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EXISTENCE OF WEAK SOLUTIONS FOR DEGENERATE SEMILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. In this study, we prove the existence of a weak solution for the degenerate semilinear elliptic Dirichlet boundary-value problem

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

in a suitable weighted Sobolev space. Here the domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is not necessarily bounded, and h is a continuous bounded nonlinearity. The theory is also extended for h continuous and unbounded.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a domain (not necessarily bounded) with boundary $\partial\Omega$. Let L be an elliptic operator in divergence form

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i u(x)) \quad \text{with } D_j = \frac{\partial}{\partial x_j},$$

with coefficients $a_{ij}/\omega \in L^\infty(\Omega)$ which are symmetric and satisfy the degenerate ellipticity condition

$$\lambda |\xi|^2 \omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \omega(x), \quad \text{a.e. } x \in \Omega, \tag{1.1}$$

for all $\xi \in \mathbb{R}^n$ and ω is an A_2-weight ($\lambda > 0, \Lambda > 0$). Let $f/\omega \in L^2(\Omega, \omega)$ and h be a real valued continuous function defined on $\mathbb{R}$. Recently Cavalheiro [2] studied the BVP

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $g_1/\omega \in L^\infty(\Omega)$, $\mu > 0$, h is a bounded continuous function and where Ω is bounded. In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occurs as spaces of solutions for elliptic and parabolic PDEs. For degenerate problems with various types of singularities in the coefficients it is natural to look for solutions in weighted Sobolev spaces; for example, see [1, 3, 4, 5, 6, 7].

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The treatment of problem (1.2) has not been effective since the usual compactness arguments for bounded domains may not extend to unbound domains. One natural approach is to approximate a solution of (1.2) by a sequence of solutions in bounded subdomains of Ω. The present work is a generalization of the work by Cavalheiro [2], for unbounded domain Ω such that, $\Omega = \cup_{i=1}^{\infty}\Omega_i$, $\Omega_i \subseteq \Omega_{i+1}$, for each $i \geq 1$. Section 2 deals with preliminaries and some basic results. Section 3 contains the existence of a sequence of solutions $\{u_i\}$ of (1.2) in each bounded subdomains Ω_i and a uniform bound for them. The main result is about the extraction of a solution for (1.2) from $\{u_i\}$. Finally section 4 deals with extension for a class of continuous function h, not necessarily bounded.

2. Preliminaries

We need the following preliminaries for the ensuing study. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an open connected set. Let $\omega : \mathbb{R}^n \to \mathbb{R}^+$ be a locally integrable non negative function with $0 < \omega < \infty$ a.e. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $c = c_{p,\omega}$ such that

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{\frac{1}{1-p}}(x)dx\Big)^{p-1} \leq c,$$

for all balls B in $\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. We assume that ω belongs to Muckenhoupt class A_p, $1 < p < \infty$ (i.e. ω is an A_p-weight). For more details on A_p-weight, we refer the reader to [9, 11, 16]. We shall denote by $L^p(\Omega,\omega)$ $(1 \leq p < \infty)$ the usual Banach space of measurable real valued functions, f, defined in Ω for which

$$\|f\|_{p,\Omega} = \Big(\int_\Omega |f(x)|^p\omega(x)dx\Big)^{1/p} < \infty \tag{2.1}$$

For $p \geq 1$ and k a non-negative integer, the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ is defined by

$$W^{k,p}(\Omega,\omega) := \{u \in L^p(\Omega,\omega) : D^\alpha u \in L^p(\Omega,\omega), 1 \leq |\alpha| \leq k\}$$

with the associated norm

$$\|u\|_{k,p,\Omega} = \|u\|_{p,\Omega} + \sum_{1\leq|\alpha|\leq k}\|D^\alpha u\|_{p,\Omega}. \tag{2.2}$$

If $\omega \in A_p$ then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\overline{\Omega})$ with respect to the norm (2.2) and the space $W_0^{k,p}(\Omega,\omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{0,k,p,\Omega} = \sum_{1\leq|\alpha|\leq k}\|D^\alpha u\|_{p,\Omega}.$$

For details we refer the reader to [4, Proposition 3.5]. We also note that $W^{k,2}(\Omega,\omega)$ and $W_0^{k,2}(\Omega,\omega)$, are Hilbert spaces. At each step, a generic constant is denoted by c or k_0 in order to avoid too many suffices. We need the following result.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain and let ω be A_2 weight. Then*

$$W_0^{1,2}(\Omega,\omega) \hookrightarrow\hookrightarrow L^2(\Omega,\omega) \tag{2.3}$$

(i.e the inclusion is compact) and there exists $C_\Omega > 0$ such that

$$\|u\|_{2,\Omega} \leq C_\Omega\|u\|_{0,1,2,\Omega}, \quad \forall u \in W_0^{1,2}(\Omega,\omega), \tag{2.4}$$

where C_Ω may be taken to depend only on $n, 2$ and the diameter of Ω.

A proof of the above statement can be found in [7, Theorem 4.6].

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be an open connected set. We say that $u \in W_0^{1,2}(\Omega, \omega)$ is a called a weak solution of (1.2) if

$$\begin{aligned} &\int_\Omega a_{ij} D_i u(x) D_j \phi(x) dx - \int_\Omega \mu u(x) g_1(x) \phi(x) dx + \int_\Omega h(u(x)) g_2(x) \phi(x) dx \\ &= \int_\Omega f(x) \phi(x) dx \end{aligned}$$

for every $\phi \in W_0^{1,2}(\Omega, \omega)$.

In section 3, we use the following result.

