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EXISTENCE OF SOLUTIONS TO QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article we use the theory of C_0-semigroup of bounded linear operators to establish the existence and uniqueness of a classical solution to a quasilinear functional differential equation considered in a Banach space.

1. Introduction

In this article we study the the existence and uniqueness of a classical solution to the following quasilinear functional differential equation, considered in a Banach space X,

$$\begin{gathered}\frac{du(t)}{dt} + A(t,u(t))u(t) = F(t,u_t), \quad t \in [0,T], \\ u_0 = \phi \quad \text{on} \quad [-\tau, 0],\end{gathered} \tag{1.1}$$

where $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0]$. For $t \in [0,T]$, we denote by $\mathcal{C}_t$ the Banach space of all continuous functions from $[-\tau, t]$ to X endowed with the supremum norm

$$\|\chi\|_{\mathcal{C}_t} = \sup_{-\tau \le \theta \le t} \|\chi(\theta)\|_X, \quad \chi \in \mathcal{C}_t.$$

The function $F(t,\psi)$ is defined on $[0,T] \times \mathcal{C}_0$ to X. Here we see that $u_t \in \mathcal{C}_0$. We assume that for $u \in C_T$, $F(\cdot, u_{(\cdot)}) : [0,T] \to X$ is a bounded L^1 function. Further we assume that there is a subset B of X such that for $(t,u) \in [0,T] \times \mathcal{C}_T$ with $u(t) \in B$ for $t \in [0,T]$, $A(t,u(t))$ is a linear operator in X. Also $\phi \in \mathcal{C}_0$ is Lipschitz continuous with Lipschitz constant L_ϕ.

Quasilinear evolution equations forms a very important class of evolution equations as many time dependent phenomena in physics, chemistry and biology can be represented by such evolution equations. For more details on the theory and applications of quasilinear evolution equations we refer to [4, 9, 11].

Kato [6] considered the quasilinear evolution equation

$$\begin{gathered}\frac{du(t)}{dt} + A(t,u(t))u(t) = G(t,u(t)), \quad t \in (0,T], \\ u(0) = u_0,\end{gathered} \tag{1.2}$$

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in a Banach space and shown the existence of a strong solution under suitable assumptions on A and G. The various cases of equation (1.2) have been treated by Amann [1] in the interpolation spaces using the theory of analytic semigroups. Bahuguna [2] has shown the existence of a classical solution of the following integrodifferential equation considered in a Banach space,

$$\begin{gathered}\frac{du(t)}{dt} + A(t, u(t))u(t) = K(u)(t) + f(t), \quad t \in [0, T], \\ u(0) = x,\end{gathered} \tag{1.3}$$

where

$$K(u)(t) = \int_0^t a(t-s)k(s, u(s))ds,$$

and $A(t, w)$ is a linear operator in X for each $(t, w) \in [0, T] \times W$, W being an open subset of X. In this paper we strengthen the result of [2] for a functional differential equation. We show the existence and uniqueness of a classical solution of (1.1).

2. Preliminaries

Let $B(X, Y)$ be the set of all bounded linear operators from X to Y. $B(X, Y)$ is a Banach space with the norm

$$\|A\|_{B(X,Y)} = \sup_{x \in X, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}.$$

We denote $B(X, X)$ by $B(X)$. Let B be a subset of Y, where Y is densely and continuously embedded in X. Since Y is continuously embedded in X so it is a subset of X too. A family $\{A(t, w), (t, w) \in [0, T] \times B\}$ of infinitesimal generators of a C_0-semigroup $S_{t,w}(s)$, $s \geq 0$ on X is called stable if there exist constants $M \geq 1$ and w, known as stability constants, such that

$$\rho(A(t, w)) \supset (w, \infty) \quad (t, w) \in [0, T] \times B,$$

where $\rho(A(t, w))$ is the resolvent set of $A(t, w)$ and

$$\Big\| \prod_{j=1}^{k} R(\lambda : A(t_j, w_j)) \Big\|_{B(X)} \leq \frac{M}{(\lambda - w)^k} \quad for\ \lambda > w$$

and every finite sequence

$$0 \leq t_1 \leq t_2 \cdots \leq t_k \leq T, \quad w_j \in B.$$

Let $S_{t,w}(s)$, $s \geq 0$ be the C_0-semigroup generated by $A(t, w)$. A subspace Y of X is called $A(t, w)$-admissible if Y is an invariant subspace of $S_{t,w}(s), s \geq 0$, and the restriction of $S_{t,w}(s)$ to Y is a C_0-semigroup in Y. We will use the following hypothesis on $A(t, w)$:

