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EXISTENCE OF SOLUTIONS FOR NONLINEAR SECOND-ORDER TWO-POINT BOUNDARY-VALUE PROBLEMS

RUI-JUAN DU

Abstract. We consider the existence of solutions for the nonlinear second-order two-point ordinary differential equations

$$u''(t) + \lambda u(t) + g(u(t)) = h(t), \quad t \in [0,1]$$
$$u(0) = u(1) = 0, \quad \text{or} \quad u'(0) = u'(1) = 0$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, and $h \in L^1(0,1)$.

1. Introduction

We consider the existence of solutions for the seconder-order two-point ordinary differential equation

$$u''(t) + \lambda u(t) + g(u(t)) = h(t), \quad t \in [0,1] \tag{1.1}$$

satisfying either

$$u(0) = u(1) = 0, \tag{1.2}$$

or

$$u'(0) = u'(1) = 0 \tag{1.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, $h \in L^1(0,1)$. The parameter $\lambda \in \mathbb{R}$ is allowed change near $m^2\pi^2 (m = 1, 2, \dots)$, the m-th eigenvalue of the linear eigenvalue problem

$$\begin{gathered} u''(t) + \lambda u(t) = 0, \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{gathered} \tag{1.4}$$

and

$$\begin{gathered} u''(t) + \lambda u(t) = 0, \quad t \in [0,1], \\ u'(0) = u'(1) = 0. \end{gathered} \tag{1.5}$$

The linear problem associated with $(1.4), (1.5)$ are

$$\begin{gathered} u''(t) + \lambda u(t) = h(t), \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{gathered} \tag{1.6}$$

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and

$$u''(t)+\lambda u(t)=h(t),\quad t\in[0,1],\\ u'(0)=u'(1)=0, \tag{1.7}$$

and the corresponding existence results are known from the linear theory. Namely, if $\lambda \neq m^2\pi^2 (m=1,2,\dots)$, then (1.4), (1.5) have a unique solution for each given h; While for $\lambda = m^2\pi^2 (m=1,2,\dots)$ a solution exists if, and only if, h satisfies the orthogonality conditions

$$\int_0^1 h(t)\phi_i(t)dt = 0 \quad (i=1,2),$$

where $\phi_1(t)=\sin m\pi t$, $\phi_2(t)=\cos m\pi t$ are the eigenfunctions associated with the eigenvalue $m^2\pi^2$. In this case, there are infinity many solutions $u(t)=u_0(t)+a\sin\pi t$, $v(t)=v_0(t)+b\cos\pi t$, $a,b\in\mathbb{R}$ with u_0, v_0 are the any particular solution of (1.4), (1.5).

A similar situation arises when introducing a sufficiently nonlinearity g. Assuming for the moment g uniformly bounded, it is easy to see that $\lambda\neq m^2\pi^2$, (1.1)-(1.2), (1.1)-(1.3) again have a solution for each given h. If $\lambda = m^2\pi^2$, there are more difficulties to hold the existence of solutions of (1.1)-(1.2), (1.1)-(1.3). In [4], only when $m=1$, (1.1)-(1.2) is solvable if h satisfies so called the Landesman-Lazer condition

$$\limsup_{t\to-\infty} g(t) < \int_0^1 h(t)\phi_1(t)dt < \liminf_{t\to+\infty} g(t)\,.$$

In [9], the authors assumed the nonlinearity $f(t,u)=g(u)-h(t)$ did not satisfy Landesman-Lazer conditions, were also proved that the boundary value problem (1.1)-(1.2) has at least one solution , but m is only allowed equal to 1.

It is not difficulty to see that when $m=2,3,\dots$, the case became more complex, there are only a few scholars to study it. In addition, in most of the papers about second-order two-point are using the same method as [4, 9]. There aren't much more method to solve those problems.

Inspired by the above results, in this paper, we try to establish the existence results of boundary value problems (1.1)-(1.2), (1.1)-(1.3), λ is allowed to change near $m^2\pi^2 (m=1,2,\dots)$, the nonlinearity g has weaker conditions than [9], and the methods are different from the methods in [9].

