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PERIODIC SOLUTIONS OF NON-AUTONOMOUS SECOND ORDER SYSTEMS WITH p-LAPLACIAN

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Abstract. We prove the existence of periodic solutions for non-autonomous second order systems with p-Laplacian. Our main tools are the minimax methods in critical point theory. Our results are new, even when $p = 2$.

1. Introduction

In this article concerns the existence of periodic solutions for the problem

$$\begin{gathered} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{gathered} \tag{1.1}$$

where $p > 1$, $T > 0$, $F : [0, T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^{\mathbb{N}}$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+), b \in L^1(0, T; \mathbb{R}^+)$, such that
$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$.

Considerable attention has been paid to the periodic solutions of problem (1.1) for $p = 2$, in recent years. The firsts to consider this problem when $p = 2$ were Berger and Schechter [12] in 1977, they proved the existence of solutions to problem (1.1) for $p = 2$ under the condition that $F(t, x) \to \pm\infty$ as $|x| \to \infty$ uniformly for a.e. $t \in [0, T]$. Subsequently, using the variational methods, many existence results are obtained, we refer the readers to [1, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15] and the references therein. However, there are few papers discussing periodic solutions for second order systems with p-Laplacian. In [2], Tian has established the existence results for problem (1.1) by the dual least action principle. Moreover, if problem (1.1) with nonlinear boundary conditions, the existence of periodic solutions has also been proved in [3] by means of the least action principle and the mountain pass lemma.

For $p = 2$, under the assumptions that there exists $h(t) \in L^1(0, T; \mathbb{R}^+)$ such that
$$|\nabla F(t, x)| \le h(t)$$

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for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, and that

$$\int_0^T F(t,x)dt \to \pm\infty \quad \text{as } |x| \to \infty,$$

Mawhin and Willem in [1] have shown that problem (1.1) admitted a periodic solution. After that, Tang in [4] generalized these results to the sublinear case. Concertely speaking, it is assumed that the nonlinearity satisfied the following restrictions

$$|\nabla F(t,x)| \le k(t)|x|^{\alpha} + p(t) \quad \text{for all } x \in \mathbb{R}^{\mathbb{N}} \text{ and a.e. } t \in [0,T], \tag{1.2}$$

$$\frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)dt \to \pm\infty \quad \text{as } |x| \to \infty, \tag{1.3}$$

here, $k(t), p(t) \in L^1(0,T;\mathbb{R}^+)$ and $\alpha \in [0,1)$. Under these conditions, periodic solutions of problem (1.1) with $p = 2$ have been obtained. In addition, Tang in [5] first introduced the local α-coercive conditions, that is, there exists $q(t) \in L^1(0,T;\mathbb{R}^+)$ and a subset E of $[0,T]$ with $\text{meas}(E) > 0$ such that

$$\frac{F(t,x)}{|x|^{2\alpha}} \le q(t) \tag{1.4}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, and that

$$\frac{F(t,x)}{|x|^{2\alpha}} \to -\infty \quad \text{as } |x| \to \infty \tag{1.5}$$

for a.e. $t \in E$, to deal with the generalized Josephson-type systems.

An interesting question naturally arises: In all the results [1, 4, 5] discussed above, the nonlinearity is required to grow at infinity at most like $|x|^{\alpha}$, is it possible to handle nonlinearity with faster increase at infinity and get the similar results of [1, 4, 5]. In the present paper, we will focus on this problem and give a positive answer. Here, we emphasize that our results are new even for $p = 2$.

We now state our main theorems.