Theorem 2.3. *Let $B, N : X \to X^*$ be operators on the real separable reflexive Banach space X.*

1. *the operator $B : X \to X^*$ is linear and continuous;*
2. *the operator $N : X \to X^*$ is demicontinuous and bounded;*
3. *$B + N$ is asymptotically linear;*
4. *for each $T \in X^*$ and for each $t \in [0, 1]$, the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S) in X.*

If $Bu = 0$ implies $u = 0$, then for each $T \in X^*$, the equation $Bu + Nu = T$ has a solution in X.

For a detailed proof of the above Theorem, we refer to [12] or to [17, Theorem 29.C].

Definition 2.4. Let $B : X \to X^*$ be an operator on the real separable reflexive Banach space X. Then, B satisfies condition (S) if

$$u_n \rightharpoonup u \text{ and } \lim_{n\to\infty} (Bu_n - Bu | u_n - u) = 0, \text{ implies } u_n \to u, \tag{2.5}$$

where $(f|x)$ denotes the value of linear functional f at x.

We need the following hypotheses for further study.

- (H1) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function;
- (H2) $\omega \in A_2$;
- (H3) Assume $g_1/\omega \in L^\infty(\Omega)$, $g_2/\omega \in L^2(\Omega, \omega)$ and $f/\omega \in L^2(\Omega, \omega)$.

Remark 2.5. If $u_k \in W_0^{1,2}(\Omega_k, \omega)$ is a solution of (2.6) (see below) on Ω_k, then, for any $k \geq i$, u_k is also a solution of (2.6) on Ω_i, which has been used in Lemma 2.6.

Lemma 2.6. *Assume* (H1)-(H3). *Let μ >0 not be an eigenvalue of*

$$\begin{gathered} Lu - \mu u(x)\omega(x) = 0 \quad \text{in } \Omega_i, \\ u = 0 \quad \text{on } \partial\Omega_i \end{gathered}$$

for $i = 1, 2, 3, \dots$ Then, the BVP

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega_i, \\ u = 0 \quad \text{on } \partial\Omega_i \end{gathered} \tag{2.6}$$

has a solution $u = u_i \in W_0^{1,2}(\Omega_i, \omega)$*. In addition, if*

$$\lambda > \mu\, C_{\Omega_i} \Big\| \frac{g_1}{\omega} \Big\|_{\infty,\Omega}, \tag{2.7}$$

then for $k \geq i$*,* $\|u_k\|_{0,1,2,\Omega_i} \leq k_0$*, where* k_0 *is independent of* k*.*

Proof. We define the operators $B_1, B_2 : W_0^{1,2}(\Omega_i, \omega) \times W_0^{1,2}(\Omega_i, \omega) \to \mathbb{R}$ by

$$B_1(u, \phi) = \int_{\Omega_i} a_{ij} D_i u(x) D_j \phi(x) dx - \int_{\Omega_i} \mu u(x) g_1(x) \phi(x) dx$$
$$B_2(u, \phi) = \int_{\Omega_i} h(u(x)) g_2(x) \phi(x) dx.$$

Also define $T : W_0^{1,2}(\Omega_i, \omega) \to \mathbb{R}$ by

$$T(\phi) = \int_{\Omega_i} f(x) \phi(x) dx.$$

A function $u = u_i \in W_0^{1,2}(\Omega_i, \omega)$ is a solution of (2.6) if

$$B_1(u, \phi) + B_2(u, \phi) = T(\phi), \quad \forall \phi \in W_0^{1,2}(\Omega_i, \omega).$$

Using the identification principle [18, Theorem 21.18], we have $W_0^{1,2}(\Omega_i, \omega) = [W_0^{1,2}(\Omega_i, \omega)]^*$ and $\langle u, v\rangle = (u|v)$, where $\langle ., .\rangle$ denotes the inner product on a Hilbert space. We define the operators $B, N : W_0^{1,2}(\Omega_i, \omega) \to W_0^{1,2}(\Omega_i, \omega)$ as

$$(Bu|\phi) = B_1(u, \phi), \quad (Nu|\phi) = B_2(u, \phi), \quad \text{for } u, \phi \in W_0^{1,2}(\Omega_i, \omega).$$

Then, problem (2.6) is equivalent to operator equation $Bu + Nu = T$, $u \in W_0^{1,2}(\Omega_i, \omega)$. The proof of the existence for (2.6) is similar to that given in [2]. The proof of the latter part of the theorem (which is not in [2]) is given below. Let $|h(t)| \leq A, t \in \mathbb{R}$. Let $u_k \in W_0^{1,2}(\Omega_k, \omega)$ be the solutions of (2.6). Then, from the hypotheses, with the help of Lemma 2.1 and from the Remark 2.5, we note that, for $k \geq i$,

$$|B_1(u_k, u_k)| \leq (c + C_{\Omega_i}|\mu| \| \frac{g_1}{\omega} \|_{\infty,\Omega_i}) \|u_k\|_{0,1,2,\Omega_i} \|u_k\|_{0,1,2,\Omega_i}$$
$$|B_2(u_k, u_k)| \leq A C_{\Omega_i} \| \frac{g_2}{\omega} \|_{2,\Omega_i} \|u_k\|_{0,1,2,\Omega_i}$$
$$|T(u_k)| \leq C_{\Omega_i} \| \frac{f}{\omega} \|_{2,\Omega_i} \|u_k\|_{0,1,2,\Omega_i},$$

where C_{Ω_i} (is the constant of Lemma 2.1) and A are constants independent of k. Also, $B_1(., .)$ is a regular Gårding form [18, p.364]. In fact, we obtain, for $k \geq i$

$$\begin{aligned} B_1(u_k, u_k) &\geq \lambda \int_{\Omega_i} |Du_k|^2 \omega dx - \mu \Big\| \frac{g_1}{\omega} \Big\|_{\infty,\Omega_i} \int_{\Omega_i} u_k^2 \omega dx \\ &= \lambda \|u_k\|_{0,1,2,\Omega_i}^2 - \mu \| \frac{g_1}{\omega} \|_{\infty,\Omega_i} \|u_k\|_{2,\Omega_i}^2 \end{aligned}$$