2. Preliminaries

In this paper, we use the following assumptions in g and h:

(H1) $g:\mathbb{R}\to\mathbb{R}$ is continuous, there is $\alpha\in[0,1)$, $c,d\in(0,+\infty)$ such that

$$|g(u)|\le c|u|^\alpha + d,\quad u\in\mathbb{R}; \tag{2.1}$$

(H2) There exists $r>0$ such that

$$ug(u)>0,\quad |u|>r; \tag{2.2}$$

(H2') There exists $r>0$ such that

$$ug(u)<0,\quad |u|>r; \tag{2.3}$$

(H3) $h:[0,1]\to\mathbb{R}$, $h\in L^1(0,1)$ satisfying

$$\int_0^1 h(t)\phi_1(t)dt = 0. \tag{2.4}$$

(H3') $h : [0,1] \to \mathbb{R}$, $h \in L^1(0,1)$ satisfying

$$\int_0^1 h(t)\phi_2(t)dt = 0. \tag{2.5}$$

Remark. For convenience, we rewrite $\lambda = \lambda_m + \bar{\lambda}$, where λ_m is the m-th eigenvalue of the linear eigenvalue problem (1.1)-(1.2), or of (1.1)-(1.3). Then the ordinary differential equation (1.1) is equivalent to

$$u''(t) + \lambda_m u(t) + \bar{\lambda} u(t) + g(u(t)) = h(t). \tag{2.6}$$

Let X, Y be the linear Banach space $C^1[0,1]$, $L^1(0,1)$, whose norms are denoted by

$$\|u\| = \max\{\|u\|_0, \|u'\|_0\}, \quad \|u\|_1 = \int_0^1 |u(s)|ds,$$

where $\|u\|_0$ denotes the max norm $\|u\|_0 = \max\{u(t), t \in [0,1]\}$.

Let $L_i : \operatorname{dom} L_i \subset X \to Y$ $(i = 1, 2)$ be linear operators defined for $u \in \operatorname{dom} L_i$ as

$$L_i u := u'' + \lambda_m, \tag{2.7}$$

where $\operatorname{dom} L_1 = \{u \in W^{2,1}(0,1) : u(0) = u(1) = 0\}$, $\operatorname{dom} L_2 = \{u \in W^{2,1}(0,1) : u'(0) = u'(1) = 0\}$.

Lemma 2.1. *Let $L_i (i = 1, 2)$ be the linear operator as defined in* (2.7)*. Then*

$$\ker L_i = \{u \in X : u(t) = \rho\phi_i(t), \rho \in \mathbb{R}\},$$
$$\operatorname{Im} L_i = \{u \in Y : \int_0^1 u(t)\phi_i(t)dt = 0\}.$$

Defined the operator $P_i : X \to \ker L_i \cap X$, $Q_i : X \to \operatorname{Im} Q_i \cap Y$,

$$(P_i u)(t) = \phi_i(t) \int_0^1 u(s)\phi_i(s)ds, \tag{2.8}$$

$$(Q_i u)(t) = u(t) - (\int_0^1 u(s)\phi_i(s)ds)\phi_i(t). \tag{2.9}$$

It is easy to check that $\operatorname{Im} P_i = \ker L_i$, $Y/\operatorname{Im} Q_i = \operatorname{Im} L_i$ $(i = 1, 2)$, and to show the following Lemma.