Theorem 1.1. *Suppose that F satisfies assumption* (A) *and the following conditions:*

- (H1) *There exists a bounded nonincreasing positive function $\omega \in C((0,\infty);\mathbb{R}^+)$ with the properties:*
  - (i) $\liminf_{t\to\infty} \frac{\omega(t)}{\omega(t^{\gamma})} > 0$ *for some $\gamma \in (0,1)$,*
  - (ii) $\omega(t) \to 0, \quad \omega(t)t \to \infty$ *as $t \to \infty$.*

  Moreover, there exist $f \in L^2(0,T;\mathbb{R}^+)$ and $g \in L^1(0,T;\mathbb{R}^+)$ such that

  $$|\nabla F(t,x)| \le f(t)[\omega(|x|)|x|]^{p-1} + g(t)$$

  for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$;
- (H2) *There exists a bounded non-increasing positive function $\omega \in C((0,\infty);\mathbb{R}^+)$ which satisfies the conditions* (i), (ii) *and*

  $$\frac{1}{[\omega(|x|)|x|]^p} \int_0^T F(t,x)dx \to -\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution in the set*

$$W_T^{1,p} := \big\{u : [0,T] \to \mathbb{R}^{\mathbb{N}},\ u \text{ is absolutely continuous},\\ u(0) = u(T), \dot u \in L^p(0,T;\mathbb{R}^{\mathbb{N}})\big\}\,.$$

This set is a Banach space with the norm

$$\|u\| := \Big(\int_0^T |u(t)|^p dt + \int_0^T |\dot u(t)|^p dt\Big)^{1/p} \quad \text{for } u \in W_T^{1,p}.$$

Remark 1.2. Obviously, Theorem 1.1 does not satisfy the corresponding conditions in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Furthermore, there are functions $F(t,x)$ that satisfy our Theorem 1.1 and do not satisfy the corresponding results in [1, 4, 5] even for $p = 2$. For example, let

$$\nabla F(t,x) = -\frac{|x|^{p-2}x}{[\ln(2+|x|^2)]^{p-1}} + d(t), \quad \forall t \in [0,T], x \in \mathbb{R}^{\mathbb{N}},$$

where $d(t) \in L^1(0,T;\mathbb{R})$. Let $\omega(|x|) = 1/\ln(2+|x|^2), \gamma = 1/2$, a straightforward computation shows that $F(t,x)$ satisfies all the conditions of our Theorem 1.1. However, it is clear that $F(t,x)$ neither satisfies (1.2), (1.3) nor (1.4), (1.5) for any $\alpha \in [0,1)$ even for $p = 2$.

Theorem 1.3. *Suppose that F satisfies assumption* (A) *and* (H1). *Moreover assume F satisfies the following conditions:*

(H2*) (1) *There exists a bounded nonincreasing and positive function ω in $C((0,\infty);\mathbb{R}^+)$ and constant $C > 0$ such that*
  (iii) $\limsup_{t\to\infty} \frac{\omega(t)}{\omega(t^\gamma)} < +\infty$ *for some $\gamma \in (0,1)$;*
  (iv) $t \le C\omega^2(t)t^2$ *as $t \to \infty$;*

(2) *There exists $r(t) \in L^1(0,T;\mathbb{R}^+)$ such that*

$$\frac{F(t,x)}{[\omega(|x|)|x|]^p} \le r(t) \quad \text{for all } x \in \mathbb{R}^{\mathbb{N}} \text{ and a.e. } t \in [0,T];$$

(3) *There exists a subset E of $[0,T]$ with meas$(E) > 0$ such that for a.e. $t \in E$*

$$\frac{F(t,x)}{[\omega(|x|)|x|]^p} \to -\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution in $W_T^{1,p}$.*

Remark 1.4. There are functions $F(t,x)$ that satisfy our Theorem 1.3 and not do not satisfy the corresponding results in [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. For example, let

$$F(t,x) = -\theta(t)\frac{|x|^p}{[\ln(2+|x|^2)]^{p-1}}, \quad \forall t \in [0,T], x \in \mathbb{R}^{\mathbb{N}},$$

where

$$\theta(t) = \begin{cases} \sin\frac{2\pi t}{T}, & t \in [0,T/2],\\ 0, & t \in [T/2,T]. \end{cases}$$

Take $\omega(|x|) = 1/\ln(2+|x|^2)$, $\gamma = 1/2$, $E = [T/6,T/4]$, by simple computation, $F(t,x)$ satisfies our Theorem 1.3. However, here $F(t,x)$ also neither satisfies (1.2), (1.3) nor (1.4), (1.5) for any $\alpha \in [0,1)$ even for $p = 2$.