Now, by Lemma 2.1, we have

$$B_1(u_k, u_k) \geq \big(\lambda - C_{\Omega_i} \mu \| \frac{g_1}{\omega} \|_{\infty,\Omega_i}\big) \|u_k\|_{0,1,2,\Omega_i}^2.$$

Since, $\lambda > C_{\Omega_i} \mu \| \frac{g_1}{\omega} \|_{\infty,\Omega_i}$, we obtain

$$\|u_k\|_{0,1,2,\Omega_i}^2 \leq \Big(\frac{1}{\lambda - C_{\Omega_i} \mu \| \frac{g_1}{\omega} \|_{\infty,\Omega_i}} \Big) B_1(u_k, u_k) \tag{2.8}$$

Also, we note that

$$|B_1(u_k,u_k)| \le C_{\Omega_i}\big\{A\|\frac{g_2}{\omega}\|_{2,\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big\}\|u_k\|_{0,1,2,\Omega_i}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned} \|u_k\|_{0,1,2,\Omega_i} &\le \frac{C_{\Omega_i}\big\{A\|\frac{g_2}{\omega}\|_{2,\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big\}}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i})} \\ &\le \frac{C_{\Omega_i}\big\{A\|\frac{g_2}{\omega}\|_{2,\Omega} + \|\frac{f}{\omega}\|_{2,\Omega}\big\}}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega})} = k_0, \end{aligned}$$

where k_0 is independent of k. Hence,

$$\|u_k\|_{0,1,2,\Omega_i} \le k_0, \quad \forall k \ge i \tag{2.10}$$

□

Corollary 2.7. *Under the hypotheses of Lemma 2.6, let M be any open bounded domain in Ω such that $M \subseteq \Omega_i$, for some i. For $k \ge i$, let u_k be a solution of*

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega_k, \\ u = 0 \quad \text{on } \partial\Omega_k \end{gathered}$$

Then, there exists a constant $k_0 > 0$ such that $\|u_k\|_{0,1,2,M} \le k_0$, where k_0 is independent of k.

The proof of this result is similar to that of Lemma 2.6 and hence omitted.

Remark 2.8. *Corollary 2.7 is needed in the main result stated in §3. Lemma 2.6 is a "modification" of the result in [2], which gives a uniform u_k, $k \ge i$ at the cost of the restriction on μ as given by (2.7).*

3. Main results

In this section, we dispense with the condition (2.7) when g_1 does not change sign. We consider a BVP

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G \end{gathered} \tag{3.1}$$

where $G \subset \mathbb{R}^n$ is an open bounded set, $n \ge 3$. The two results are related to the cases when $g_1 > 0$ with $\mu < 0$ and $g_1 < 0$ with $\mu > 0$. These results are similar to that found in [2] but with suitable changes.

Proposition 3.1. *Let $G \subset \Omega$ be an open bounded set in $\mathbb{R}^n$, $n \ge 3$. Suppose that* (H1)–(H3) *hold. Let $g_1 > 0$ and $\mu < 0$, then the BVP*

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G \end{gathered} \tag{3.2}$$

has a solution $u \in W_0^{1,2}(G,\omega)$.

Proof. As in Lemma 2.6, the basic idea is to reduce the problem (3.2) to an operator equation $Bu + Nu = T$ with the help of the Theorem 2.3. To do proceed, we define

B, N, and T with Ω_i replaced by G, as in Lemma 2.6 and after a little bit of computation, we have

$$|B_1(u,\phi)| \le (c + C_G|\mu|\|\frac{g_1}{\omega}\|_{\infty,G})\|u\|_{0,1,2,G}\|\phi\|_{0,1,2,G}$$
$$|B_2(u,\phi)| \le C_G A\|\frac{g_2}{\omega}\|_{2,G}\|\phi\|_{0,1,2,G}$$
$$|T(\phi)| \le C_G\|\frac{f}{\omega}\|_{2,G}\|\phi\|_{0,1,2,G}$$

where c (a generic constant), A are constants depending on n, p and the constant C_G comes from Lemma 2.1. With these preliminaries, (3.2) is equivalent to

$$Bu + Nu = T, \quad u \in W_0^{1,2}(G,\omega).$$

The compact embedding of $W_0^{1,2}(G,\omega) \hookrightarrow\hookrightarrow L^2(G,\omega)$, shows that $B_1(.,.)$ is a strict regular Gårding form. Also, $\mu < 0$ and $g_1 > 0$ yields

$$B_1(u,u) = \int_G a_{ij}D_iu(x)D_ju(x)dx - \int_G \mu u^2(x)g_1(x)dx \ge \lambda\|u\|^2_{0,1,2,G} \tag{3.3}$$

Next, we also show that $B+N$ is asymptotically linear and N strongly continuous. The proof is similar to the one in [2] and we omit the same for brevity. Since μ is not an eigenvalue of

$$\begin{gathered} Lu - \mu u(x)\omega(x) = 0 \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G, \end{gathered} \tag{3.4}$$

$Bu = 0$ implies $u = 0$. By Theorem 2.3, $Bu+Nu = T$ has a solution $u \in W_0^{1,2}(G,\omega)$ which equivalently shows the BVP (3.2) has a solution $u \in W_0^{1,2}(G,\omega)$. □

We consider the boundary-value problem

$$\begin{gathered} Lu - \mu u g_1 + h(u)g_2 = f \quad \text{in } \Omega_i, \\ u = 0 \quad \text{on } \partial\Omega_i \end{gathered} \tag{3.5}$$

where $\Omega_i \subseteq \mathbb{R}^n,\ n \ge 3$ is an open bounded set, for $i \ge 1$.

