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EXISTENCE OF SOLUTIONS AND CONVERGENCE RESULTS FOR DYNAMIC INITIAL VALUE PROBLEMS USING LOWER AND UPPER SOLUTIONS

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Abstract. In this article, we study the existence of solutions to first order non-linear initial value problems within the field of "dynamic equations on time scales". We employ the method of upper and lower solutions and Schauder's fixed point theorem. We also provide sufficient conditions under which the upper and lower solutions converge uniformly to a solution. Some examples are given to illustrate the new results.

1. Introduction

Dynamic equations on time scales had been introduced in 1988 as generalised forms of mathematical modelling which can incorporate the structure of differential or difference equations or both at the same time, see [12, 13, 1].

This article considers the dynamic initial value problem

$$x^{\nabla} = f(t, x), \quad \text{for all } t \in [0, a]_{\kappa,\mathbb{T}}; \tag{1.1}$$

$$x(0) = 0. \tag{1.2}$$

Here x^{∇} is the "nabla" derivative of x introduced in [8, p.77]. Here $f : [0, a]_{\kappa,\mathbb{T}} \times [l, u] \subset \mathbb{R}^2 \to \mathbb{R}$ is a left-Hilger-continuous and possibly non-linear function and l, u are continuous on $[0, a]_{\mathbb{T}} = [0, a] \cap \mathbb{T}$ for an arbitrary time scale $\mathbb{T}$. The subscript κ refers to $[0, a]_{\mathbb{T}}$ less any right scattered minimum points in it [9, p.331]. The term "left-Hilger-continuous" is used in accordance with the term "left-dense-continuous" (or ld-continuous) [9, Definition 8.43] and will be defined in the next section.

Our results show that (1.1), (1.2) has at least one solution which is bounded above by some function, u, and is bounded below by another function, l, where u, l are upper and lower solutions to (1.1), (1.2). The results follow some notions of La Salle [16] extended to the time scale setting. In this way, our results exhibit a broader span of modelling a system described as a first order initial value problem, no matter if the system has a discrete or a continuous domain or a hybrid of both. We apply our ideas to establish non-negative solutions to (1.1), (1.2).

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In addition, we establish sufficient conditions under which: solutions to (1.1), (1.2) established within $[l, u]$ are unique; and l and u approximate solutions to (1.1), (1.2). Then we establish an error estimate on the i-th approximation.

The motivation for using upper and lower solutions in our results are due to the wide use of this method to establish existence results for a variety of first and second order initial and boundary value problems, see [3, 4, 5, 6, 7, 8, 9, 10, 14, 21]. In this work, we use this method to determine: existence of solutions to (1.1), (1.2); and establishing successive approximations converging to a solution of the above IVP.

It had been shown in [6] that existence results involving lower and upper solutions in the time scale setting can be proved with less restrictions using nabla derivatives than using delta derivatives. We use nabla derivatives in this work to allow the solution to assume maximal values at the right end point of a given interval of existence, $[l, u]$, using the maximum principle. In this way, our results are different from the existence and uniqueness results for the first order IVPs involving delta derivatives proved in [20] using fixed point theorems and in [19] using the method of successive approximations. Our results are also different in context and methodology from the existence and uniqueness results using lower and upper solutions for the first order delta IVPs proved in [15, Theorem 4.1.2].

This paper is organised in the following manner. In Section 2, a brief introduction to the time scale calculus concerning nabla derivatives is presented. For more details, see [8, pp.77–81] and [9, Chapter 1, Chapter 8].

In Section 3, we define lower and upper solutions to the dynamic IVP (1.1), (1.2) and establish existence and uniqueness of solutions to (1.1), (1.2) within lower and upper solutions to the IVP.

In Section 4, we show that $l(t), u(t)$ are zero approximations to solutions of (1.1), (1.2) established in Section 3, for all $t \in [0, a]_{\mathbb{T}}$. We also prove that an upper bound exists on the error of the i-th approximation on $[0, a]_{\mathbb{T}}$ which approaches to zero for a unique solution.

2. Preliminaries

A time scale, denoted by $\mathbb{T}$, is a non-empty closed subset of $\mathbb{R}$. Thus, $\mathbb{N}$, $\mathbb{Z}$, $[0, 1]$, $[-1, 0] \cup [2, 5]$ and the Cantor set are examples of time scales. A dynamic equation on time scales models a phenomenon that may be continuous at one time and discrete at another. Hence, those dynamic equations that demonstrate a completely continuous phenomenon (or alternatively a completely discrete phenomenon) are equivalent to a differential equation (alternatively a difference equation).

For any point $t \in \mathbb{T}$, the left and right movements are measured in terms of left and right "jump operators", named "$\rho(t)$" and "$\sigma(t)$" respectively. These operators are defined as

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \text{for all } t \in \mathbb{T};$$
$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{for all } t \in \mathbb{T}.$$

We can see from the above inequalities that $\rho(t)$ and $\sigma(t)$ would coincide with t in continuous intervals of $\mathbb{T}$. Within discrete intervals of $\mathbb{T}$, the functions

$$\nu(t) := t - \rho(t), \quad \text{for all } t \in \mathbb{T}_\kappa;$$
$$\mu(t) := \sigma(t) - t, \quad \text{for all } t \in \mathbb{T}^\kappa,$$

where $\mathbb{T}^\kappa$ refers to $\mathbb{T}$ less any left-scattered maximum points in it [12, p. 27], describe a measure of the step size between two consecutive points. The point t is considered "left-scattered" when $\nu(t) > 0$ and "left-dense" when $\nu(t) = 0$. Similar relationships hold between the appearance of t to the right and $\mu(t)$. Our results in this work concern the behaviour of points to the left of t. Hence, further ideas regard left-dense or left-scattered points only.