2. Preliminaries

For convenience, we will denote various positive constants as C_i, $i = 0, 1, 2, \dots$. Define functional φ on $W_T^{1,p}$ by

$$\varphi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t, u(t))dt \tag{2.1}$$

for $u \in W_T^{1,p}$. It follows from assumption (A) that functional φ is continuously differentiable on $W_T^{1,p}$, moreover, one has

$$(\varphi'(u), v) = \int_0^T [(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))]dt$$

for all $u, v \in W_T^{1,p}$. It is well known that the solutions to problem (1.1) correspond to the critical points of the functional φ.

For $u \in W_T^{1,p}$, let $\bar{u} := \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u}(t) := u(t) - \bar{u}$, then we have

$$\|\tilde{u}\|_\infty \le C_0\|\dot{u}\|_{L^p} \quad \text{(Sobolev's inequality)},$$
$$\|\tilde{u}\|_{L^p} \le C_0\|\dot{u}\|_{L^p} \quad \text{(Wirtinger's inequality)},$$

where $\|\tilde{u}\|_\infty := \max_{0\le t\le T} |\tilde{u}(t)|$.

To proof of our main theorems, we need the following auxiliary results.

Lemma 2.1 ([6]). *Suppose that G satisfies assumption (A) and E is a measurable subset of $[0, T]$. Assume that*

$$G(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$, there exists subset E_δ of E with $\operatorname{meas}(E \backslash E_\delta) < \delta$ such that

$$G(t, x) \to +\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2.2 ([5]). *For every constant $\beta > 0$, there exists a constant $m_\beta > 0$ such that*

$$meas\{t \in [0, T] | |\bar{u}| < m_\beta\|\bar{u}\|\} < \beta \quad \text{for all } \bar{u} \ne 0.$$

The proof of the above lemma follows from the first part in [5, Lemma 3], we omit the details.

Lemma 2.3. *Suppose that (H1) holds, then there exists a non-increasing positive function $\tilde{\omega}(t) \in C((0, \infty); \mathbb{R}^+)$ which satisfies the following conditions:*

(a) $\tilde{\omega}(t) \to 0$, $\tilde{\omega}(t)t \to \infty$ *as* $t \to \infty$;

(b) $\|\nabla F(t, u(t))\|_{L^1} \le \|f\|_{L^2}[\tilde{\omega}(\|u\|)\|u\|]^{p-1} + \|g\|_{L^1}$;

(c) *If (H2) holds, then*

$$\frac{1}{[\tilde{\omega}(\|\bar{u}\|)\|\bar{u}\|]^p}\int_0^T F(t, \bar{u})dt \to -\infty \quad \text{as } \|\bar{u}\| \to \infty;$$

(d) *If (H2*) holds, moreover assume that* $\limsup_{\|u\|\to\infty} \frac{\|\tilde{u}\|}{\tilde{\omega}(\|u\|)\|u\|} < +\infty$, *then*

$$\frac{1}{[\tilde{\omega}(\|u\|)\|u\|]^p}\int_0^T F(t, u(t))dt \to -\infty \quad \text{as } \|u\| \to \infty.$$