Corollary 3.2. *Let the hypotheses of Proposition 3.1 hold for Ω_i in place of G, for $i \ge 1$. Then, there exists $u_i \in W_0^{1,2}(\Omega_i,\omega)$ which satisfies (3.5) and in addition, for $k \ge i$,*

$$\|u_k\|_{0,1,2,\Omega_i} \le k_0, \tag{3.6}$$

where k_0 is a constant independent of k.

The proof of the above corollary is similar to the later part of the Lemma 2.6 and hence omitted. With suitable changes in the proof of Proposition 3.1, we arrive at the following result.

Theorem 3.3. *Let the hypotheses of Proposition 3.1 hold, except that $g_1 < 0$ and $\mu > 0$. Let μ not be an eigenvalue of*

$$\begin{gathered} Lu - \mu u(x)\omega(x) = 0 \quad \textit{in } G, \\ u = 0 \quad \textit{on } \partial G \end{gathered} \tag{3.7}$$

Then the (3.2) has a solution $u \in W_0^{1,2}(G,\omega)$.

Corollary 3.4. *Let the hypotheses of Proposition 3.1 hold for Ω_i in place of G, for $i \geq 1$. Then, there exists $u_i \in W_0^{1,2}(\Omega_i, \omega)$ which satisfies* (3.5) *and in addition, for $k \geq i$,*

$$\|u_k\|_{0,1,2,\Omega_i} \leq k_0,$$

where k_0 is a constant independent of k.

The proof of the above corollary is similar to Corollary 3.2 and hence omitted.

Theorem 3.5. *Let $\Omega = \cup_{i=1}^{\infty} \Omega_i, \Omega_i \subseteq \Omega_{i+1}$ be open bounded domains in Ω. Let $\mu > 0$ not be an eigenvalue of*

$$\begin{gathered} Lu - \mu u(x)\omega(x) = 0 \quad \text{in } \Omega_i, \\ u = 0 \quad \text{on } \partial\Omega_i \end{gathered} \tag{3.8}$$

for $i = 1, 2, 3, \dots$ and in addition the condition $\lambda > C_{\Omega_i} \mu \|\frac{g_1}{\omega}\|_{\infty,\Omega}$ be fulfilled. Under the hypotheses (H1)-(H3), (1.2) *has a weak solution $u \in W_0^{1,2}(\Omega, \omega)$.*

Proof. A part of this proof follows from [10, 14, 15]. Let $\{u_k\}$ be the sequence of solutions for (3.5) in $W_0^{1,2}(\Omega_k, \omega), (k \geq 1)$. Let $\tilde{u}_k($ for $k \geq 1)$ denote the extension of u_k by zero outside Ω_k, which we continue to denote it by u_k. From (2.10), we have

$$\|u_k\|_{0,1,2,\Omega_l} \leq k_0, \text{ for } k \geq l.$$

Then, $\{u_k\}$ has a subsequence $\{u_{k_m^1}\}$ which converges weakly to u^1, as $m \to \infty$, in $W_0^{1,2}(\Omega_1, \omega)$. Since $\{u_{k_m^1}\}$ is bounded in $W_0^{1,2}(\Omega_2, \omega)$, it has a convergent subsequence $\{u_{k_m^2}\}$ converging weakly to u^2 in $W_0^{1,2}(\Omega_2, \omega)$. By induction, we have $\{u_{k_m^{l-1}}\}$ has a subsequence $\{u_{k_m^l}\}$ which weakly converges to u^l in $W_0^{1,2}(\Omega_l, \omega)$, i.e in short, we have $u_{k_m^l} \rightharpoonup u^l$ in $W_0^{1,2}(\Omega_l, \omega)$, $l \geq 1$. Define $u : \Omega \to \mathbb{R}$ by

$$u(x) := u^l(x), \quad \text{for } x \in \Omega_l.$$

(Here there is no confusion occurs since $u^l(x) = u^m(x)$ for $x \in \Omega$ for any $m \geq l$).

Let M be any fixed (but arbitrary) bounded domain such that $M \subseteq \Omega$. Then there exists an integer l such that $M \subseteq \Omega_l$. We note that, the diagonal sequence $\{u_{k_m^m}; m \geq l\}$ converges weakly to $u = u^l$ in $W_0^{1,2}(M, \omega)$, as $m \to \infty$.