- (H1) There is a subset B in X such that the family $\{A(t, w), (t, w) \in [0, T] \times B\}$ is stable.
- (H2) Y is $A(t, w)$-admissible for all (t, w) in $[0, T] \times B$ and the family $\{\tilde{A}(t, w), (t, w) \in [0, T] \times B\}$ of parts of $A(t, w)$ in Y is stable in Y.
- (H3) For $(t, w) \in [0, T] \times B$, $A(t, w)$ is a bounded linear operator from Y to X and $A(\cdot, w)$ is continuous in $B(Y, X)$ i.e. $A(\cdot, w) \in C([0, T], B(Y, X))$ also $D(A(t, w)) \supset Y$.

(H4) There exists a positive constant L_A such that

$$\|A(t,w_1)-A(t,w_2)\|_{B(Y,X)} \le L_A\|w_1-w_2\|_Y$$

for all $(t,w_1),(t,w_2)\in[0,T]\times B$.

Next we define an evolution family as follows.

Definition 2.1. A two parameter family of bounded linear operators $U(t,s), t \ge s \ge 0$, on X is called an evolution system if

(i) $U(s,s)=I$ and $U(t,r)U(r,s)=U(t,s)$, $t\ge r\ge s\ge 0$;
(ii) $(t,s)\to U(t,s)$ is strongly continuous for $t\ge s\ge 0$.

If $u\in C([0,T],X)$ and the family $\{A(t,w),(t,w)\in[0,T]\times X\}$ of operators satisfies (H1)–(H4) then there exists an evolution system $U_u(t,s)$ ([10, Theorem 4.6]) in X satisfying:

(i) $\|U_u(t,s)\|_{B(X)} \le Me^{\delta(t-s)}$ for $t\ge s\ge 0$, where M and δ are the stability constants;
(ii) $\frac{\partial^+}{\partial t}U_u(t,s)w|_{t=s}=A(s,u(s))w$ for $w\in Y$;
(iii) $\frac{\partial^+}{\partial s}U_u(t,s)w|_{t=s}=-U_u(t,s)A(s,u(s))w$ for $w\in Y$.

Moreover there exists a constant $C_0>0$ such that for every $u,v\in C([0,T],X)$ with values in B and every $y\in Y$ we have

$$\|U_u(t,s)y-U_v(t,s)y\|_X \le C_0\|y\|_Y\int_s^t \|u(\xi)-v(\xi)\|_X\,d\xi.$$

Now we mention some additional hypotheses.

(H5) For each $u\in C(\mathbb{R},X)$, we have

$$U_u(t,s)Y\subset Y,\quad s,t\in\mathbb{R}, s\le t,$$

and $U_u(t,s)$ is strongly continuous in Y.

(H6) Every closed convex and bounded subset of Y is also closed in X.

(H7) There exists a constant $L_F>0$ such that

$$\|F(t,\phi_1)-F(s,\phi_2)\|_X \le L_F(|t-s|+\|\phi_1-\phi_2\|_{\mathcal{C}_0})$$

for all $(t,\phi_1),(s,\phi_2)\in[0,T]\times\mathcal{C}_0$.

We note that the condition (H6)) is always satisfied if X and Y are reflexive Banach spaces.

Definition 2.2. A function $u\in\mathcal{C}_T$ with values in B satisfying

$$u(t)=U_u(t,0)\phi(0)+\int_0^t U_u(t,s)F(s,u_s)ds,\quad t\in[0,T]$$
$$u_0=\phi\quad\text{on } [-\tau,0],$$

is called a mild solution to (1.1) on $[0,T]$.

Definition 2.3. A function $u\in\mathcal{C}_T$ such that $u(t)\in Y\cap B$ for $t\in(0,T]$ and $u\in\mathcal{C}^1((0,T],X)$ satisfying the equation (1.1) in X is called a classical solution to (1.1) on $[0,T]$. Where $\mathcal{C}^1([0,T],X)$, space of all continuously differentiable functions from $[0,T]$ to X and Y is a $A(t,w)$-admissible subspace of X.