All continuous functions on a time scale are ld-continuous [9, Theorem 8.43]. The term "left-Hilger-continuous" introduced in the previous section for f in (1.1) is used in equivalence with the term "ld-continuous" for a function of two or more variables, the first of which should be from an arbitrary time scale. This is a more generalised definition and we have introduced this particular term for functions of several variables, to avoid confusion with ld-continuous functions of one variable.

Definition 2.1 (Left-Hilger-continuous functions)**.** A mapping $f : [a, b]_{\kappa,\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called left-Hilger-continuous at a point (t, x) if: f is continuous at each (t, x) where t is left-dense; and the limits

$$\lim_{(s,y)\to(t^+,x)} f(s, y) \quad \text{and} \quad \lim_{y\to x} f(t, y)$$

both exist and are finite at each (t, x) where t is right-dense.

The following definitions and theorem [9, Section 8.4] describe nabla differentiable functions and their properties for a generalised time scale and will be fundamental to our results in this work.

Definition 2.2 (The nabla derivative)**.** Let $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$. Define $x^\nabla(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood N of t with

$$|[x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s]| \le \epsilon|\rho(t) - s|, \text{ for all } s \in N.$$

We call $x^\nabla(t)$ the nabla derivative of $x(t)$ for all $t \in \mathbb{T}_\kappa$ and say that x is nabla differentiable on $\mathbb{T}_\kappa$.

Theorem 2.3. *Let $\mathbb{T}$ be an arbitrary time scale and consider a function $h : \mathbb{T}_\kappa \to \mathbb{R}$. Then the following hold for all $t \in \mathbb{T}_\kappa$:*

(1) *if h is nabla differentiable at t, then h is continuous at t;*
(2) *if h is continuous at t and t is left-scattered, then h is nabla differentiable at t and*
$$h^\nabla(t) := \frac{h(t) - h(\rho(t))}{\nu(t)};$$
(3) *if t is left-dense, then h is nabla differentiable at t such that*
$$h^\nabla(t) := lim_{s\to t}\frac{h(t) - h(s)}{t - s},$$
provided the limit on the right hand side exists and is finite;
(4) *if h is nabla differentiable at t then*
$$h^\rho(t) := h(t) - \nu(t)h^\nabla(t),$$
where $h^\rho = h \circ \rho$.

Hence, if $\mathbb{T} = \mathbb{R}$ then $x^\nabla = x'$, while if $\mathbb{T} = \mathbb{Z}$ then $x^\nabla = \nabla x(t) = x(t) - x(t-1)$.

Definition 2.4 (The nabla integral)**.** Let $h : \mathbb{T} \to \mathbb{R}$. A function $H : \mathbb{T} \to \mathbb{R}$ will be a nabla anti-derivative of h if $H^\nabla(t) = h(t)$ holds for all $t \in \mathbb{T}_\kappa$. Let $t_0 \in \mathbb{T}$ with $t_0 < t$ then the Cauchy nabla integral of h is defined as

$$\int_{t_0}^{t} h(s)\nabla s := H(t) - H(t_0), \quad \text{for all } t \in \mathbb{T}.$$

3. Existence results

In this section, we define lower and upper solutions to (1.1), (1.2). We also prove that (1.1), (1.2) has a solution on $[0, a]_\mathbb{T}$ that lies within the interval $[l, u]$, where $l(t), u(t)$ act respectively as lower and upper solutions to (1.1), (1.2) for all $t \in [0, a]_\mathbb{T}$, using Schauder's fixed point theorem.

Definition 3.1. Let l, u be nabla differentiable functions on $[0, a]_{\kappa,\mathbb{T}}$. We call l a lower solution to (1.1), (1.2) on $[0, a]_\mathbb{T}$ if

$$l^\nabla(t) \le f(t, l(t)), \quad \text{for all } t \in [0, a]_{\kappa,\mathbb{T}}; \tag{3.1}$$

$$l(0) = 0. \tag{3.2}$$

Similarly, we call u an upper solution to (1.1), (1.2) on $[0, a]_\mathbb{T}$ if

$$u^\nabla(t) \ge f(t, u(t)), \quad \text{for all } t \in [0, a]_{\kappa,\mathbb{T}}; \tag{3.3}$$

$$u(0) = 0. \tag{3.4}$$

Definition 3.2. A solution of (1.1), (1.2) is a nabla differentiable function $x : \mathbb{T}_\kappa \to \mathbb{R}$ that satisfies (1.1) and (1.2) and the point $(t, x(t)) \in [0, a]_\mathbb{T} \times [l, u]$, where l, u are continuous on $[0, a]_\mathbb{T}$.