What remains is to show that u is the required weak solution. It is sufficient to show that u is a weak solution of (1.2) for an arbitrary bounded domain M in Ω. Since $u_{k_m^m} \rightharpoonup u^l$ in $W_0^{1,2}(M, \omega)$, we have

$$\int_M \nabla(u_{k_m^m} - u).\nabla\phi\omega dx \to 0, \quad \text{as } m \to \infty,$$

implies

$$\int_M D_i(u_{k_m^m} - u) D_j\phi\omega dx \to 0, \quad \text{as } m \to \infty.$$

From (1.1), for a constant c, we have $|a_{ij}| \leq c\omega$.

$$\begin{aligned} \int_M a_{ij} D_i(u_{k_m^m} - u) D_j\phi \, dx &\leq \int_M |a_{ij}||D_i(u_{k_m^m} - u)||D_j\phi| dx \\ &\leq c\|D_i(u_{k_m^m} - u)\|_{2,M} \|D_j\phi\|_{2,M} \to 0, \end{aligned} \tag{3.9}$$

as $m \to \infty$. Also, by Lemma 2.1, $u_{k_m^m} \to u$ in $L^2(M,\omega)$. We have

$$
\begin{aligned}
\Big|\int_M (u_{k_m^m} - u) g_1 \phi dx\Big| &\leq \int_M |(u_{k_m^m} - u)||g_1||\phi| dx \\
&\leq \int_M |(u_{k_m^m} - u)||\frac{g_1}{\omega}||\phi|\omega\, dx \\
&\leq \|\frac{g_1}{\omega}\|_{\infty,M}\|u_{k_m^m} - u\|_{2,M}\|\phi\|_{2,M}.
\end{aligned}
$$

So we have now

$$
\mu \int_M u_{k_m^m} g_1 \phi dx \to \mu \int_M u g_1 \phi dx\,. \tag{3.10}
$$

A little computation shows that

$$
\int_M h(u_k(x)) \to \int_M h(u(x)), \tag{3.11}
$$

which follows from dominated convergence theorem, if needed through a subsequence. Since M is an arbitrary bounded domain in Ω, it follows from (3.9), (3.10) and (3.11),

$$
\begin{aligned}
&\int_\Omega a_{ij} D_i u(x) D_j \phi(x) dx - \int_\Omega \mu u(x)\phi(x) g_1(x) dx + \int_\Omega h(u(x))\phi(x) g_2(x) \\
&= \int_\Omega f(x)\phi(x) dx
\end{aligned}
$$

which completes the proof of the theorem. $\square$

Theorem 3.6. *Let $\Omega = \cup_{i=1}^\infty \Omega_i$, $\Omega_i \subseteq \Omega_{i+1}$ be open bounded domains in Ω. Let $g_1 > 0$ and $\mu < 0$. Under hypotheses* (H1)-(H3)*,* (1.2) *has a weak solution $u \in W_0^{1,2}(\Omega,\omega)$.*

The proof is similar to that of Theorem 3.6 and hence omitted. We remark that the above theorem is also true when $g_1 < 0$ and $\mu > 0$ is not an eigenvalue of (3.8).

4. Extensions

In section 3, the nonlinearity h is assumed to be continuous and bounded. In this section, we extend these results for a class of functions h which are continuous only. Generalized Hölder's inequality comes handy for establishing suitable estimates. Below, we consider the problem

$$
\begin{gathered}
Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.1}
$$

where $\Omega \subseteq \mathbb{R}^n, n \geq 3$ is an open and connected set and $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = |t|^\epsilon, 0 < \epsilon < 1$. We establish the existence of weak solution in a bounded domain G.

Again, we consider the cases $g_1 < 0$ and $g_1 > 0$ separately. Although the proofs are similar to the ones in section 3, we restrict ourselves to sketch the differences wherever needed. The result of [2] is not applicable here since h is not bounded. We collect the common hypotheses for convenience.

- (H1') Suppose that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = |t|^\epsilon, t \in \mathbb{R}, 0 < \epsilon < 1$;
- (H2') $g_1/\omega \in L^\infty(\Omega)$, $g_2/\omega \in L^\infty(\Omega)$ and $f/\omega \in L^2(\Omega,\omega)$, where ω is an A_2 weight.

Theorem 4.1. *Let $G \subset \mathbb{R}^n$, $n \geq 3$ be any open bounded set. Let the hypotheses* (H1'), (H2') *hold. Let $g_1 > 0$ and $\mu < 0$ then the problem*

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G \end{gathered} \tag{4.2}$$

has a solution $u \in W_0^{1,2}(G, \omega)$.

Proof. We give only a sketch of the proof as it is similar to the proof of Proposition 3.1. From the hypotheses and by Lemma 2.1 and for $u \in W_0^{1,2}(G, \omega)$, we note that

$$\begin{gathered} |B_1(u, \phi)| \leq \big(c + C_G |\mu| \|\frac{g_1}{\omega}\|_{\infty, G}\big) \|u\|_{0,1,2,G} \|\phi\|_{0,1,2,G}, \\ |T(\phi)| \leq C_G \|\frac{f}{\omega}\|_{\infty, G} \|\phi\|_{0,1,2,G}, \end{gathered} \tag{4.3}$$

where c is a generic constant and the constant C_G comes from Lemma 2.1. Again, by Lemma 2.1 and generalized Hölder's inequality [8, p.67], we have

$$|B_2(u, \phi)| \leq \int_G |h(u(x))| |\phi(x)| |\frac{g_2}{\omega}| \omega \, dx \leq \|u\|_{2,G}^{\epsilon} \|\phi\|_{2,G} \|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon}, G}.$$

We also observe that B_1 satisfies condition (S) by a similar argument as in [2] (also refer to [17, Proposition 27.12]). We observe that

$$|(Nu|\phi)| = |B_2(u, \phi)| \leq C_G \|u\|_{0,1,2,G}^{\epsilon} \|\phi\|_{0,1,2,G} \|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon}, G}$$

which implies

$$\|Nu\| \leq C_G \|u\|_{0,1,2,G}^{\epsilon} \|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon}, G} \leq c C_G \|u\|_{0,1,2,G}^{\epsilon},$$