3. Main result

In this section we prove the existence and uniqueness result for a classical solution to (1.1). Let $\tilde{\phi} \in \mathcal{C}_T$ be given by $\tilde{\phi}(t) = \phi(t)$ for $t \in [-\tau, 0]$ and $\tilde{\phi}(t) = \phi(0)$ for $t \in [0, T]$. Denote

$$\begin{aligned} B_r(\phi(0)) &= \{x \in X : \|x - \phi(0)\|_X \le r\}, \\ B_{2r}(\tilde{\phi_0}) &= \{\chi \in \mathcal{C}_0 : \|\chi - \tilde{\phi}_0\|_{\mathcal{C}_0} \le 2r\}. \end{aligned}$$

Theorem 3.1. *Let B and V be open subsets of X and $\mathcal{C}_0$, respectively, and the family $\{A(t,w)\}$ of linear operators for $t \in [0,T]$ and $w \in B_r(\phi(0))$ satisfy assumptions* (H1)-(H6) *and $A(t,w)\phi(0) \in Y$ with*

$$\|A(t,w)\phi(0)\|_Y \le C$$

for all $(t,w) \in [0,T] \times B$. Suppose $F(t,u_t)$ satisfies (H7). *Then there exists a unique local classical solution of* (1.1).

Proof. From assumption (H5) for $t \ge s$, $t, s \in [0,T]$ and $u \in \mathcal{C}([0,T];X)$ with values in B, we have

$$\|U_u(t,s)\|_{B(Y)} \le C_1.$$

Take $r > 0$ such that $B_r(\phi(0)) \subset B$ and $B_{2r}(\tilde{\phi_0}) \subset V$. Choose

$$T_0 = \min\Big\{T, \frac{r}{2C_1C\|\phi(0)\|_X}, \frac{r}{L_F}, \frac{r}{2C_1(2L_Fr + N)}, \frac{1}{n\Lambda}, \frac{r}{L_\phi}\Big\}$$

where $\Lambda = C_0\|\phi(0)\|_X + C_1L_F + C_0(2L_Fr + N)\frac{T_0}{2}$, $n > 1$ is any natural number and $\|F(s,u_0)\|_X \le N$, where N is a positive constant.

Define the set

$$S = \{\psi \in \mathcal{C}_{T_0} : \psi_0 = \phi, \text{ for } t \in [-\tau, 0], \psi(t) \in B_r(\phi(0)), \ t \in [0, T_0]\}.$$

We easily deduce that S is a closed, convex and bounded subset of $\mathcal{C}_{T_0}$. Take $\psi \in S$. Now for $\theta \in [-\tau, 0]$ we have the following two cases.

Case 1: If $t + \theta \le 0$ we have

$$\begin{aligned} \|\psi_t(\theta) - \tilde{\phi}_0(\theta)\|_X &= \|\psi(t+\theta) - \tilde{\phi}(\theta)\|_X \\ &= \|\phi(t+\theta) - \phi(\theta)\|_X \quad \text{(by the definition of } S) \\ &\le L_\phi T_0 \le r. \end{aligned}$$

Case 2: If $t + \theta \ge 0$ we have

$$\begin{aligned} \|\psi_t(\theta) - \tilde{\phi}_0(\theta)\|_X &= \|\psi(t+\theta) - \tilde{\phi}(\theta)\|_X \\ &\le \|\psi(t+\theta) - \phi(0)\|_X + ||\phi(0) - \phi(\theta)\|_X \\ &\le r + L_\phi(-\theta) \quad (\text{since } \psi(t+\theta) \in B_r(\phi(0))) \\ &\le r + L_\phi t \\ &\le r + L_\phi T_0 \le 2r \quad (\text{since } -\theta \le t \le T_0). \end{aligned}$$

Thus, for $\psi \in S$, $\psi_t \in B_{2r}(\phi)$. Define $G : S \to S$ by

$$Gu(t) = \begin{cases} U_u(t,0)\phi(0) + \int_0^t U_u(t,s)F(s,u_s)ds, & t \in [0,T_0], \\ \phi(t), & t \in [-\tau, 0]. \end{cases}$$

First we show that G is well defined and $Gu(0) = \phi(0)$. For $t \geq 0$, we have

$$Gu(t) - \phi(0) = U_u(t,0)\phi(0) - \phi(0) + \int_0^t U_u(t,s)F(s,u_s)ds.$$

Taking the norm, we get

$$\|Gu(t) - \phi(0)\|_X \leq \|U_u(t,0)\phi(0) - \phi(0)\|_X + \int_0^t \|U_u(t,s)F(s,u_s)\|_X ds.$$

Integrating (iii), we obtain

$$U_u(t,0)\phi(0) - \phi(0) = \int_0^t U_u(t,s)A(s,u(s))\phi(0)ds.$$

Thus we have

$$\begin{aligned} \|U_u(t,0)\phi(0) - \phi(0)\|_X &\leq \int_0^t \|U_u(t,s)A(s,u(s))\|_X \|\phi(0)\|_X ds \\ &\leq C_1 C T_0 \|\phi(0)\|_X \leq \frac{r}{2}. \end{aligned} \tag{3.1}$$

