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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FRACTIONAL SEMILINEAR MIXED VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this article we study the fractional semilinear mixed Volterra-Fredholm integrodifferential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f\Big(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds, \int_{t_0}^{T} h(t,s,x(s))ds\Big),$$

where $t \in [t_0, T]$, $t_0 \geq 0$, $0 < \alpha < 1$, and f is a given function. We prove the existence and uniqueness of solutions to this equation, with a nonlocal condition.

1. Introduction

The problem of existence and uniqueness of solution of fractional differential equations have been considered by many authors; see for example [1, 2, 3, 7, 8, 9, 10, 11]). In particular, fractional differential equations with nonlocal conditions have been studied by N'Guerekata [3], Balachandran, and Park [7], Furati and Tatar [8], and by many others. In [10], the authors investigated the existence for a semilinear fractional differential equation with kernels in the nonlinear function by using the Banach fixed point theorem. The nonlocal Cauchy problem is discussed by authors in [7] using the fixed point concepts. Tidke [4] studied the non-fractional mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions using Leray-Schauder theorem. Motivated by these works, we study the existence of solutions for nonlocal fractional semilinear integrodifferential equations in Banach spaces by using fractional calculus and a Banach fixed point theorem.

Consider the fractional semilinear integrodifferential equation

$$\begin{gathered}\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds, \int_{t_0}^{T} h(t,s,x(s))ds), \\ x(t_0) = x_0 \in X.\end{gathered} \tag{1.1}$$

where $t \in J = [t_0, T]$, $t_0 \geq 0, 0 < \alpha < 1$, $x \in Y = C(J, X)$ is a continuous function on J with values in the Banach space X and $\|x\|_Y = \max_{t\in J} \|x(t)\|_X$, and the

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nonlinear functions $f : J \times X \times X \times X \to X$, $k : D \times X \to X$, and $h : D_0 \times X \to X$ are continuous. Here $D = \{(t,s) \in \mathbb{R}^2 : t_0 \le s \le t \le T\}$, and $D_0 = J \times J$. The operator $\frac{d^\alpha}{dt^\alpha}$ denotes the Caputo fractional derivative of order α. For brevity let

$$Kx(t) = \int_{t_0}^{t} k(t,s,x(s))ds, \quad Hx(t) = \int_{t_0}^{T} h(t,s,x(s))ds.$$

and we use the common norm $\|\cdot\|$.

The paper is organized as follows. In section 2, some definitions, lemmas, and assumptions are introduced to be used in the sequel. Section 3 will involve the main results and proofs of existence problem of (1.1), together with a nonlocal condition.

2. Preliminaries

In this section, present some definitions and lemmas to be used later.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. A function $f \in C_\mu$, $\mu \ge -1$ is said to be fractional integrable of order $\alpha > 0$ if

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds < \infty,$$

where $t_0 \ge 0$; and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.3. The fractional derivative in the Caputo sense is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{1-\alpha}\Big(\frac{df(t)}{dt}\Big) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds$$

for $0 < \alpha \le 1$, $t_0 \ge 0$, $f' \in C_{-1}$.

The properties of the above operators can be found in [6] and the general theory of fractional differential equations can be found in [5].

Next we introduce the so-called "Mild Solution" for fractional integrodifferential equation (1.1) (see [10, Definition 1.3]).

Definition 2.4. A continuous solution $x(t)$ of the integral equation

$$x(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} T(t-s) f(s, x(s), Kx(s), Hx(s)) ds \quad (2.1)$$

is called a mild solution of (1.1).

To proceed, we need the following assumptions:

- (A1) $T(\cdot)$ is a C_0-semigroup generated by the operator A on X which satisfies $M = \max_{t\in J} \|T(t)\|$.
- (A2) f is a continuous function and there exist positive constants L_1, L_2, and L such that

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le L_1(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|)$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in Y$, $L_2 = \max_{t\in J} \|f(t,0,0,0)\|$, and $L = \max\{L_1, L_2\}$.

(A3) k is a continuous function and there exist positive constants N_1, N_2, and N such that
$$\|k(t,s,x_1) - k(t,s,x_2)\| \le N_1\|x_1 - x_2\|$$
for all $x_1, x_2 \in Y$, $N_2 = \max_{(t,s)\in D} \|k(t,s,0)\|$, and $N = \max\{N_1, N_2\}$.