So

$$\frac{\|Nu\|}{\|u\|_{0,1,2,G}} \leq \frac{c C_G \|u\|_{0,1,2,G}^{\epsilon}}{\|u\|_{0,1,2,G}} \to 0 \quad \text{as } \|u\|_{0,1,2,G} \to \infty. \tag{4.4}$$

This shows that $B + N$ is asymptotically linear. Also, $u \in L^2(\Omega, \omega)$ implies $h(u) \in L^{\frac{2}{\epsilon}}(\Omega, \omega)$ and define the Nemyckii operator

$$h_u : L^2(\Omega, \omega) \to L^{\frac{2}{\epsilon}}(\Omega, \omega) \tag{4.5}$$

by $h_u(x) = h(u(x))$; we have h_u is continuous (by [13, Theorem 2.1]). Let $u_n \rightharpoonup u$ in $W_0^{1,2}(G, \omega)$, then

$$\begin{aligned} |(Nu_n|\phi) - (Nu|\phi)| &\leq \int_G |h(u_n) - h(u)| |\frac{g_2}{\omega}| |\phi| \omega dx \\ &\leq C_G \|h(u_n) - h(u)\|_{\frac{2}{\epsilon}, G} \|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon}, G} \|\phi\|_{0,1,2,G}. \end{aligned}$$

Hence we have

$$\|Nu_n - Nu\| \to 0 \quad \text{as } n \to \infty \tag{4.6}$$

By a similar argument as in [2], the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S). If $\mu < 0$ is not an eigenvalue of the linear problem

$$\begin{gathered} Lu - \mu u(x) \omega(x) = 0 \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G \end{gathered}$$

shows that the operator equation $Bu + Nu = T$ has a solution $u \in W_0^{1,2}(G, \omega)$, which completes the proof. $\square$

An immediate consequence is the following result.

Corollary 4.2. *Let Ω be any open set in $\mathbb{R}^n$ such that $\Omega = \cup_{i=1}^{\infty}\Omega_i, \Omega_i \subseteq \Omega_{i+1}, \Omega_i$ is an open bounded subset of $\mathbb{R}^n$ for each $i = 1, 2, 3..$ Let the hypotheses of Theorem 4.1 hold. Let $g_1 > 0$ and $\mu < 0$ then, the problem*

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega_i, \\ u = 0 \quad \text{on } \partial\Omega_i \end{gathered} \tag{4.7}$$

has a solution $u = u_i \in W_0^{1,2}(\Omega_i, \omega)$, for $i = 1, 2..$ in addition $\|u_k\|_{0,1,2,\Omega_i} \le k_0$ for all $k \ge i$, where k_0 is independent of k.

Remark 4.3. Theorem 4.1 and Corollary 4.2 hold if $g_1 < 0$ and $\mu > 0$ with the remaining intact. But when $\mu > 0$ and g_1 changes sign, we need additional conditions on μ and g_1(stated below) to obtain a uniform bound k_0 for $\|u_k\|$, $k = 1, 2.$ where k_0 is independent of k. This uniform boundedness is essential to establish the existence of solution when Ω is not necessarily bounded. We state these results below in Theorem 4.4 and Corollary 4.5 but we give a sketch of the proof. We note that in (4.4 the required asymptotic linearity of $B + N$ is a consequence of ϵ lying between 0 and 1.

Theorem 4.4. *Let G be an open bounded set in $\mathbb{R}^n, n \ge 3$. Let the hypotheses* (H1'), (H2') *hold. Also, let $\mu > 0$ not be an eigenvalue of (3.4). Then the BVP*

$$\begin{gathered} Lu - \mu u g_1 + h(u) g_2 = f \quad \text{in } G, \\ u = 0 \quad \text{on } \partial G \end{gathered} \tag{4.8}$$

has a solution $u \in W_0^{1,2}(G, \omega)$.

The proof is omitted since it is along the same lines of the proof of Theorem 4.1. As a consequence of Theorem 4.4, we have the following result.

Corollary 4.5. *In addition to the hypotheses of Theorem 4.4, let $\lambda > C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega}$. Then* (4.7) *has a solution $u = u_i \in W_0^{1,2}(\Omega_i, \omega)$, for $i = 1, 2\dots$ and in addition*

$$\|u_k\|_{0,1,2,\Omega_i} \le k_0, \quad \text{for all } k \ge i.$$

where k_0 is a constant independent of k.

Proof. The proof for existence of solutions $u = u_i \in W_0^{1,2}(\Omega_i, \omega)$ for (4.7) is similar to the proof of Theorem 4.1 and hence omitted. We note that on Ω_i, for $k \ge i$,

$$\begin{aligned} &(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i})\|u_k\|_{0,1,2,\Omega_i}^2 \\ &\le C_{\Omega_i}\big\{\|u_k\|_{0,1,2,\Omega_i}^{\epsilon}\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big\}\|u_k\|_{0,1,2,\Omega_i}, \end{aligned}$$

where C_{Ω_i} is independent of k. Since $\lambda > C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i}$, we obtain

$$\|u_k\|_{0,1,2,\Omega_i} \le \frac{C_{\Omega_i}\big(\|u_k\|_{0,1,2,\Omega_i}^{\epsilon}\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big)}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i})} \tag{4.9}$$