Proof. Let $A := \{t \in [0,T] | |u(t)| \geq (T^{-1/p}\|u\|)^{1/2}\}$, for $u(t) \neq 0$, by (H_1) and the Hölder's inequality, we have

$$
\begin{aligned}
&\|\nabla F(t,u(t))\|_{L^1}\\
&\leq \int_0^T [f(t)(\omega(|u(t)|)|u(t)|)^{p-1} + g(t)]dt\\
&\leq \|f\|_{L^2}\Big[\int_0^T [\omega(|u(t)|)|u(t)|]^{2(p-1)}dt\Big]^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\Big[\int_A [\omega(|u(t)|)|u(t)|]^{2(p-1)}dt + \int_{[0,T]\setminus A} [\omega(|u(t)|)|u(t)|]^{2(p-1)}dt\Big]^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\Big[\int_0^T \Big(\omega((T^{-1/p}\|u\|)^{1/2})|u(t)|\Big)^{2(p-1)} dt\\
&\quad + (\sup_{s>0}\omega(s))^{2(p-1)}\int_0^T (T^{-1/p}\|u\|)^{p-1}dt\Big]^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\left[C_1\Big(\omega((T^{-1/p}\|u\|)^{1/2})\|u\|\Big)^{2(p-1)} + C_2\|u\|^{p-1}\right]^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\left[C_3\Big(\omega^2((T^{-1/p}\|u\|)^{1/2})\|u\|^2 + \|u\|\Big)\right]^{\frac{p-1}{2}} + \|g\|_{L^1}.
\end{aligned}
$$

Take $\tilde{\omega}(t) := \big[C_3\big(\omega^2((T^{-1/p}t)^{1/2}) + \frac{1}{t}\big)\big]^{1/2}, t > 0$, then

$$
\|\nabla F(t,u(t))\|_{L^1} \leq \|f\|_{L^2}[\tilde{\omega}(\|u\|)\|u\|]^{p-1} + \|g\|_{L^1}.
$$

Obviously, $\tilde{\omega}(t)$ satisfies (a) due to the properties of $\omega(t)$.

Next, we come to check condition (c). Note that $\liminf_{t\to\infty}\omega(t)/\omega(t^\gamma) > 0$ for $\gamma \in (0,1)$, we define

$$
\xi := \liminf_{t\to\infty} \frac{\omega^2(T^{-1/p}t)}{\omega^2((T^{-1/p}t)^{1/2})} > 0,
$$

if (H2) holds, for any $M > 0$, we get

$$
\int_0^T F(t,u(t))dt \leq -M[\omega(|u(t)|)|u(t)|]^p + C_4, \tag{2.2}
$$

which implies that for $\bar{u} \neq 0$

$$
\begin{aligned}
\frac{\int_0^T F(t,\bar{u})dt}{[\tilde{\omega}(\|\bar{u}\|)\|\bar{u}\|]^p} &\leq \frac{-M\big[\omega(|\bar{u}|)|\bar{u}|\big]^p + C_4}{\big[C_3\big(\omega^2\big((T^{-1/p}\|\bar{u}\|)^{1/2}\big)\|\bar{u}\|^2 + \|\bar{u}\|\big)\big]^{p/2}}\\
&= \frac{-M\big[\omega(T^{-1/p}\|\bar{u}\|)T^{-1/p}\|\bar{u}\|\big]^p + C_4}{\big[C_3\big(\omega^2\big((T^{-1/p}\|\bar{u}\|)^{1/2}\big)\|\bar{u}\|^2 + \|\bar{u}\|\big)\big]^{p/2}}.
\end{aligned} \tag{2.3}
$$

By the definition of ξ, there exists $R > 0$ such that

$$
\frac{\omega^2(T^{-1/p}t)t^2}{\omega^2\big((T^{-1/p}t)^{1/2}\big)t^2 + t} \geq \frac{\xi}{2} \quad \text{as } t \geq R. \tag{2.4}
$$

Therefore,

$$\begin{aligned} &\frac{1}{[\tilde{\omega}(\|\bar{u}\|)\|\bar{u}\|]^p}\int_0^T F(t,\bar{u})dt \\ &\le \frac{-MT^{-1}\big[\frac{\xi}{2}\left(\omega^2((T^{-1/p}\|\bar{u}\|)^{1/2})\|\bar{u}\|^2+\|\bar{u}\|\right)\big]^{p/2}+C_4}{\big[C_3\left(\omega^2((T^{-1/p}\|\bar{u}\|)^{1/2})\|\bar{u}\|^2+\|\bar{u}\|\right)\big]^{p/2}} \\ &\le -C_5M \quad \text{as } \|\bar{u}\|\to\infty, \end{aligned} \tag{2.5}$$

condition (c) holds.