All ld-continuous functions are nabla integrable [9, Theorem 8.45]. The following lemma establishes equivalence of (1.1), (1.2) as nabla integral equations. The result is nabla equivalent of ideas in [20, Lemma 2.1] for the "delta" case. Therefore, the proof is omitted.

Lemma 3.3. *Consider the dynamic IVP* (1.1), (1.2)*. Let $f : [0, a]_{\kappa,\mathbb{T}} \times [l, u] \to \mathbb{R}$ be a left-Hilger-continuous function. Then a function $x \in C([0, a]_\mathbb{T}; \mathbb{R})$ solves* (1.1), (1.2) *if and only if it satisfies the nabla integral equation*

$$x(t) = \int_0^t f(s, x(s))\nabla s, \quad \textit{for all } t \in [0, a]_\mathbb{T}. \tag{3.5}$$

The following definition and the next two theorems are the keys to our proof for the existence of solutions to (1.1), (1.2).

Definition 3.4. [22, p.54] Let U, V be Banach spaces and $F : A \subseteq U \to V$. We say F is compact on A if:

- F is continuous on A;
- for every bounded set B of A, $F(B)$ is relatively compact in V.

The next theorem from [18, Theorem 1.3] is stated in the context of $\mathbb{T} \subseteq \mathbb{R}$. The proof is, therefore, omitted.

Theorem 3.5 (Arzela-Ascoli theorem on $\mathbb{T}$)**.** *Let $D \subseteq C([a, b]_\mathbb{T}; \mathbb{R})$. Then D is relatively compact if and only if it is bounded and equicontinuous.*

Theorem 3.6 (Schauder's fixed point theorem [17, p.67] [22, p.57])**.** *Let X be a normed linear space and D be a closed, bounded and convex subset of X. If $F : D \to D$ is a compact map then F has at least one fixed point.*

Define an infinite strip

$$S_{\kappa,\infty} := \{(t,p) : t \in [0,a]_{\kappa,\mathbb{T}} \text{ and } -\infty < p < \infty\}.$$

Let $g : S_{\kappa,\infty} \to \mathbb{R}$ be a left-Hilger-continuous function. Our next theorem concerns the existence of solutions to the initial value problem

$$x^{\nabla} = g(t,x), \quad \text{for all } t \in [0,a]_{\kappa,\mathbb{T}}; \tag{3.6}$$

$$x(0) = 0 \tag{3.7}$$

in $S_{\kappa,\infty}$. We prove this result by using Schauder's fixed point theorem.

Theorem 3.7. *Consider the initial value problem* (3.6), (3.7) *with g left-Hilger-continuous on $S_{\kappa,\infty}$. If g is uniformly bounded on $S_{\kappa,\infty}$ then* (3.6), (3.7) *has at least one solution, x, such that the point $(t, x(t))$ lies in the infinite strip*

$$S_{\infty} := \{(t,p) : t \in [0,a]_{\mathbb{T}} \textit{ and } -\infty < p < \infty\}.$$

Proof. From Lemma 3.3, a solution of (3.6), (3.7) is given by

$$x(t) := \int_0^t g(s, x(s))\nabla s, \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{3.8}$$

Since g is uniformly bounded on $S_{\kappa,\infty}$, there exists $M > 0$ such that

$$|g(t,p)| \le M, \quad \text{for all } (t,p) \in S_{\kappa,\infty}. \tag{3.9}$$

Define $K := Ma$ and consider the Banach space $(C([0,a]_{\mathbb{T}};\mathbb{R}), |\cdot|_0)$ [20, Lemma 3.3]. Let $D \subset C([0,a]_{\mathbb{T}};\mathbb{R})$ defined by

$$D := \{x \in C([0,a]_{\mathbb{T}};\mathbb{R});\ |x|_0 \le K\}.$$

Then D is closed, bounded and convex. We show that a compact map $F : D \to D$ exists and Schauder's theorem applies.

Define

$$[Fx](t) := \int_0^t g(s, x(s))\nabla s, \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{3.10}$$

See F is well defined on $C([0,a]_{\mathbb{T}};\mathbb{R})$ as g is left-Hilger continuous on $S_{\kappa,\infty}$.

We show that $F : D \to D$ is a compact map. For this, we show that the following properties hold for F:

(i) F is continuous on D;
(ii) for every bounded subset B of D, $F(B)$ is relatively compact in $C([0,a]_{\mathbb{T}};\mathbb{R})$,

and verify Definition 3.4.