Also, we have

$$\begin{aligned} \int_0^t \|U_u(t,s)F(s,u_s)\|_X ds &\leq C_1 \int_0^t (\|F(s,u_s) - F(s,u_0)\|_X + \|F(s,u_0)\|_X)ds \\ &\leq C_1 \int_0^t (\|F(s,u_s) - F(s,\phi)\|_X + \|F(s,\phi)\|_X)ds \\ &\leq C_1 \int_0^t (L_F\|u_s - \phi\|_X + N)ds \\ &\leq C_1(2L_F r + N)T_0 \leq \frac{r}{2}, \end{aligned}$$

using the result that for $u \in S$, $u_s \in B_{2r}(\phi)$. Thus, for $u \in S$ and $t \geq 0$, we get

$$\|Gu(t) - \phi(0)\|_X \leq r.$$

So G is well defined. For $u, v \in S$, we consider

$$\begin{aligned} Gu(t) - Gv(t) = U_u(t,0)\phi(0) - U_v(t,0)\phi(0) \\ + \int_0^t (U_u(t,s)F(s,u_s) - U_v(t,s)F(s,v_s))ds. \end{aligned}$$

Let

$$\begin{aligned} I_1 &= \|U_u(t,0)\phi(0) - U_v(t,0)\phi(0)\|_X \\ &\leq C_0\|\phi(0)\| \int_0^t \|u(s) - v(s)\|_X ds \\ &\leq C_0\|\phi(0)\|_X \|u - v\|_{\mathcal{C}_{T_0}} T_0. \end{aligned} \tag{3.2}$$

Also let

$$
\begin{aligned}
I_2 &= \Big\| \int_0^t (U_u(t,s)F(s,u_s) - U_v(t,s)F(s,v_s))ds \Big\|_X \\
&\le \int_0^t \Big(\|(U_u(t,s)F(s,u_s) - U_u(t,s)F(s,v_s)\|_X \\
&\quad + \|U_u(t,s)F(s,v_s) - U_v(t,s)F(s,v_s))\|_X \Big) ds \\
&\le C_1 L_F \int_0^t \|u_s - v_s\|_{\mathcal{C}_0} ds + C_0 \int_0^t \|F(s,v_s)\|_X \int_s^t \|u(\xi) - v(\xi)\|_X d\xi ds \\
&\le C_1 L_F \int_0^t \sup_\theta \|u(s+\theta) - v(s+\theta)\|_X ds \\
&\quad + C_0(2L_F r + N) \int_0^t \int_s^t \|u(\xi) - v(\xi)\|_X d\xi ds \\
&\le C_1 L_F T_0 \|u - v\|_{\mathcal{C}_{T_0}} + C_0(2L_F r + N) \int_0^t \int_0^s \|u(\xi) - v(\xi)\|_X d\xi ds \\
&\le C_1 L_F T_0 \|u - v\|_{\mathcal{C}_{T_0}} + C_0(2L_F r + N) \|u - v\|_{\mathcal{C}_{T_0}} \frac{T_0^2}{2} \\
&\le (C_1 L_F + C_0(2L_F r + N)) \frac{T_0^2}{2} \|u - v\|_{\mathcal{C}_{T_0}}.
\end{aligned} \tag{3.3}
$$

Hence from (3.2) and (3.3) we get

$$
\begin{aligned}
I_1 + I_2 &= \|Gu(t) - Gv(t)\|_X \\
&\le \Big(C_0 \|\phi(0)\|_X T_0 + (C_1 L_F + C_0(2L_F r + N)) \frac{T_0^2}{2} \Big) \|u - v\|_{\mathcal{C}_{T_0}} \\
&\le \Lambda T_0 \|u - v\|_{\mathcal{C}_{T_0}} \\
&\le \frac{1}{n} \|u - v\|_{\mathcal{C}_{T_0}}.
\end{aligned} \tag{3.4}
$$

Thus G is a contraction from S to S. So, by the Banach contraction mapping theorem, G has a unique fixed point $u \in S$ which satisfies the integral equation. Hence it is a mild solution of (1.1). Now, we consider the following evolution equation

$$
\begin{gathered}
\frac{dv(t)}{dt} + A(t, u(t))v(t) = F(t, u_t), \quad t \in [0, T_0], \\
u(0) = \phi(0).
\end{gathered} \tag{3.5}
$$

Denote $\tilde{A}(t) = A(t, u(t))$ and $\tilde{F}(t) = F(t, u_t)$, then equation (3.5) can be written as

$$
\begin{gathered}
\frac{dv(t)}{dt} + \tilde{A}(t)v(t) = \tilde{F}(t), \quad t \in [0, T_0], \\
u(0) = \phi(0),
\end{gathered} \tag{3.6}
$$

where u is the unique fixed point of G in S.