Finally, we show that (d) is true. From Lemma 2.2, for all $\bar{u}\neq 0$, we have

$$\operatorname{meas}\big\{t\in[0,T]|\omega(|\bar{u}|)|\bar{u}|<m_\beta\omega(|\bar{u}|)\|\bar{u}\|=m_\beta\omega(T^{-1/p}\|\bar{u}\|)\|\bar{u}\|\big\}<\beta.$$

Consequently, for all $\bar{u}\neq 0$, let $B:=\{t\in[0,T]|\omega(|\bar{u}|)|\bar{u}|\ge m_\beta\omega(T^{-1/p}\|\bar{u}\|)\|\bar{u}\|\}$, then we have $\operatorname{meas}([0,T]\backslash B)<\beta$.

By (H2*)(3) and Lemma 2.1, there exists subset E_δ of E with $\operatorname{meas}(E\backslash E_\delta)<\delta$ such that

$$\frac{F(t,x)}{[\omega(|x|)|x|]^p}\to-\infty \quad \text{as } |x|\to\infty \tag{2.6}$$

uniformly for all $t\in E_\delta$. Then, we find

$$\operatorname{meas}(B\cap E_\delta)\ge\operatorname{meas}(E_\delta)-\operatorname{meas}([0,T]\backslash B)\ge\operatorname{meas}(E)-\delta-\beta>0 \tag{2.7}$$

for δ and β small enough. By (2.6), for every $\eta>0$, there exists $L>0$ such that

$$\frac{F(t,x)}{[\omega(|x|)|x|]^p}\le-\eta$$

for all $|x|\ge L$ and a.e. $t\in E_\delta$. Furthermore, applying assumption $(H_2^*)(2)$ yields

$$F(t,x)\le-\eta[\omega(|x|)|x|]^p+r^*(t)$$

for all $x\in\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in E_\delta$, where $r^*(t):=(\sup_{s>0}\omega(s))^pL^pr(t)$. Noting the definition of $\tilde{\omega}(t)$, $0<\liminf_{t\to\infty}\frac{\omega(t)}{\omega(t^\gamma)}\le\limsup_{t\to\infty}\frac{\omega(t)}{\omega(t^\gamma)}<+\infty$ and assumption $(H_2^*)(1)(\mathrm{iv})$, we deduce that there exist $C_6,C_7>0$ such that

$$\begin{aligned} C_6\omega(T^{-1/p}\|\bar{u}\|)\|\bar{u}\| &\le \tilde{\omega}(\|\bar{u}\|)\|\bar{u}\| \\ &= [C_3(\omega^2((T^{-1/p}\|\bar{u}\|)^{1/2})\|\bar{u}\|^2+\|\bar{u}\|)]^{1/2} \\ &\le C_7\omega(T^{-1/p}\|\bar{u}\|)\|\bar{u}\| \quad \text{as } \|\bar{u}\|\to\infty, \end{aligned}$$