To show that F is continuous on D, we define

$$B_K(0) := \{p \in \mathbb{R} : |p| \le K\}.$$

See $B_K(0)$ is closed and bounded and hence compact in $\mathbb{R}$. Therefore, g is bounded and uniformly left-Hilger-continuous on $[0,a]_{\mathbb{T}} \times B_K(0)$. Thus, for every $\epsilon_1 > 0$ there exists a $\delta_1 = \delta_1(\epsilon_1)$ such that for $(t,x_1), (t,x_2) \in [a,b]_{\kappa,\mathbb{T}} \times B_K(0)$, we have

$$|g(t,x_1) - g(t,x_2)| < \epsilon_1 \quad \text{whenever } |x_1 - x_2| < \delta_1. \tag{3.11}$$

Let x_i be a convergent sequence in D with $x_i \to x$ for all i. Then for every $\delta_1 > 0$ there exists $N > 0$ such that

$$|x_i - x| < \delta_1, \quad \text{for all } i \geq N.$$

We show that the sequence $F_i := Fx_i$ is uniformly convergent in $\mathbb{R}$. Let $\epsilon_0 := \epsilon_1 a$. We see that

$$\begin{aligned}
|Fx_i - Fx|_0 &= \sup_{t\in[0,a]_{\mathbb{T}}} |Fx_i(t) - Fx(t)| \\
&\leq \sup_{t\in[0,a]_{\mathbb{T}}} \Big| \int_0^t (g(s, x_i(s)) - g(s, x(s)))\nabla s\Big| \\
&\leq \sup_{t\in[0,a]_{\mathbb{T}}} \Big| \int_0^t |g(s, x_i(s)) - g(s, x(s))|\nabla s\Big| \\
&< \epsilon_1 a \quad \text{whenever } |x_i - x| < \delta_1 \\
&= \epsilon_0,
\end{aligned}$$

for all $i \geq N$. Thus F_i are uniformly convergent on D and hence are uniformly continuous on D. We show that $F : D \to D$: See for all $x \in D$, we have

$$\begin{aligned}
|Fx|_0 &:= \sup_{t\in[0,a]_{\mathbb{T}}} |Fx(t)| \\
&\leq \sup_{t\in[0,a]_{\mathbb{T}}} \int_0^t |g(s, x(s))|\nabla s \\
&\leq Ma = K.
\end{aligned} \tag{3.12}$$

Thus, F is in D.

Next, we show that for every bounded subset B of D, $F(B)$ is relatively compact on $C[0,a]_{\mathbb{T}}$ using the Arzela-Ascoli theorem. Let B be an arbitrary bounded subset of D. Assume $x \in B$. Then we see from (3.12) that we have $|Fx|_0 \leq K$ for all $t \in [0,a]_{\mathbb{T}}$. Thus F is uniformly bounded on B.

We also see that for any given $\epsilon > 0$ we can define $\delta := \epsilon/M$ and for $t_1, t_2 \in [0,a]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
|[Fx](t_1) - [Fx](t_2)| &= \Big| \int_{t_1}^{t_2} g(s, x(s))\nabla s\Big| \\
&\leq \Big| \int_{t_1}^{t_2} |g(s, x(s))|\nabla s\Big| \\
&\leq M\ |t_1 - t_2| < \epsilon
\end{aligned}$$

whenever $|t_1 - t_2| < \delta$. Hence, F is equicontinuous. By the Arzela-Ascoli theorem, $F(B)$ is relatively compact in $C([a,b]_{\mathbb{T}}; \mathbb{R})$.

From (i) and (ii) above, we see that $F : D \to D$ is a compact map. We also see that F satisfies the conditions of Schauder's theorem and, so, has at least one fixed point in D given by (3.8). Hence, (3.6), (3.7) has at least one solution, x, such the point $(t, x(t)) \in S_\infty$. □

The above result ensures existence of a solution to (3.6), (3.7) when the function g is bounded in an infinite domain $S_{\kappa,\infty}$ and considers this as a sufficient condition for the existence of a solution to the above IVP in $S_{\kappa,\infty}$. However, the result does not ensure the existence if the domain is restricted.

In the next result, we strengthen the condition in the above theorem by restricting the solution to (3.6), (3.7) within a lower and an upper solution to (1.1), (1.2). Hence we prove the existence of a solution to (1.1), (1.2) within the region

$$S := \{(t,p) : t \in [0,a]_{\mathbb{T}} \text{ and } l(t) \le p \le u(t)\},$$

where l, u are, respectively, lower and upper solutions to (1.1), (1.2). To prove this, we define a modified function g in terms of f in (1.1) and prove that g is uniformly bounded on $S_{\kappa,\infty}$ and use Theorem 3.7. We also prove that the solution, x, to the IVP (3.6), (3.7) satisfies $l(t) \le x(t) \le u(t)$ for all $t \in [0,a]_{\mathbb{T}}$, so that x must also be a solution to the original unmodified problem (1.1), (1.2).

Define

$$S_\kappa := \{(t,p) : t \in [0,a]_{\kappa,\mathbb{T}} \text{ and } l(t) \le p \le u(t)\}.$$

Theorem 3.8. *Let $f : S_\kappa \to \mathbb{R}$ be a left-Hilger-continuous function. If l, u are, respectively, lower and upper solutions to* (1.1), (1.2), *then* (1.1), (1.2) *has at least one solution, x, such that $l(t) \le x(t) \le u(t)$ for all $t \in [0,a]_{\mathbb{T}}$.*

Proof. Consider the IVP (3.6), (3.7), where $g(t,p)$ is defined on $S_{\kappa,\infty}$ such that for all $t \in [0,a]_{\kappa,\mathbb{T}}$,

$$g(t,p) := \begin{cases} f(t,l(t)) + \dfrac{l(t)-p}{1+(l(t)-p)^2}, & \text{if } p < l(t); \\ f(t,p), & \text{if } l(t) \le p \le u(t); \\ f(t,u(t)) - \dfrac{p-u(t)}{1+(p-u(t))^2}, & \text{if } p > u(t). \end{cases} \tag{3.13}$$