Now we show that $F(\cdot, \chi) \in \mathcal{C}_{T_0}$ for $t, s \in [0, T_0]$. By assumption (H7) we have

$$
\|F(t,\chi) - F(s,\chi)\|_X \le L_F |t - s|.
$$

Hence for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|t - s| \le \delta$, implies $\|F(t,\chi) - F(s,\chi)\|_X \le \epsilon$.

Thus, $F(t,\chi) \in \mathcal{C}_{T_0}$ for a fixed χ. Hence from Pazy [10, Theorem 5.5.2], we get a unique function $v \in \mathcal{C}^1((0,T_0],X)$ satisfying (3.6) in X and v given by

$$v(t) = U_u(t,0)\phi(0) + \int_0^t U_u(t,s)F(s,u_s)ds, \quad t \in [0,T_0].$$

Where $U_u(t,s), 0 \le s \le t \le T_0$ is the evolution system generated by the family $\{A(t,u(t))\}$, $t \in [0,T_0]$. The uniqueness of v implies that $v \equiv u$ on $[0,T_0]$. Thus u is a unique local classical solution of (1.1). $\square$

4. Example

Let us consider the equation

$$\frac{du(t)}{dt} + A(t,u(t))u(t) = K(u)(t), \quad t \in [0,T], \tag{4.1}$$

where

$$K(u)(t) = \int_0^t k(t-s)f(s,u(s))ds$$

and $A(t,u(t))$ satisfies all the required conditions of Theorem 3.1. Further let $k:[0,T] \to \mathbb{R}$ and $f:[0,T]\times B \to X$ be continuous functions, where B is a subset of X. We also assume that $f(\cdot,u(\cdot)):[0,T] \to X$ is a bounded function and there exists a constant $L_f \ge 0$ such that

$$\|f(t,u(s)) - f(s,v(s))\|_X \le L_f(|t-s| + \|u(s)-v(s)\|_X).$$

If we put $t-s=-\eta$ in the second term on the right hand side of (4.1) to obtain

$$\begin{aligned}\int_0^t k(t-s)f(s,u(s))ds &= \int_{-t}^0 k(-\eta)f(t+\eta,u(t+\eta))d\eta \\ &= \int_{-t}^0 k(-\eta)f(t+\eta,u_t(\eta))d\eta,\end{aligned}$$

then (4.1) can be rewritten as

$$\frac{du}{dt} + A(t,u(t))u(t) = F(t,u_t), \tag{4.2}$$

where $F:[0,T]\times\mathcal{C}_0 \to X$ given by

$$F(t,\phi) = \int_{-t}^0 k(-\eta)f(t+\eta,\phi(\eta))d\eta.$$

here k is bounded on $[0,T]$; i.e., $\sup_{t\in[0,T]}|k(t)| \le M_2 < \infty$, for some positive constant M_2. For (t,ϕ), $(s,\psi) \in [0,T]\times\mathcal{C}_0$, we have

$$\begin{aligned}&\|F(t,\phi) - F(s,\psi)\|_X \\ &\le \Big\| \int_{-t}^0 k(-\eta)f(t+\eta,\phi(\eta))d\eta - \int_{-s}^0 k(-\eta)f(s+\eta,\psi(\eta))d\eta \Big\|_X \\ &\le \int_{-t}^{-s} |k(-\eta)|\|f(t+\eta,\phi(\eta))\|_X d\eta \\ &\quad + \int_{-s}^0 |k(-\eta)|\|f(t+\eta,\phi(\eta)) - f(s+\eta,\psi(\eta))\|_X d\eta \\ &\le M_2M_1|t-s| + M_2TL_f(|t-s| + \|\phi(\eta)-\psi(\eta)\|_X)\end{aligned}$$

$$\begin{aligned}&\le M_2M_1|t-s| + M_2TL_f(|t-s| + \|\phi-\psi\|_{\mathcal{C}_0})\\&\le L_F(|t-s| + \|\phi-\psi\|_{\mathcal{C}_0}),\end{aligned}$$

where $L_F = M_2M_1 + TM_2L_f$ and $\|f(t,\phi)\|_X \le M_1$ for some positive constant M_1.

Thus all the conditions of theorem 3.1 are satisfied, so we may apply the results established in the earlier sections to ensure the existence and uniqueness of the solution.

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