which implies

$$\begin{aligned} \int_{B\cap E_\delta}F(t,\bar{u})dt &\le -\eta\int_{B\cap E_\delta}[\omega(|\bar{u}|)|\bar{u}|]^pdt+\int_{B\cap E_\delta}r^*(t)dt \\ &\le -\eta\int_{B\cap E_\delta}[\omega(T^{-1/p}\|\bar{u}\|)m_\beta\|\bar{u}\|]^pdt+\int_{B\cap E_\delta}r^*(t)dt \\ &\le -\eta\int_{B\cap E_\delta}\frac{{m_\beta}^p}{C_7}(\tilde{\omega}(\|\bar{u}\|)\|\bar{u}\|)^pdt+\int_{B\cap E_\delta}r^*(t)dt \\ &\le -\eta C_8[\tilde{\omega}(\|\bar{u}\|)\|\bar{u}\|]^p+\int_{B\cap E_\delta}r^*(t)dt \quad \text{as } \|\bar{u}\|\to\infty, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\int_{[0,T]\setminus(B\cap E_\delta)} F(t,\bar u)dt &\le \int_{[0,T]\setminus(B\cap E_\delta)} r(t)[\omega(|\bar u|)|\bar u|]^p dt\\ &\le \int_0^T r(t)[\omega(T^{-1/p}\|\bar u\|)T^{-1/p}\|\bar u\|]^p dt\\ &\le \int_0^T \frac{T^{-1}}{C_6}(\tilde\omega(\|\bar u\|)\|\bar u\|)^p r(t)dt\\ &\le C_9[\tilde\omega(\|\bar u\|)\|\bar u\|]^p\int_0^T r(t)dt \quad \text{as } \|\bar u\|\to\infty.\end{aligned} \tag{2.9}$$

So, it follows from (2.8) and (2.9) that

$$\limsup_{\|\bar u\|\to\infty}\frac{1}{[\tilde\omega(\|\bar u\|)\|\bar u\|]^p}\int_0^T F(t,\bar u)dt \le -\eta C_8 + C_9\int_0^T r(t)dt,$$

which implies that

$$\frac{1}{[\tilde\omega(\|\bar u\|)\|\bar u\|]^p}\int_0^T F(t,\bar u)dt \to -\infty \quad \text{as } \|\bar u\|\to\infty. \tag{2.10}$$

On the other hand, from (b) and the assumptions of (d), we get

$$\begin{aligned}\Big|\int_0^T F(t,u(t))dt - \int_0^T F(t,\bar u)dt\Big| &\le \int_0^T\int_0^1 |\nabla F(t,\bar u + s\tilde u)||\tilde u|\,ds\,dt\\ &\le \|\tilde u\|_\infty[\|f\|_{L^2}(\tilde\omega(\|u\|)\|u\|)^{p-1} + \|g\|_{L^1}]\\ &\le C_{10}[\tilde\omega(\|u\|)\|u\|]^{p-1}\|\tilde u\| + C_{11}\|\tilde u\|.\end{aligned} \tag{2.11}$$

As a consequence, note $\limsup_{\|u\|\to\infty}\frac{\|\tilde u\|}{\tilde\omega(\|u\|)\|u\|} < +\infty$, we then have

$$C_{12} := \limsup_{\|u\|\to\infty}\Big|\frac{1}{[\tilde\omega(\|u\|)\|u\|]^p}\Big[\int_0^T F(t,u(t))dt - \int_0^T F(t,\bar u)dt\Big]\Big| < +\infty. \tag{2.12}$$

In addition, for $\|u\|\to\infty$, one knows

$$1 = \frac{\|u\|}{\|u\|} = \frac{\|\bar u\| + \|\tilde u\|}{\|u\|} = \frac{\|\bar u\|}{\|u\|} + \frac{\|\tilde u\|}{\tilde\omega(\|u\|)\|u\|}\cdot\tilde\omega(\|u\|) = \frac{\|\bar u\|}{\|u\|}. \tag{2.13}$$

This, in conjunction with (2.10)-(2.12), gives

$$\begin{aligned}&\limsup_{\|u\|\to\infty}\frac{1}{[\tilde\omega(\|u\|)\|u\|]^p}\int_0^T F(t,u(t))dt\\ &\le \limsup_{\|u\|\to\infty}\frac{1}{[\tilde\omega(\|u\|)\|u\|]^p}\int_0^T F(t,\bar u)dt + C_{12}\\ &= \limsup_{\|\bar u\|\to\infty}\frac{1}{[\tilde\omega(\|\bar u\|)\|\bar u\|]^p}\int_0^T F(t,\bar u)dt + C_{12}\to-\infty,\end{aligned} \tag{2.14}$$

which completes the proof. $\square$

3. Proofs of theorems

Now, we give the proofs of the main results.

