough we have $J(u) \ge \|u\|^{p^-}$, on the other hand, the fact that $\beta < p^- < p_2^*(x)$ gives that there exists $C' > 0$ such that $|u|_\beta \le C'\|u\|$. Hence,

$$\Psi(u) \ge \frac{1}{p^+}\|u\|^{p^-} - C_1\|u\|^\beta - C_2,$$

and this approaches $+\infty$ as $\|u\| \to +\infty$. Since Ψ is weakly lower semi-continuous, Ψ admits a minimum point u in X. Then u is a weak solution of (1.1). This completes the proof. □

Proof of theorem 1.3. For the proof of the above theorem, we will use the Mountain Pass Theorem. We start by the following lemmas.

Lemma 4.1. *Under assumption* (H1)–(H2)*, the functional Ψ satisfies the Palais Smale condition (P.S).*

Proof. Let $(u_n)_n$ be a (P.S) sequence for the functional Ψ: $\Psi(u_n)$ bounded and $\Psi'(u_n) \to 0$. Let us show that $(u_n)_n$ is bounded in X. Using hypothesis (H2), since $\Psi(u_n)$ is bounded, we have

$$\begin{aligned} C_1 &\ge \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} - \lambda|u|^{p(x)})dx - \int_\Omega \frac{u_n}{\theta} f(x,u_n)dx + C_2 \\ &\ge \frac{1}{p^+}J(u_n) - \int_\Omega \frac{u_n}{\theta} f(x,u_n)dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\langle \Psi'(u_n), u_n\rangle = \int_\Omega (|\Delta u_n|^{p(x)} - \lambda|u_n|^{p(x)})dx - \int_\Omega f(x,u_n)u_n dx,$$

which implies

$$C_1 \ge (\frac{1}{p^+} - \frac{1}{\theta})J(u_n) + \frac{1}{\theta}\langle \Psi'(u_n), u_n\rangle + C_2. \tag{4.1}$$

Suppose, by contradiction that $(u_n)_n$ unbounded in X, so $\|u_n\| \ge 1$ for rather large values of n and it results that

$$\|u_n\|^{p^-} \le J(u_n) \le \|u_n\|^{p^+}$$

for rather large values of n. Furthermore, $\Psi'(u_n) \to 0$ assure that there exists $C_3 > 0$ such that

$$-C_3\|u_n\| \le \langle \Psi'(u_n), u_n\rangle \le C_3\|u_n\|$$

for rather large values of n. Consequently,

$$C_1 \ge a(u_n) := \big(\frac{1}{p^+} - \frac{1}{\theta}\big)\|u_n\|^{p^-} - \frac{C_3}{\theta}\|u_n\| + C_2.$$

Since $p^- > 1$ and $(\frac{1}{p^+} - \frac{1}{\theta}) > 0$, we have $a(u_n) \to +\infty$ as $n \to +\infty$, what is a contradiction. So $(u_n)_n$ is a bounded sequence in X. □

Lemma 4.2. *There exist $r, C > 0$ such that $\Psi(u) \geq C$ for all $u \in X$ such that $\|u\| = r$.*

Proof. Conditions (H1) and (H3) assure that

$$|F(x,s)| \leq \varepsilon |s|^{p^+} + C(\varepsilon)|s|^{\alpha(x)} \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}.$$

For $\|u\|$ small enough, we have

$$\begin{aligned} \Psi(u) &\geq \frac{1}{p^+} J(u) - \int F(x,u)dx, \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \varepsilon \int |u|^{p^+} - C(\varepsilon)\int |u|^{\alpha(x)}. \end{aligned} \tag{4.2}$$

By condition (H1), it follows that

$$p^- \leq p \leq p^+ < \alpha^- \leq \alpha < p_2^*$$

then $X \subset L^{p^+}(\Omega)$ $X \subset L^{\alpha(x)}(\Omega)$, with a continuous and compact embedding, what implies the existence of $C_4, C_5 > 0$ such that

$$\|u\|_{L^{p^+}} \leq C_4\|u\| \quad \text{and} \quad \|u\|_{L^{\alpha(x)}} \leq C_5\|u\|$$

for all $u \in X$. Since $\|u\|$ is small enough, we deduce

$$\int |u|^{\alpha(x)} \leq \max\left(\|u\|_{L^{\alpha(x)}}^{\alpha^-}, \|u\|_{L^{\alpha(x)}}^{\alpha^+}\right) \leq C_6\|u\|^{\alpha^-}.$$