We first show that g is left-Hilger-continuous and uniformly bounded on $S_{\kappa,\infty}$ and Theorem 3.7 applies. Note that f is left-Hilger-continuous on the compact region S_κ and so it is bounded on S_κ. Thus, there exists $M_1 > 0$ such that $|f(t,p)| \le M_1$ for all $(t,p) \in S_\kappa$. We also see that for $l(t) > p \in \mathbb{R}$, we have

$$\Big|\frac{l(t)-p}{1+(l(t)-p)^2}\Big| < 1, \quad \text{for all } t \in [0,a]_{\mathbb{T}},$$

and so

$$f(t,l(t)) + \Big|\frac{l(t)-p}{1+(l(t)-p)^2}\Big| < 1 + M_1, \quad \text{for all } t \in [0,a]_{\kappa,\mathbb{T}}.$$

Let $M := 1 + M_1$. Then from (3.13), we obtain

$$|g(t,p)| \le M, \quad \text{for all } (t,p) \in S_{\kappa,\infty}. \tag{3.14}$$

Hence g is uniformly bounded on $S_{\kappa,\infty}$. In addition, the left-Hilger-continuity of f on S_κ and the ld-continuity of l, u, p on $[0,a]_{\mathbb{T}}$ show that the right hand side of (3.13) is left-Hilger-continuous on $S_{\kappa,\infty}$ and, so, we have g left-Hilger-continuous on $S_{\kappa,\infty}$. By Theorem 3.7, the modified IVP (3.6), (3.7) has a solution, x, such that the graph $(t,x(t)) \in S_\infty$ for all $t \in [0,a]_{\mathbb{T}}$.

Next, we prove that $l(t) \le x(t) \le u(t)$ for all $t \in [0,a]_{\mathbb{T}}$. We split the inequality $l(t) \le x(t) \le u(t)$ into two parts and first show that

$$l(t) \le x(t), \quad \text{for all } t \in [0,a]_{\mathbb{T}}, \tag{3.15}$$

using the contradiction method.

Let $r(t) := l(t) - x(t)$ for all $t \in [0,a]_{\mathbb{T}}$. Assume there exists a point $t_1 \in [0,a]_{\mathbb{T}}$ such that $l(t_1) > x(t_1)$. See $t_1 \neq 0$ as $x(0) = 0 = l(0)$ from (1.2) and (3.2). Without loss of generality, we may assume that

$$r(t_1) = \max_{t \in [0,a]_{\mathbb{T}}} r(t) > 0. \tag{3.16}$$

Thus, $r(t)$ is non-decreasing at $t = t_1$ and, so, $r^{\nabla}(t_1) \geq 0$.

On the other hand, since $x(t_1) < l(t_1)$ we see that using (3.6), (3.13) and (3.1), we obtain

$$\begin{aligned}
0 \leq r^{\nabla}(t_1) &= l^{\nabla}(t_1) - x^{\nabla}(t_1) \\
&= l^{\nabla}(t_1)) - g(t_1, x(t_1)) \\
&= l^{\nabla}(t_1)) - f(t_1, l(t_1)) - \frac{l(t_1) - x(t_1)}{1 + (l(t_1) - x(t_1))^2} \\
&< l^{\nabla}(t_1)) - f(t_1, l(t_1)) \leq 0,
\end{aligned}$$

which is a contradiction. Hence $l(t) \leq x(t)$ for all $t \in [0,a]_{\mathbb{T}}$. It is very similar to show that $u(t) \geq x(t)$ for all $t \in [0,a]_{\mathbb{T}}$ as in the above case. We omit the details.

Thus, we have $l(t) \leq x(t) \leq u(t)$ for all $t \in [0,a]_{\mathbb{T}}$. Hence, from (3.13), $x(t)$ is a solution to (1.1), (1.2) for all $t \in [0,a]_{\mathbb{T}}$ and the point $(t, x(t)) \in S$ for all $t \in [0,a]_{\mathbb{T}}$. This completes the proof. $\square$

The following example illustrates the above theorem.

Example 3.9. Consider the Riccati initial value problem

$$x^{\nabla}(t) = f(t,x) := x^2 - t, \quad \text{for all } t \in [0,1]_{\kappa,\mathbb{T}}; \tag{3.17}$$

$$x(0) = 0. \tag{3.18}$$

We claim that there exists at least one solution, x, to the above IVP such that $-t \leq x(t) \leq t$ for all $t \in [0,1]_{\mathbb{T}}$.

Proof. We see that the right hand side of (3.17) is a composition of a continuous function t and a continuous function x^2 and hence, is continuous on $[0,1]_{\mathbb{T}} \times \mathbb{R}$. So our f is left-Hilger-continuous on $[0,1]_{\kappa,\mathbb{T}} \times \mathbb{R}$. Let us define

$$l(t) := -t, \quad \text{for all } t \in [0,1]_{\mathbb{T}}.$$

Then we see that $l(0) = 0$ and for all $t \in [0,1]_{\mathbb{T}}$, we have

$$f(t, l(t)) = t^2 - t \geq -1 = l^{\nabla}(t).$$

Thus, our l satisfies (3.1), (3.2) and is a lower solution to (3.17), (3.18).