Proof of Theorem 1.1. First, we prove that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence for φ, that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. It follows from Wirtinger's inequality that

$$\|\dot{u}_n\|_{L^p} \leq \|\tilde{u}_n\| \leq (C_0+1)^{1/p}\|\dot{u}_n\|_{L^p} \tag{3.1}$$

for all n. By virtue of the properties of $\tilde{\omega}(t)$, one has

$$\tilde{\omega}(\|\bar{u}+\tilde{u}\|) \leq \min\{\tilde{\omega}(\|\bar{u}\|), \tilde{\omega}(\|\tilde{u}\|)\}, \tag{3.2}$$

which implies

$$\begin{aligned}
&\Big|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt\Big| \\
&\leq \|\tilde{u}_n\|_\infty \left[\|f\|_{L^2}(\tilde{\omega}(\|u_n\|)\|u_n\|)^{p-1} + \|g\|_{L^1}\right] \\
&\leq \|\tilde{u}_n\|_\infty \left[\|f\|_{L^2}(\tilde{\omega}(\|\bar{u}_n+\tilde{u}_n\|)\|\bar{u}_n+\tilde{u}_n\|)^{p-1} + \|g\|_{L^1}\right] \\
&\leq \|\tilde{u}_n\|_\infty \left[\|f\|_{L^2}(\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\| + \tilde{\omega}(\|\tilde{u}_n\|)\|\tilde{u}_n\|)^{p-1} + \|g\|_{L^1}\right] \\
&\leq C_{13}\|\tilde{u}_n\|\,[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^{p-1} + C_{13}\|\tilde{u}_n\|\,[\tilde{\omega}(\|\tilde{u}_n\|)\|\tilde{u}_n\|]^{p-1} + C_{14}\|\tilde{u}_n\|
\end{aligned} \tag{3.3}$$

for all n. Thus, by (3.1) and (3.3), we get

$$\begin{aligned}
\|\tilde{u}_n\| &\geq (\varphi'(u_n), \tilde{u}_n) \\
&= \int_0^T |\dot{u}_n(t)|^p dt + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt \\
&\geq C_{15}\|\tilde{u}_n\|^p - C_{13}\|\tilde{u}_n\|\,[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^{p-1} \\
&\quad - C_{13}\|\tilde{u}_n\|\,[\tilde{\omega}(\|\tilde{u}_n\|)\|\tilde{u}_n\|]^{p-1} - C_{14}\|\tilde{u}_n\|.
\end{aligned} \tag{3.4}$$

Assume that $\{\|\tilde{u}_n\|\}$ is unbounded; that is, $\|\tilde{u}_n\| \to \infty$ as $n \to \infty$. Since $\tilde{\omega}(t) \to 0$ as $t \to \infty$, it follows from (3.4) that we can find a constant $C_{16} > 0$ such that

$$C_{16}\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\| \geq \|\tilde{u}_n\|, \tag{3.5}$$

which implies

$$[C_3(\omega^2((T^{-1/p}\|\bar{u}_n\|)^{1/2})\|\bar{u}_n\|^2 + \|\bar{u}_n\|)]^{1/2} = \tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\| \to \infty \quad \text{as } n \to \infty.$$