Replacing in (4.2), it results that

$$\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \varepsilon C_4^{p^+}\|u\|^{p^+} - C_7\|u\|^{\alpha^-},$$

with C_i are positives constants. Let us choose $\varepsilon > 0$ such that $\varepsilon C_4^{p^+} \leq \frac{1}{2p^+}$, we obtain

$$\begin{aligned} \Psi(u) &\geq \frac{1}{2p^+}\|u\|^{p^+} - C_7\|u\|^{\alpha^-} \\ &\geq \|u\|^{p^+}\left(\frac{1}{2p^+} - C_7\|u\|^{\alpha^- - p^+}\right). \end{aligned}$$

Since $p^+ < \alpha^-$, the function $t \mapsto (\frac{1}{2p^+} - C_7 t^{\alpha^- - p^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $C > 0$ such that

$$\Psi(u) \geq C \quad \forall u \in X : \|u\| = r.$$

The proof is complete. □

Proof of theorem 1.3. To apply the Mountain Pass Theorem, we must prove that

$$\Psi(tu) \to -\infty \quad \text{as } t \to +\infty,$$

for a certain $u \in X$. From condition (H2), we obtain

$$F(x,s) \geq c|s|^\theta \quad \text{for all } (x,s) \in \overline{\Omega} \times \mathbb{R}.$$

Let $u \in X$ and $t > 1$ we have

$$\begin{aligned}\Psi(tu) &= \int \frac{t^{p(x)}}{p(x)}[|\Delta u|^{p(x)} - \lambda |u|^{p(x)}]dx - \int F(x,tu)dx \\ &\le t^{p^+} \int \frac{1}{p(x)}[|\Delta u|^{p(x)} - \lambda |u|^{p(x)}]dx - ct^{\theta} \int |u|^{\theta} dx.\end{aligned}$$

The fact $\theta > p^+$, implies

$$\Psi(tu) \to -\infty \quad \text{as } t \to +\infty.$$

It follows that there exists $e \in X$ such that $\|e\| > r$ and $\Psi(e) < 0$. According to the Mountain Pass Theorem, Φ admits a critical value $\mu \ge C$ which is characterized by

$$\mu = \inf_{h \in \Lambda} \sup_{t \in [0,1]} \Phi(h(t))$$

where

$$\Lambda = \{h \in C([0,1], X) : h(0) = 0 \text{ and } h(1) = e\}.$$

This completes the proof. □

Proof of theorem 1.4. We use the Bartsch's fountain theorem [3]. The space X is a Banach reflexive and separable, then there exists $\{e_i\} \subset X$ and $\{f_i t\} \subset X'$ such that

$$X = \overline{\langle e_i, i \in \mathbb{N}^* \rangle}, \quad X' = \overline{\langle f_i, i \in \mathbb{N}^* \rangle}, \quad \langle e_i, f_j \rangle = \delta_{i,j},$$

where $\delta_{i,j}$ denotes the Kroneker symbol. For $k \in \mathbb{N}^*$. Put

$$\begin{gathered} X_k = \mathbb{R} e_k, \quad Y_k = \overset{k}{\underset{i=1}{\oplus}} X_i, \quad Z_k = \overset{\infty}{\underset{i=k}{\oplus}} X_i, \\ \beta_k = \sup\{|u|_{\alpha(x)} / \|u\| = 1, \quad u \in Z_k\}. \end{gathered}$$

□

We have the following lemma.

Lemma 4.3. *If $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, then* $\lim_{k \to +\infty} \beta_k = 0$.