In a similar way, the function $u(t) := t$ is an upper solution to (3.17), (3.18) for all $t \in [0,1]_{\mathbb{T}}$. By Theorem 3.8, there is at least one solution, x, to (3.17), (3.18) such that $-t \leq x(t) \leq t$ for all $t \in [0,1]_{\mathbb{T}}$. $\square$

Our next result gives a sufficient condition for uniqueness of solution to (1.1), (1.2). We show that the solution, x, of the above IVP established in Theorem 3.8 is the only solution satisfying $l(t) \leq x(t) \leq u(t)$ for all $t \in [0,a]_{\mathbb{T}}$.

Theorem 3.10. *Let f be left-Hilger-continuous on S_κ. Assume l, u are, respectively, lower and upper solutions of* (1.1), (1.2). *If there exists $L > 0$ such that f satisfies*

$$|f(t,p) - f(t,q)| \leq L|p - q|, \quad \textit{for all } (t,p),(t,q) \in S_\kappa, \tag{3.19}$$

then the solution x of (1.1), (1.2) *brought forward under the conditions of Theorem 3.8 is the unique solution satisfying $l(t) \leq x(t) \leq u(t)$ for all $t \in [0,a]_{\mathbb{T}}$.*

Proof. Let x, y be solutions of (1.1), (1.2) such that the points $(t, x(t)), (t, y(t)) \in S_\kappa$. Then, using (3.5), we obtain for all $t \in [0,a]_{\mathbb{T}}$,

$$\begin{aligned} |x(t) - y(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \nabla s \\ &\leq L \int_0^t |x(s) - y(s)| \nabla s, \end{aligned} \tag{3.20}$$

where we employed (3.19) in the last step. Define

$$k(t) := |x(t) - y(t)|, \quad \text{for all } t \in [0,a]_{\mathbb{T}}.$$

See, $L > 0$ and so $L \in \mathcal{L}^+$ [7, p.225]. Applying Gronwall's inequality concerning nabla derivatives [7, Theorem 2.7] (taking $f(t) = 0$ and $p(t) = L$) to (3.20), we obtain

$$k(t) \leq 0, \quad \text{for all } t \in [0,a]_{\mathbb{T}}.$$

But $k(t) = |x(t) - y(t)|$ and so, is non-negative for all $t \in [0,a]_{\mathbb{T}}$. Thus, $x(t) = y(t)$ for all $t \in [0,a]_{\mathbb{T}}$. □

The next corollary establishes existence of a unique, non-negative and bounded solution of the IVP (1.1), (1.2) on $[0,a]_{\mathbb{T}}$.

Corollary 3.11. *Let $f : S_\kappa \to \mathbb{R}$ be a left-Hilger-continuous function satisfying* (3.19)*. Let l, u be lower and upper solutions to* (1.1), (1.2)*. If $l(t) = 0$ for all $t \in [0,a]_{\mathbb{T}}$, then the IVP* (1.1), (1.2) *has a unique, bounded and non-negative solution, $x(t)$, for all $t \in [0,a]_{\mathbb{T}}$.*

The proof of the above corollary follows from Theorem 3.10, as $0 \leq x(t) \leq u(t)$ for all $t \in [0,a]_{\mathbb{T}}$. The following example illustrates this result.

Example 3.12. Consider the dynamic initial value problem

$$x^\nabla(t) = f(t, x) := \rho(t) + x^3, \quad \text{for all } t \in [0,1]_{\kappa,\mathbb{T}}; \tag{3.21}$$

$$x(0) = 0. \tag{3.22}$$

We claim that the above IVP has a unique non-negative solution x such that $0 \leq x(t) \leq 1$ for all $t \in [0,1]_{\mathbb{T}}$.

Proof. See $f(t, p) = \rho(t) + p^3$ for all $(t, p) \in [0,1]_{\kappa,\mathbb{T}} \times \mathbb{R}$. Since $\rho(t)$ and p^3 are everywhere ld-continuous functions and so is their composition, our f is left-Hilger-continuous on $[0,1]_{\kappa,\mathbb{T}} \times \mathbb{R}$. We define

$$l(t) := 0, \quad \text{and} \quad u(t) := t^2, \quad \text{for all } t \in [0,1]_{\mathbb{T}}.$$

Then we see that $l(t) \leq u(t)$ for all $t \in [0,a]_{\mathbb{T}}$ with $l(0) = 0 = u(0)$. It is evident that l satisfies (3.1) and so, is a lower solution to (3.21), (3.22). We also see that, for all $t \in [0,1]_{\mathbb{T}}$

$$f(t, u(t)) = \rho(t) + t^6 \leq \rho(t) + t = u^\nabla(t).$$

Thus, our u satisfies (3.3) and is an upper solution to (3.21), (3.22). By Theorem 3.8, there exists a solution, x, to (3.21), (3.22) such that $0 \leq x(t) \leq t^2 \leq 1$, for all $t \in [0,1]_{\mathbb{T}}$. Moreover, for all $t \in [0,1]_{\mathbb{T}}$, we have

$$\Big|\frac{\partial f}{\partial p}\Big| = |3p^2| \leq 3t^4 \leq 3.$$

Thus, f has bounded partial derivatives in $[0,1]_{\mathbb{T}} \times [0,1]$ and satisfies (3.19) for $L = 3$ (see [2, Lemma 3.2.1], [11, p.248]). By Corollary 3.11, x is the unique solution to (3.21), (3.22) such that $0 \le x(t) \le 1$ for all $t \in [0,1]_{\mathbb{T}}$. □

4. Convergence results

In this section, we establish conditions under which lower and upper solutions to (1.1), (1.2) approximate the existing solutions of (1.1), (1.2). We also establish error estimates on he ith approximation.