Lemma 2.2. *Let $X_{P_i} = \ker L_i$, $X_{I-P_i} = \ker P_i$, $Y_{Q_i} = \operatorname{Im} L_i$, $Y_{I-P_i} = \operatorname{Im} Q_i$. Then*

$$X = X_{P_i} \oplus X_{I-P_i}, \quad Y = Y_{I-Q_i} \oplus Y_{Q_i}.$$

It is easy to check that the restriction of L_i to X_{I-P_i} is a bijection from X_{I-P_i} onto $\operatorname{Im} L_i$ $(i = 1, 2)$. We define $K_i : \operatorname{Im} L_i \to X_{I-P_i}$ by

$$K_i = L_i|_{X_{I-P_i}}^{-1}. \tag{2.10}$$

Define the nonlinear operator $G : X \to Y$ by

$$(Gu)(t) = g(u(t)) \quad t \in [0,1].$$

It is easy to check that $G : X \to Y$ is completely continuous. Obviously (1.1)-(1.2), (1.1)-(1.3) are equivalent to

$$L_i u + \bar{\lambda} u + Gu = h, \quad u \in D(L_i). \tag{2.11}$$

Since $\ker L_i = \operatorname{span}\{\phi_i(t)\}$ $(i = 1, 2)$, we see that each $x \in X$ can be uniquely decomposed as

$$x(t) = \rho\phi_i(t) + v(t) \quad t \in [0, 1],$$

for some $\rho \in \mathbb{R}$ and $v \in X_{I-P_i}$.

For $y \in Y$, we also have the decomposition

$$y(t) = \tau\phi_i(t) + w(t), \quad t \in [0, 1],$$

with $\tau \in \mathbb{R}$, $w \in Y_{Q_i}$ $(i = 1, 2)$.

Lemma 2.3. *The boundary-value problems* (2.11) *are equivalent to the system*

$$\begin{gathered} L_i v(t) + \bar{\lambda} v(t) + Q_i G(\rho\phi_i(t) + v(t)) = h(t), \\ \bar{\lambda}\int_0^1 (\phi_i(t))^2 dt + \int_0^1 \phi_i(t) G(\rho\phi_i(t) + v(t)) dt = 0. \end{gathered} \tag{2.12}$$

3. Main results

Theorem 3.1. *Assume* (H1), (H2), (H3). *Then there exists* $\lambda_+ > 0$ *such that* (1.1)-(1.2) *has at least one solutions in* $C^1[0, 1]$ *if* $\lambda \in [0, \lambda_+]$.

Theorem 3.2. *Assume* (H1), (H2'), (H3). *Then there exists* $\lambda_- < 0$ *such that* (1.1)-(1.2) *has at least one solutions in* $C^1[0, 1]$ *if* $\lambda \in [\lambda_-, 0]$.

Theorem 3.3. *Assume* (H1), (H2), (H3'). *Then there exists* $\lambda_+ > 0$ *such that* (1.1)-(1.3) *has at least one solutions in* $C^1[0, 1]$ *if* $\lambda \in [0, \lambda_+]$.

Theorem 3.4. *Assume* (H1), (H2'), (H3'). *Then there exists* $\lambda_- < 0$ *such that* (1.1)-(1.3) *has at least one solutions in* $C^1[0, 1]$ *if* $\lambda \in [\lambda_-, 0]$.

In this article, we prove only Theorem 3.1; the other theorems can be proved by using the similarly method.

Lemma 3.5. *Assume* (H1), (H2), (H3). *Then there exists* $M > 0$, *such that any solution* $u \in D(L_1)$ *of* (2.11) *satisfies* $\|u\| < M$, *as long as*

$$0 \le \bar{\lambda} \le \delta := \frac{1}{2\|K_1 J_1\|_{Y_{Q_1} \to X_{I-P_1}}} \tag{3.1}$$

where $J_1 : X \to Y$ *is defined by* $(J_1 u)(t) = u(t), t \in [0, 1]$.

Proof. We divide the proof into two steps.