Proof. It is clear that $0 < \beta_{k+1} \le \beta_k$, so, β_k converges to $\beta \ge 0$. Let $u_k \in Z_k$ such that

$$\|u_k\| = 1 \quad \text{and} \quad 0 \le \beta_k - |u_k|_{\alpha(x)} < \frac{1}{k}.$$

Then, there exists a subsequence, noted also by $(u_k)_k$, such that $u_k \rightharpoonup u$ in X and

$$\langle f_i, u \rangle = \lim_{k \to +\infty} \langle f_i, u_k \rangle = 0$$

for all $i \in \mathbb{N}^*$. Thus, $u = 0$ and $u_k \rightharpoonup 0$ in X. According to the remark 3.1, there is a compact embedding of X into $L^{\alpha(x)}(\Omega)$, which assure that $u_k \to 0$ in $L^{\alpha(x)}(\Omega)$. Hence, it results that $\beta_k \to 0$. □

Proof of theorem 1.4. From conditions (H2) and (H4), Ψ is an even function satisfies the Palais-Smale condition. We will prove that for k large enough, there exists $\rho_k > \gamma_k > 0$ such that

(A1) $b_k := \inf\{\Psi(u) / u \in Z_k, \|u\| = \gamma_k\} \to +\infty$ as $k \to +\infty$,
(A2) $a_k := \max\{\Psi(u) / u \in Y_k, \|u\| = \rho_k\} \le 0$.

The assertion of theorem 1.4 is then obtained by the fountain theorem.

(A1): For $u \in Z_k$ such that $\|u\| = \gamma_k > 1$, we have by the condition (H1)

$$\begin{aligned}\Psi(u) &= \int \frac{1}{p(x)} \Big[|\Delta u|^{p(x)} - \lambda |u|^{p(x)}\Big] dx - \int F(x,u)dx \\ &\geq \frac{1}{p^+} J(u) - B \int |u|^{\alpha(x)} dx - \|A\|_{L^1} \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - C_2 \int |u|^{\alpha(x)} dx - C_1.\end{aligned}$$

If $|u|_{\alpha(x)} \leq 1$ then $\int |u|^{\alpha(x)} dx \leq |u|^{\alpha^-}_{\alpha(x)} \leq 1$. However, if $|u|_{\alpha(x)} > 1$ then $\int |u|^{\alpha(x)} dx \leq |u|^{\alpha^+}_{\alpha(x)} \leq (\beta_k \|u\|)^{\alpha^+}$. So, we conclude that

$$\begin{aligned}\Psi(u) &\geq \begin{cases} \frac{1}{p^+} \|u\|^{p^-} - (C_2 + C_1) & \text{if } |u|_{\alpha(x)} \leq 1 \\ \frac{1}{p^+} \|u\|^{p^-} - C_2(\beta_k \|u\|)^{\alpha^+} - C_1 & \text{if } |u|_{\alpha(x)} > 1 \end{cases} \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - C_2(\beta_k \|u\|)^{\alpha^+} - C_3,\end{aligned}$$

For $\gamma_k = (C_2 \alpha^+ \beta_k^{\alpha^+})^{1/(p^- - \alpha^+)}$, it follows that

$$\Psi(u) \geq \gamma_k^{p^-} (\frac{1}{p^+} - \frac{1}{\alpha^+}) - C_3.$$

Since $\beta_k \to 0$ and $p^- \leq p^+ < \alpha^+$, we have $\gamma_k \to +\infty \quad$ as $k \to +\infty$. Consequently,

$$\Psi(u) \to +\infty \quad \text{as } \|u\| \to +\infty, u \in Z_k$$

and the assertion (A1) is true.

(A2): Condition (H2) implies

$$F(x,s) \geq C_1 |s|^\theta - C_2.$$

Let $u \in Y_k$ such that $\|u\| = \rho_k > \gamma_k > 1$. Then

$$\begin{aligned}\Psi(u) &\leq \frac{1}{p^-} J(u) - \int F(x,u)dx \\ &\leq \frac{1}{p^-} \|u\|^{p^+} - C_1 \int |u|^\theta dx - C_3.\end{aligned}$$

Note that the space Y_k has finite dimension, then all norms are equivalents and we obtain

$$\Psi(u) \leq \frac{1}{p^-} \|u\|^{p^+} - C_4 \|u\|^\theta - C_3.$$

Finally

$$\Psi(u) \to -\infty \quad \text{as } \|u\| \to +\infty, u \in Y_k$$

because $\theta > p^+$. The assertion (A2) is then satisfied and the proof of theorem 1.4 is complete. □

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Abdelrachid El Amrouss

University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco

E-mail address: elamrouss@fso.ump.ma, elamrouss@hotmail.com

Fouzia Moradi

University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco

E-mail address: f.moradi@hotmail.com

Mimoun Moussaoui

University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco

E-mail address: moussaoui@est.ump.ma