Let $f : S_\kappa \to \mathbb{R}$ be left-Hilger-continuous. Define $F : C([0,a]_{\mathbb{T}};\mathbb{R}) \to C([0,a]_{\mathbb{T}};\mathbb{R})$ by

$$[Fp](t) = \int_0^t f(s,p(s))\nabla s, \quad \text{for all } t \in [0,a]_{\mathbb{T}}.$$

Then F is well-defined on $C([0,a]_{\mathbb{T}};\mathbb{R})$. Under the conditions of Theorem 3.8, a fixed point x of F will be a solution to (1.1), (1.2) such that $l(t) \le x(t) \le u(t)$ for all $t \in [0,a]_{\mathbb{T}}$, where l, u are, respectively, lower and upper solutions of (1.1), (1.2).

Consider an iterative scheme defined as

$$[F^0 p](t) := [Fp](t) = \int_0^t f(s,p(s))\nabla s, \quad \text{for all } t \in [0,a]_{\mathbb{T}}; \tag{4.1}$$

$$F^i := F[F^{i-1}], \quad \text{for all } i \ge 1. \tag{4.2}$$

It had been shown in [19, pp.78–79] that, in general, the continuity of a function f alone is not sufficient for a sequence or subsequences of successive approximations to converge to a solution on a compact rectangle. In our next result, we show that the successive approximations defined in (4.1), (4.2) provide a sequence of functions that converge to a solution to (1.1), (1.2).

We assume f to be non-decreasing on S_κ and prove that if x is a solution to (1.1), (1.2) such that $l(t) \le x(t) \le u(t)$ for all $t \in [0,a]_{\mathbb{T}}$, then $l(t)$ and $u(t)$ approximate $x(t)$ for all $t \in [0,a]_{\mathbb{T}}$. We also show that an upper bound on the error of the ith approximation will be $[F^i u](t) - [F^i l](t)$ for all $t \in [0,a]_{\mathbb{T}}$. The next definition describes zero approximation to the solution of (1.1), (1.2) (see [16, p.724] for the ODE case).

Definition 4.1. Let x be a solution to (1.1), (1.2) and $y : \mathbb{T} \to \mathbb{R}$ be a ld-continuous function. We call $y(t)$ a zero approximation to $x(t)$ for all $t \in [0,a]_{\mathbb{T}}$ if, $\{F^i y\}$ converges uniformly to x on $[0,a]_{\mathbb{T}}$.

Theorem 4.2. *Let $f : S_\kappa \to \mathbb{R}$ be left-Hilger-continuous and l, u are lower and upper solutions to* (1.1), (1.2). *If f is non-decreasing in the second argument on S_κ, that is, for $p \le q$, we have*

$$f(t,p) \le f(t,q), \quad \text{for all } (t,p),(t,q) \in S_\kappa; \tag{4.3}$$

then $l(t)$ and $u(t)$ will be the zero approximations to a solution x of (1.1), (1.2) *for all $t \in [0,a]_{\mathbb{T}}$.*

Moreover, for $m, n \ge 0$, the sequence F^i given by (4.1), (4.2) *satisfies*

$$[F^m l](t) \le [F^{m+1} l](t) \le [F^{n+1} u](t) \le [F^n u](t), \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{4.4}$$

Proof. We show that $l(t), u(t)$ satisfy Definition 4.1 and (4.4) holds for all $t \in [0,a]_{\mathbb{T}}$. We see from (4.1) that for $p = u$, we obtain for all $t \in [0,a]_{\mathbb{T}}$

$$[Fu](t) = \int_0^t f(s,u(s))\nabla s \le \int_0^t u^\nabla(s))\nabla s = u(t). \tag{4.5}$$

Similarly, for $p = l$, we obtain

$$l(t) \leq [Fl](t) \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{4.6}$$

Since f is non-decreasing in the second variable and is left-Hilger-continuous on S_κ, it follows from (4.1), (4.6), and (4.3) that, for all $t \in [0,a]_{\mathbb{T}}$, we have

$$\begin{aligned}(t) = [Fl](t) &= \int_0^t f(s, l(s))\nabla s \\ &\leq \int_0^t f(s, [Fl](s))\nabla s \\ &= [F^1 l](t).\end{aligned} \tag{4.7}$$

Proceeding in this way, we obtain

$$[Fl](t) \leq [F^1 l](t) \leq [F^2 l](t) \leq [F^3 l](t) \leq \dots, \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{4.8}$$