Case 1: If $\|u_k\|_{0,1,2,\Omega_i} \le 1$, then from (4.9), we have

$$
\begin{aligned}
\|u_k\|_{0,1,2,\Omega_i} &\le \frac{C_{\Omega_i}\big(\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big)}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i})} \\
&\le \frac{C_{\Omega_i}\big(\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega} + \|\frac{f}{\omega}\|_{2,\Omega}\big)}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega})} = c^*,
\end{aligned}
$$

where c^* is a constant independent of k. Hence, we obtain

$$\|u_k\|_{0,1,2,\Omega_i} \le c^*, \text{ for all } k \ge i.$$

Case 2: If $\|u_k\|_{0,1,2,\Omega_i} > 1$, from (4.9), we have

$$
\begin{aligned}
\|u_k\|_{0,1,2,\Omega_i} &\le \frac{C_{\Omega_i}\big(\|u_k\|^{\epsilon}_{0,1,2,\Omega_i}\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega_i} + \|\frac{f}{\omega}\|_{2,\Omega_i}\big)}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega_i})} \\
&\le \frac{C_{\Omega_i}\big(\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},\Omega} + \|\frac{f}{\omega}\|_{2,\Omega}\big)\|u_k\|^{\epsilon}_{0,1,2,\Omega_i}}{(\lambda - C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega})}
\end{aligned}
$$

where C_{Ω_i} is independent of k. This implies

$$\|u_k\|^{1-\epsilon}_{0,1,2,\Omega_i} \le c, 0 < \epsilon < 1, \quad \|u_k\|_{0,1,2,\Omega_i} \le c^{\frac{1}{1-\epsilon}} = c',$$

where c and c' are constants independent of k. Since $\Omega_i \subseteq \Omega_{i+1}, \forall\, i \ge 1$, we have

$$\|u_k\|_{0,1,2,\Omega_i} \le c', \quad \text{for all } k \ge i.$$

Let $k_0 = \max\{c^*, c'\}$. Hence, we have

$$\|u_k\|_{0,1,2,\Omega_i} \le k_0, \quad \text{for all } k \ge i, \tag{4.10}$$

where k_0 is independent of k. $\square$

Now we state the main result of this section.

Theorem 4.6. *Let $\Omega = \cup_{i=1}^{\infty}\Omega_i, \Omega_i \subseteq \Omega_{i+1}$ be open bounded domains in Ω. Let $\mu > 0$ not be an eigenvalue of*

$$
\begin{gathered}
Lu - \mu u(x)\omega(x) = 0 \quad \text{in } \Omega_i, \\
u = 0 \quad \text{on } \partial\Omega_i
\end{gathered}
\tag{4.11}
$$

for $i = 1, 2, 3, \dots$ and in addition let $\lambda > C_{\Omega_i}\mu\|\frac{g_1}{\omega}\|_{\infty,\Omega}$. Under hypotheses (H1'), (H2')*,* (4.1) *has a weak solution $u \in W_0^{1,2}(\Omega, \omega)$.*

Proof. Let $\{u_k\}$ be the sequence of solutions for (4.7) in $W_0^{1,2}(\Omega_k, \omega), (k \ge 1)$. Let $\tilde{u}_k$ (for $k \ge 1$) denote the extension of u_k by zero outside Ω_k, which we continue to denote it by u_k. From (4.10), we have

$$\|u_k\|_{0,1,2,\Omega_l} \le k_0, \quad \text{for } k \ge l.$$

Then, $\{u_k\}$ has a subsequence $\{u_{k_m^1}\}$ which converges weakly to u^1, as $m \to \infty$, in $W_0^{1,2}(\Omega_1, \omega)$. Since $\{u_{k_m^1}\}$ is bounded in $W_0^{1,2}(\Omega_2, \omega)$, it has a convergent subsequence $\{u_{k_m^2}\}$ converging weakly to u^2 in $W_0^{1,2}(\Omega_2, \omega)$. By induction, we have $\{u_{k_m^{l-1}}\}$ has a subsequence $\{u_{k_m^l}\}$ which weakly converges to u^l in $W_0^{1,2}(\Omega_l, \omega)$, i.e in short, we have $u_{k_m^l} \rightharpoonup u^l$ in $W_0^{1,2}(\Omega_l, \omega),\ l \ge 1$. Define $u : \Omega \to \mathbb{R}$ by

$$u(x) := u^l(x), \quad \text{for } x \in \Omega_l.$$

(Here there is no confusion occurs since $u^l(x) = u^m(x)$ for $x \in \Omega$ for any $m \ge l$).

Let M be any fixed (but arbitrary) bounded domain such that $M \subseteq \Omega$. Then there exists an integer l such that $M \subseteq \Omega_l$. We note that, the diagonal sequence $\{u_{k_m^m}; m \geq l\}$ weakly converges to $u = u^l$ in $W_0^{1,2}(M,\omega)$, as $m \to \infty$.

What remains is to show that u is the required weak solution. It is sufficient to show that u is a weak solution of (4.1) for an arbitrary bounded domain M in Ω. Since $u_{k_m^m} \rightharpoonup u^l$ in $W_0^{1,2}(M,\omega)$, we have

$$\int_M \nabla(u_{k_m^m} - u).\nabla\phi\omega dx \to 0, \quad \text{as } m \to \infty,$$

which implies

$$\int_M D_i(u_{k_m^m} - u)D_j\phi\omega dx \to 0, \quad \text{as } m \to \infty.$$

From (1.1), for a constant c, we have $|a_{ij}| \leq c\omega$.

$$\begin{aligned}\int_M a_{ij}D_i(u_{k_m^m} - u)D_j\phi dx &\leq \int_M |a_{ij}||D_i(u_{k_m^m} - u)||D_j\phi|dx \\ &\leq c\|D_i(u_{k_m^m} - u)\|_{2,M}\|D_j\phi\|_{2,M} \to 0, \quad \text{as } m \to \infty.\end{aligned} \tag{4.12}$$