(A4) h is a continuous function and there exist positive constants C_1, C_2, and C such that
$$\|h(t,s,x_1) - h(t,s,x_2)\| \le C_1\|x_1 - x_2\|$$
for all $x_1, x_2 \in Y$, $C_2 = \max_{(t,s)\in D_0} \|h(t,s,0)\|$, and $C = \max\{C_1, C_2\}$.

3. Existence of solutions

In this section, we prove the main results on the existence of solutions to (1.1). Firstly, we obtain the following estimates.

Lemma 3.1. *If* (A3), (A4) *are satisfied, then the estimates*
$$\|Kx(t)\| \le (t - t_0)(N_1\|x\| + N_2)$$
$$\|Kx_1(t) - Kx_2(t)\| \le N_1(t - t_0)\|x_1 - x_2\|$$
and
$$\|Hx(t)\| \le (T - t_0)(C_1\|x\| + C_2)$$
$$\|Hx_1(t) - Hx_2(t)\| \le C_1(T - t_0)\|x_1 - x_2\|$$
are satisfied for any $t \in J$, and $x, x_1, x_2 \in Y$.

Proof. By (A3), we have
$$\begin{aligned}
\|Kx(t)\| &\le \int_{t_0}^{t} \|k(t,s,x(s))\| ds \\
&= \int_{t_0}^{t} \|k(t,s,x(s)) - k(t,s,0) + k(t,s,0)\| ds \\
&\le \int_{t_0}^{t} \|k(t,s,x(s)) - k(t,s,0)\| ds + \int_{t_0}^{t} \|k(t,s,0)\| ds \\
&\le N_1(t - t_0)\|x\| + N_2(t - t_0) \le (T - t_0)(N_1\|x\| + N_2).
\end{aligned}$$
On the other hand,
$$\begin{aligned}
\|Kx_1(t) - Kx_2(t)\| &\le \int_{t_0}^{t} \|k(t,s,x_1(s)) - k(t,s,x_2(s))\| ds \\
&\le N_1 \int_{t_0}^{t} \|x_1(s) - x_2(s)\| ds \\
&\le N_1\,(t - t_0)\,\|x_1 - x_2\|.
\end{aligned}$$
Similarly, for the other estimates, we use assumption (A4), to get
$$\|Hx(t)\| \le \int_{t_0}^{T} \|h(t,s,x(s))\| ds \le (T - t_0)(C_1\|x\| + C_2)$$
and
$$\|Kx_1(t) - Kx_2(t)\| \le C_1(T - t_0)\|x_1 - x_2\|.$$
□

The existence result for (1.1) and its proof is as follows.

Theorem 3.2. *If* (A1)-(A4) *are satisfied, and*

$$q\Gamma(\alpha+1) \geq ML\Big(1 + C(T-t_0) + \frac{N}{\alpha+1}(T-t_0)\Big)(T-t_0)^\alpha, \quad 0<q<1,$$

then the fractional integrodifferential equation (1.1) *has a unique solution.*

Proof. We use the Banach contraction principle to prove the existence and uniqueness of the mild solution to (1.1). Let $B_r = \{x \in Y : \|x\| \leq r\} \subseteq Y$, where $r \geq (1-q)^{-1}(M\|x_0\| + q)$, and define the operator Ψ on the Banach space Y by

$$\Psi x(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}T(t-s)f(s,x(s),Kx(s),Hx(s))ds.$$

Firstly, we show that the operator Ψ maps B_r into itself. For this, by using (A1), and triangle inequality, we have

$$\begin{aligned}
&\|\Psi x(t)\| \\
&\leq M\|x_0\| + \frac{1}{\Gamma(\alpha)}\|\int_{t_0}^t (t-s)^{\alpha-1}T(t-s)f(s,x(s),Kx(s),Hx(s))ds\| \\
&\leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s))\|ds \\
&\leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s)) \\
&\quad - f(s,0,0,0) + f(s,0,0,0)\|ds \\
&\leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|f(s,x(s),Kx(s),Hx(s)) - f(s,0,0,0)\|ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|f(s,0,0,0)\|ds.
\end{aligned}$$

Now, if (A2) is satisfied, then

$$\begin{aligned}
\|\Psi x(t)\| &\leq M\|x_0\| + \frac{ML_1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left(\|x(s)\| + \|Kx(s)\| + \|Hx(s)\|\right)ds \\
&\quad + \frac{ML_2}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}ds \\
&\leq M\|x_0\| + \frac{ML_1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|x(s)\|ds + \frac{ML_1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|Kx(s)\|ds \\
&\quad + \frac{ML_1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|Hx(s)\|ds + \frac{ML_2}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}ds.
\end{aligned}$$