Thus, the sequence $\{F^i l\}$ is non-decreasing. In a similar way, using (4.3), (4.1) and (4.5), we obtain

$$[Fu](t) \geq [F^1 u](t) \geq [F^2 u](t) \geq \dots, \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{4.9}$$

Now since $l(t) \leq u(t)$ for all $t \in [0,a]_\kappa$, we can write using (4.8) and (4.9) that for all $t \in [0,a]_\kappa$

$$[F^n l](t) \leq [F^{n+1} l](t) \leq [F^{n+1} u](t) \leq [F^n u](t). \tag{4.10}$$

We further see that

$$[Fl](0) = 0 = [Fu](0). \tag{4.11}$$

We show that the sequence $\{F^i l\}$ converges uniformly to the fixed point x ($x(0) = 0$). Define

$$w(t) := [Fu](t) - [Fl](t), \quad \text{for all } t \in [0,a]_{\mathbb{T}}.$$

See, $w(t) \geq 0$ for all $t \in [0,a]_{\mathbb{T}}$. Since f is non-decreasing in the second variable on S_κ, it follows from (4.1) that

$$w^\nabla(t) = f(t, u(t)) - f(t, l(t)) \geq 0, \quad \text{for all } t \in [0,a]_{\kappa,\mathbb{T}}.$$

It is clear from (4.10) that for $n > m \geq 0$, we have

$$[F^m l](t) \leq [F^n l](t) \leq [F^n u](t),$$

and for $n < m$, we have

$$[F^m l](t) \leq [F^m u](t) \leq [F^n u](t).$$

Hence for any $m, n \geq 0$, we have the inequality

$$[F^m l](t) \leq [F^{m+1} l](t) \leq [F^{n+1} u](t) \leq [F^n u](t), \quad \text{for all } t \in [0,a]_{\mathbb{T}}.$$

The boundedness and equicontinuity of each $F^i l$ can be established in the same way as in Theorem 3.7. Hence, as $i \to \infty$, $F^i l$ converges uniformly on $[0,a]_{\mathbb{T}}$ to a fixed point x. Similarly, $\{F^i u\}$ converges uniformly on $[0,a]_{\mathbb{T}}$ to a fixed point x. Thus $l(t)$ and $u(t)$ are zero approximations to $x(t)$ with $w^i(t) := [F^i u](t) - [F^i l](t)$ as an upper bound on the error of the i-th approximation for all $t \in [0,a]_{\mathbb{T}}$. If the solution is unique, then $w^i(t) \to 0$ for all $i \geq 1$ for all $t \in [0,a]_{\mathbb{T}}$. This completes the proof. $\square$

Example 4.3. Consider the dynamic IVP

$$x^{\nabla}(t) = f(t,x) := x^3 - t, \quad \text{for all } t \in [0,1]_{\kappa,\mathbb{T}}; \tag{4.12}$$

$$x(0) = 0. \tag{4.13}$$

We claim that $l(t) = -t$ and $u(t) = t$ are zero approximations to the solution of (4.12), (4.13) for all $t \in [0,1]_{\mathbb{T}}$. Moreover, for all $t \in [0,1]_T$, the sequence F^i given by

$$F^0(t) := [Fx](t) = \int_0^t (x^3 - s)\nabla s,$$

$$F^i := F[F^{i-1}], \quad \text{for all } i \geq 1.$$

satisfies (4.4) for any $m, n \geq 0$.

Proof. We see that $f(t,p) = p^3 - t$ for all $(t,p) \in [0,1]_{\kappa,\mathbb{T}} \times \mathbb{R}$. Since t and p^3 are everywhere ld-continuous functions and so is their composition, our f is left-Hilger-continuous on $[0,1]_{\kappa,\mathbb{T}} \times \mathbb{R}$. We further see that $l(0) = 0 = u(0)$ and for all $t \in [0,1]_{\mathbb{T}}$

$$f(t, l(t)) = -t(t^2 + 1) \geq -1 = l^{\nabla}(t).$$

Thus, l satisfies (3.1) and so, is a lower solution to (4.12), (4.13). In a similar way, we have u satisfying (3.3) and so, is an upper solution to (4.12), (4.13). By Theorem 3.8, there exists a solution, x, to (4.12), (4.13) such that $-t \leq x(t) \leq t$, for all $t \in [0,1]_{\mathbb{T}}$.

Next, we see that for $p \leq q$, we have

$$f(t,p) = p^3 - t \leq q^3 - t = f(t,q), \quad \text{for all } (t,p),(t,q) \in [0,1]_{\kappa,\mathbb{T}} \times [-t,t].$$

Thus, f is non-decreasing with respect to the second argument on $[0,1]_{\mathbb{T},\kappa} \times [-t,t]$ and so, by Theorem 4.2, the functions $-t$ and t are zero approximations to the solution x of (4.12), (4.13). We further see that for $x = l$, we have for all $t \in [0,1]_{\mathbb{T}}$,

$$[Fl](t) = \int_0^t -(s^3 + s)\nabla s \geq -t = l(t)\,.$$

This leads to (4.7) and then to (4.8). We obtain (4.9) in a similar way. Thus, (4.4) holds for any $m, n \geq 0$. ☐

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