Also, by Lemma 2.1, $\{u_{k_m^m}\} \to u$ in $L^2(M,\omega)$. We have

$$\begin{aligned}\Big|\int_M (u_{k_m^m} - u)g_1\phi\, dx\Big| &\leq \int_M |(u_{k_m^m} - u)||g_1||\phi|dx \leq \int_M |(u_{k_m^m} - u)||\frac{g_1}{\omega}||\phi|\omega dx \\ &\leq \|\frac{g_1}{\omega}\|_{\infty,M}\|u_{k_m^m} - u\|_{2,M}\|\phi\|_{2,M}.\end{aligned}$$

So we have

$$\mu\int_M u_{k_m^m}g_1\phi dx \to \mu\int_M ug_1\phi dx \tag{4.13}$$

By (4.5) and generalized Hölder's inequality, we obtain

$$\int_M |h(u_{k_m^m}) - h(u)||\frac{g_2}{\omega}||\phi|\omega dx \leq \|h(u_{k_m^m}) - h(u)\|_{\frac{2}{\epsilon},M}\|\frac{g_2}{\omega}\|_{\frac{2}{1-\epsilon},M}\|\phi\|_{2,M}.$$

Hence, we have

$$\int_M h(u_k(x))g_2\phi dx \to \int_M h(u(x))g_2\phi dx\,. \tag{4.14}$$

Since M is an arbitrary bounded domain in Ω, it follows from (4.12), (4.13) and (4.14), that

$$\begin{aligned}&\int_\Omega a_{ij}D_iu(x)D_j\phi(x)dx - \int_\Omega \mu u(x)\phi(x)g_1(x)dx + \int_\Omega h(u(x))\phi(x)g_2(x) \\ &= \int_\Omega f(x)\phi(x)dx\end{aligned}$$

which completes the proof of the theorem. $\square$

Theorem 4.7. *Let $\Omega = \cup_{i=1}^\infty \Omega_i, \Omega_i \subseteq \Omega_{i+1}$ be open bounded domains in Ω. Let $g_1 > 0$ and $\mu < 0$. Under the hypotheses (H_1')-(H_2'), (4.1) has a weak solution $u \in W_0^{1,2}(\Omega,\omega)$.*

The proof is similar to Theorem 4.6 and hence omitted. Above theorem is also true when, $g_1 < 0$ and $\mu > 0$ is not an eigenvalue of (4.11).

Remark 4.8. The main results Theorem 3.5 and Theorem 4.6 hold, if h is continuous, $|h(t) - h(s)| \leq c|t - s|^\epsilon$, $0 < \epsilon < 1$ and $h(0) = 0$.

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References

[1] Cavalheiro, A. C.; *An approximation theorem for solutions of degenerate elliptic equations*, Proc. of the Edinburgh Math.Soc. 45, 363-389, (2002).

[2] Cavalheiro, A. C.; *Existence of solutions in weighted Sobolev spaces for some degenerate semilinear elliptic equations*, Applied Mathematics Letters, 17, 387-391, (2004).

[3] Chanillo, S. and Wheeden, R. L.; *Weighted Poincaré and Sobolev inequalities and estimates for the Peano maximal functions*, Am. J. Math. 107, 1119-1226, (1985).

[4] Chiadò Piat, V. and Serra Cassano, F.; *Relaxation of degenerate variational integrals*, Nonlinear Anal. 22, 409-429, (1994).

[5] Fabes, E., Jerison, D. and Kenig, C.; *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier(Grenoble), 32(3), 151-182, (1982).

[6] Fabes, E., Kenig, C. and Serapioni, R; *The local regularity of solutions of degenerate elliptic equations*, Comm. in P.D.E, 7(1), 77-116, (1982).

[7] Franchi, B. and Serapioni, R.; *Pointwise estimates for a class of strongly degenerate elliptic operators: A geometrical approach*, Ann. Scuola Norm. Sup. Pisa, 14, 527-568, (1987).

[8] Kufner, A., John, O. and Fučik, S.; *Functions Spaces*, Noordhoff,Leyden, (1977).

[9] Garcia-Cuerva, J. and Rubio de Francia, J. L.; *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies,Amsterdam, 116, (1985).

[10] Graham-Eagle, J.; *Monotone methods for semilinear elliptic equations in unbounded domains*, J. Math. Anal. Appl.137, 122-131, (1989).

[11] Heinonen, J., Kilpeläinen, T. and Martio, O.; *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, (1993).

[12] Hess, P.; *On the Fredholm alternative for nonlinear functional equations in Banach spaces*, Proc. Amer. Math. Soc, 33, 55-61, (1972).

[13] Krasnolsel'skii, M. A.; *Topological Methods in the Theory of Nonlinear Integral Equations*, GITTL, Moscow, (1956).

[14] Noussair, E. S. and Swanson, C. A.; *Global positive solutions of semilinear elliptic problems*, Pacific J. Math. 115, 177-192, (1984).

[15] Noussair, E. S. and Swanson, C. A.; *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal.Appl.75, 121-133, (1980).

[16] Turesson, B. O.; *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lec. Notes in Math. 1736, Springer-Verlag (2000), Berlin.

[17] Zeidler, E.; *Nonlinear Functional anlysis and its Applications*, Part II/B, Springer-Verlag, New York, (1990).

[18] Zeidler, E.; *Nonlinear Functional anlysis and its Applications*, Part II/A, Springer-Verlag, New York (1990).

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