Using Lemma 3.1, we have

$$\begin{aligned}
&\|\Psi x(t)\| \\
&\leq M\|x_0\| + \frac{ML_1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\|x\| \\
&\quad + \frac{ML_1}{\Gamma(\alpha)}(N_1\|x\| + N_2)\int_{t_0}^t (t-s)^{\alpha-1}(s-t_0)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{ML_1}{\Gamma(\alpha+1)}(T-t_0)(C_1\|x\|+C_2)(t-t_0)^\alpha+\frac{ML_2}{\Gamma(\alpha+1)}(t-t_0)^\alpha\\
&\le M\|x_0\|+\frac{ML_1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\|x\|+\frac{ML_1}{\Gamma(\alpha+2)}(t-t_0)^{\alpha+1}(N_1\|x\|+N_2)\\
&+\frac{ML_1}{\Gamma(\alpha+1)}(T-t_0)(C_1\|x\|+C_2)(t-t_0)^\alpha+\frac{ML_2}{\Gamma(\alpha+1)}(t-t_0)^\alpha\\
&=M\|x_0\|+\frac{ML_1N_2}{\Gamma(\alpha+2)}(t-t_0)^{\alpha+1}+\frac{ML_1C_2}{\Gamma(\alpha+1)}(T-t_0)(t-t_0)^\alpha+\frac{ML_2}{\Gamma(\alpha+1)}(t-t_0)^\alpha\\
&+\frac{ML_1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\Big(1+\frac{N_1}{\alpha+1}(t-t_0)+C_1(T-t_0)\Big)\|x\|,
\end{aligned}
$$

if $x\in B_r$, we have

$$
\begin{aligned}
\|\Psi x(t)\| &\le M\|x_0\|+\frac{ML}{\Gamma(\alpha+1)}\left(1+\frac{N}{\alpha+1}(T-t_0)+C(T-t_0)\right)(T-t_0)^\alpha\\
&\quad+\frac{MLr}{\Gamma(\alpha+1)}\left(1+\frac{N}{\alpha+1}(T-t_0)+C(T-t_0)\right)(T-t_0)^\alpha\\
&\le M\|x_0\|+q+qr\\
&\le (1-q)r+qr=r.
\end{aligned}
$$

Thus $\Psi B_r\subset B_r$. Next, we prove that Ψ is a contraction mapping. For this, let $x_1,x_2\in Y$. Applying (A1) and (A2), we have

$$
\begin{aligned}
&\|\Psi x_1(t)-\Psi x_2(t)\|\\
&=\|\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}T(t-s)f(s,x_1(s),Kx_1(s),Hx_1(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}T(t-s)f(s,x_2(s),Kx_2(s),Hx_2(s))ds\|\\
&\le\frac{M}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}\|f(s,x_1(s),Kx_1(s),Hx_1(s))\\
&\quad-f(s,x_2(s),Kx_2(s),Hx_2(s))\|ds\\
&\le\frac{ML_1}{\Gamma(\alpha)}\Big(\int_{t_0}^t(t-s)^{\alpha-1}\|x_1(s)-x_2(s)\|ds+\int_{t_0}^t(t-s)^{\alpha-1}\|Kx_1(s)-Kx_2(s)\|ds\\
&\quad+\int_{t_0}^t(t-s)^{\alpha-1}\|Hx_1(s)-Hx_2(s)\|ds\Big)
\end{aligned}
$$

then using (A3), (A4) and Lemma 3.1, one gets

$$
\begin{aligned}
&\|\Psi x_1(t)-\Psi x_2(t)\|\\
&\le\frac{ML_1}{\Gamma(\alpha)}\|x_1-x_2\|\Big(\int_{t_0}^t(t-s)^{\alpha-1}ds+N_1\int_{t_0}^t(t-s)^{\alpha-1}(s-t_0)ds\\
&\quad+C_1\int_{t_0}^t(t-s)^{\alpha-1}(T-t_0)ds\Big)\\
&\le\frac{ML_1}{\Gamma(\alpha)}\Big(\frac{(t-t_0)^\alpha}{\alpha}+\frac{N_1\Gamma(\alpha)(t-t_0)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{C_1(T-t_0)(t-t_0)^\alpha}{\alpha}\Big)\|x_1-x_2\|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{ML_1}{\Gamma(\alpha+1)}\Big(1 + C_1(T-t_0) + \frac{N_1}{\alpha+1}(t-t_0)\Big)(t-t_0)^{\alpha}\|x_1 - x_2\| \\
&\le \frac{ML}{\Gamma(\alpha+1)}\Big(1 + C(T-t_0) + \frac{N}{\alpha+1}(T-t_0)\Big)(T-t_0)^{\alpha}\|x_1 - x_2\| \\
&\le q\|x_1 - x_2\|.
\end{aligned}
$$