Step I. Obviously $(L_1 + \bar{\lambda} J_1)|_{X_{I-P_1}} : X_{I-P_1} \to Y_{Q_1}$ is invertible for $\bar{\lambda} \le \delta$. Moreover, by (3.1),

$$\begin{aligned} \|(L_1 + \bar{\lambda} J_1)|_{X_{I-P_1}}^{-1}\|_{Y_{Q_1} \to X_{I-P_1}} &= \|L_1|_{X_{I-P_1}}^{-1} (I + \bar{\lambda} K_1 J_1)^{-1}\|_{Y_{Q_1} \to X_{I-P_1}} \\ &= \|K_1\|_{Y_{Q_1} \to X_{I-P_1}} \|(I + \bar{\lambda} K_1 J_1)^{-1}\|_{X_{I-P} \to X_{I-P_1}} \\ &\le 2\|K_1\|_{Y_{Q_1} \to X_{I-P_1}}. \end{aligned}$$

Let $u(t) = \rho\phi_1(t) + v(t) = \rho\sin m\pi t + v(t)$ is a solution of (2.11) for some $\rho \neq 0$. Then

$$\begin{aligned}
\|v\| &= \|(L_1 + \bar\lambda J_1)|^{-1}_{X_{I-P_1}} Q_1(h - g(\rho\sin m\pi t + v(t)))\| \\
&\leq \|(L_1 + \bar\lambda J_1)|^{-1}_{X_{I-P_1}}\|_{Y_{Q_1}\to X_{I-P_1}} \|Q_1\|_{Y\to Y_{Q_1}} \\
&\quad \times \big[\|h\|_1 + c(|\rho|\|\sin m\pi t\|_1 + \|v\|_1)^\alpha + d\big] \\
&\leq 2\|K_1\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}}[\|h\|_1 + c(|\rho|\|\sin m\pi t\| + \|v\|)^\alpha + d] \\
&\leq 2\|K_1\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}}[\|h\|_1 + c(|\rho|m\pi + \|v\|)^\alpha + d] \\
&= 2\|K\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}}[\|h\|_1 + cm\pi|\rho|^\alpha(1 + \frac{\|v\|}{m\pi|\rho|})^\alpha + d] \\
&\leq 2\|K_1\|_{Y_{Q_1}\to X_{I-P}}\|Q_1\|_{Y\to Y_{Q_1}}[\|h\|_1 + cm\pi|\rho|^\alpha(1 + \frac{\alpha\|v\|}{m\pi|\rho|}) + d] \\
&= 2\|K_1\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}} \\
&\quad \times \big[\|h\|_1 + cm\pi|\rho|^\alpha(1 + \frac{\alpha}{(m\pi|\rho|)^{1-\alpha}}\cdot\frac{\|v\|}{(m\pi|\rho|)^\alpha}) + d\big]
\end{aligned}$$

Hence,

$$\frac{\|v\|}{(m\pi|\rho|)^\alpha} \leq \frac{c_0}{(m\pi|\rho|)^\alpha} + c_1 + \frac{\alpha c_1}{(m\pi|\rho|)^{1-\alpha}}\cdot\frac{\|v\|}{(m\pi|\rho|)^\alpha}.$$

where

$$\begin{gathered}
c_0 = 2\|K_1\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}}(\|h\|_1 + d), \\
c_1 = 2c\|K_1\|_{Y_{Q_1}\to X_{I-P_1}}\|Q_1\|_{Y\to Y_{Q_1}}.
\end{gathered}$$

If

$$|\rho| \geq \frac{(2\alpha c_1)^{-\frac{1}{1-\alpha}}}{m\pi} := \tilde c,$$

then

$$\frac{\|v\|}{(m\pi|\rho|)^\alpha} \leq \frac{2c_0}{(m\pi\tilde c)^\alpha} + 2c_1 := \bar c. \tag{3.2}$$

Step II. If we assume that the conclusion of the lemma is false, we obtain a sequence $\{\bar\lambda_n\}$ with $0 \leq \bar\lambda_n \leq \delta, \bar\lambda_n \to 0$ and a sequence of corresponding solutions $\{u_n\} : u_n = \rho_n\phi_1(t)dt + (t), \rho_n \in \mathbb{R}, v_n \in X_{I-P_1}, n \in N$, such that

$$\|u_n\| \to +\infty.$$

From (3.2)

$$\|v\| \leq \bar c(m\pi)^\alpha(|\rho|)^\alpha := \hat c|\rho|^\alpha. \tag{3.3}$$

we conclude that $|\rho_n| \to +\infty$. We may assume that $\rho_n \to +\infty$, the other case can be treated in the same way. Then for all $n \in N$, we get that $\rho_n \geq \tilde c$.

Now, from (2.11) we obtain

$$\bar\lambda_n\rho_n\int_0^1(\sin m\pi t)^2dt + \int_0^1 \sin m\pi t g(\rho_n\sin m\pi t + v_n(t))dt = 0. \tag{3.4}$$

Since $\bar\lambda_n \geq 0$, $\int_0^1 \bar\lambda_n\rho_n(\sin m\pi t)^2dt \geq 0$, for all $n \in N$, so we have

$$\int_0^1 \sin m\pi t g(\rho_n\sin m\pi t + v_n(t))dt \leq 0. \tag{3.5}$$

Let $I^+ := \{t : t \in [0,1], \sin \pi t > 0\}$, $I^- := \{t : t \in [0,1], \sin \pi t < 0\}$. It is easy to see that $I^+ \cap I^- \neq 0$, and

$$\min\{|\sin m\pi t| t \in I^+ \cap I^-\} > 0. \tag{3.6}$$

Combining (3.6) and (3.3),we conclude

$$\lim_{\rho_n \to +\infty} \min\{\rho_n \sin m\pi t + v_n(t) | t \in I^+\} = +\infty. \tag{3.7}$$

$$\lim_{\rho_n \to +\infty} \min\{\rho_n \sin m\pi t + v_n(t) | t \in I^-\} = -\infty. \tag{3.8}$$

Applying (3.4),(3.7) and (3.8) and (H2), we conclude

$$\begin{aligned} \int_0^1 \sin m\pi t g(\rho_n \sin m\pi t + v_n(t)) dt &= \int_{t \in I^+} \sin m\pi t g(\rho_n \sin m\pi t + v_n(t)) dt \\ &\quad + \int_{t \in I^-} \sin m\pi t g(\rho_n \sin m\pi t + v_n(t)) dt > 0 \end{aligned}$$

hold for some n large enough. This contradicts (3.5). □

Similarly, we obtain the following result.

Lemma 3.6. *Assume* (H1), (H2'), (H3). *Then there exists $M' > 0$, such that any solution $u \in D(L_1)$ of* (2.11) *satisfies*

$$\|u\| < M',$$

as long as $-\delta \le \lambda \le 0$, where δ and J_1 as lemma 3.5

Proof of Theorem 3.1. Consider the linear operator $L : X \to Y$, defined for $u \in \operatorname{dom} L$ by

$$Lu = L_1 u + \bar{\lambda} u = \lambda_m u + \bar{\lambda} u,$$

and the family maps $T_\mu : X \to Y$ $(0 \le \mu \le 1)$,

$$(T_\mu u)(t) = \mu(h(t) - g(u(t))), \quad t \in [0,1].$$

where $\operatorname{dom} L := \{u \in W^{2,1}(0,1) : u(0) = u(1) = 0\}$. Observe that L is invertible with, let $K : Y \to X$, then $K = L^{-1}$, and

$$u(t) = K(G(u(t)) - h(t)), \quad t \in [0,1]. \tag{3.9}$$

If

$$R = \{u \in X : \|u\| \le M + 1\},$$

we can define a compact homotopy $H_\mu : R \to \operatorname{dom} L$,

$$H_\mu = L^{-1} \circ (T_\mu u) \circ J_1.$$

We can see that the fixed points of H_μ are exactly the solution of (1.1)-(1.2), and the choice of R enables us to say that the homotopy H_μ is fixed-point free on the boundary of R. since $H_0 = 0$, by the Leray-Schauder theory [3], we obtain that H_1 has a fixed point and so there is a solution to (1.1)-(1.2). □

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