Therefore Ψ has a unique fixed point $x = \Psi(x) \in B_r$, which is a solution of (2.1), and hence is a mild solution of (1.1). □

The last result in this article is to prove the existence and uniqueness of solutions to (1.1), but with nonlocal condition of the form

$$
x(t_0) + g(x) = x_0, \tag{3.1}
$$

where $g : Y \to X$ is a given function that satisfies the condition

(A5) g is a continuous function and there exists a positive constant G such that

$$
\|g(x) - g(y)\| \le G\|x - y\|, \quad \text{for } x, y \in Y.
$$

Theorem 3.3. *If* (A1)-(A5) *are satisfied, and*

$$
q \ge M\Big(G + \frac{L}{\Gamma(\alpha+1)}\Big(1 + C(T-t_0) + \frac{N}{\alpha+1}(T-t_0)\Big)(T-t_0)^{\alpha}\Big), \quad 0 < q < 1,
$$

then the fractional integrodifferential equation (1.1) *with nonlocal condition* (3.1) *has a unique solution.*

Proof. We want to prove that the operator $\Phi : Y \to Y$ defined by

$$
\begin{aligned}
\Phi x(t) &= T(t-t_0)(x_0 - g(x)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T(t-s)f(s, x(s), Kx(s), Hx(s))ds
\end{aligned} \tag{3.2}
$$

has a fixed point. This fixed point is then a solution of (1.1) and (3.1). For this, choose $r \ge (1-q)^{-1}\big(M\big(\|x_0\| + \|g(0)\|\big) + q\big)$. The proof is similar to the proof of Theorem 3.2 and hence we write it briefly. Let $x \in B_r$, then by assumptions, we have

$$
\begin{aligned}
&\|\Phi x(t)\| \\
&\le M\Big(\|x_0\| + \|g(0)\|\Big) + \frac{ML}{\Gamma(\alpha+1)}\Big(1 + \frac{N}{\alpha+1}(T-t_0) + C(T-t_0)\Big)(T-t_0)^{\alpha} \\
&\quad + M\Big(G + \frac{L}{\Gamma(\alpha+1)}\Big(1 + C(T-t_0) + \frac{N}{\alpha+1}(T-t_0)\Big)(T-t_0)^{\alpha}\Big)r \\
&\le M\big(\|x_0\| + \|g(0)\|\big) + q + qr \\
&\le (1-q)r + qr = r.
\end{aligned}
$$

Thus $\Phi B_r \subset B_r$. Next, we prove that Φ is a contraction. For this, let $x_1, x_2 \in Y$, one can show that

$$
\begin{aligned}
&\|\Phi x_1(t) - \Phi x_2(t)\| \\
&\le MG\|x_1 - x_2\| + \frac{ML_1}{\Gamma(\alpha)}\|x_1 - x_2\|\int_{t_0}^{t}(t-s)^{\alpha-1}ds \\
&\quad + \frac{ML_1N_1}{\Gamma(\alpha)}\|x_1 - x_2\|\int_{t_0}^{t}(t-s)^{\alpha-1}(s-t_0)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{ML_1C_1}{\Gamma(\alpha)}\|x_1 - x_2\| \int_{t_0}^{t} (t-s)^{\alpha-1}(T-t_0)ds \\
&\leq MG\|x_1 - x_2\| + \frac{ML_1}{\Gamma(\alpha)}\Big(\frac{(t-t_0)^\alpha}{\alpha} + \frac{N_1\Gamma(\alpha)(t-t_0)^{\alpha+1}}{\Gamma(\alpha+2)} \\
&+ \frac{C_1(T-t_0)(t-t_0)^\alpha}{\alpha}\Big)\|x_1 - x_2\| \\
&\leq M\Big(G + \frac{L}{\Gamma(\alpha+1)}\Big(1 + C(T-t_0) + \frac{N}{\alpha+1}(T-t_0)\Big)(T-t_0)^\alpha\Big)\|x_1 - x_2\| \\
&\leq q\|x_1 - x_2\|.
\end{aligned}
$$

Therefore Φ has a unique fixed point $x = \Phi(x) \in B_r$, which is a solution of (3.2), and hence is a mild solution of (1.1) with condition (3.1). □

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