Since ω is bounded, this leads to

$$\|\bar{u}_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.6}$$

On the other hand, $\|\bar{u}_n + s\tilde{u}_n\| \geq \|\bar{u}_n\|, s \in [0,1]$, by Lemma 2.3 (b) and (3.5), we see that

$$
\begin{aligned}
&\int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]dt \\
&\leq \Big| \int_0^T \int_0^1 (\nabla F(t,\bar{u}_n + s\tilde{u}_n(t)), \tilde{u}_n(t))\, ds\, dt \Big| \\
&\leq \|\tilde{u}_n\|_\infty \int_0^T \int_0^1 |\nabla F(t,\bar{u}_n + s\tilde{u}_n(t))|\, ds\, dt \\
&\leq \|f\|_{L^2} \left[\tilde{\omega}(\|\bar{u}_n + s\tilde{u}_n\|)\|\bar{u}_n + s\tilde{u}_n\|\right]^{p-1} \|\tilde{u}_n\|_\infty + \|g\|_{L^1}\|\tilde{u}_n\|_\infty \\
&\leq \|f\|_{L^2}[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\| + \tilde{\omega}(\|\bar{u}_n\|)\|\tilde{u}_n\|]^{p-1}\|\tilde{u}_n\|_\infty + \|g\|_{L^1}\|\tilde{u}_n\|_\infty \\
&\leq C_{17}\|\tilde{u}_n\|[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^{p-1} + C_{17}\|\tilde{u}_n\|[\tilde{\omega}(\|\bar{u}_n\|)\|\tilde{u}_n\|]^{p-1} + C_{18}\|\tilde{u}_n\| \\
&\leq C_{19}[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p + C_{20}[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p[\tilde{\omega}(\|\bar{u}_n\|]^{p-1} + C_{21}\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|,
\end{aligned}
\tag{3.7}
$$

which implies that

$$
\begin{aligned}
\varphi(u_n) &= \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]dt + \int_0^T F(t,\bar{u}_n)dt \\
&\leq C_{22}[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p + C_{20}[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p[\tilde{\omega}(\|\bar{u}_n\|]^{p-1} \\
&\quad + C_{21}\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\| + \int_0^T F(t,\bar{u}_n)dt \\
&\leq [\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p \Big[C_{22} + C_{20}[\tilde{\omega}(\|\bar{u}_n\|]^{p-1} + \frac{C_{21}}{[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^{p-1}} \\
&\quad + \frac{1}{[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p}\int_0^T F(t,\bar{u}_n)dt\Big] \to -\infty \quad \text{as } \|\bar{u}_n\| \to \infty.
\end{aligned}
\tag{3.8}
$$

This contradicts the boundedness of $\varphi(u_n)$. So

$$
\{\|\tilde{u}_n\|\} \text{ is bounded.} \tag{3.9}
$$

Suppose that $\{\|\bar{u}_n\|\}$ is unbounded and $\{\|\tilde{u}_n\|\}$ is bounded. With the similar manner above, we deduce that

$$
\begin{aligned}
\varphi(u_n) &= \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]dt + \int_0^T F(t,\bar{u}_n)dt \\
&\leq C_{23}\|\tilde{u}_n\|^p + C_{17}\|\tilde{u}_n\|[\tilde{\omega}(\|\bar{u}_n\|)\|\tilde{u}_n\|]^{p-1} + C_{18}\|\tilde{u}_n\| + \int_0^T F(t,\bar{u}_n)dt \\
&\leq C_{24} + C_{25}[\tilde{\omega}(\|\bar{u}_n\|)\|\tilde{u}_n\|]^{p-1} + C_{26}[\tilde{\omega}(\|\bar{u}_n\|)]^{p-1} + \int_0^T F(t,\bar{u}_n)dt \\
&\leq [\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p \Big[\frac{C_{24}}{[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p} + \frac{C_{25}}{\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|} + \frac{C_{26}}{\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|^p} \\
&\quad + \frac{1}{[\tilde{\omega}(\|\bar{u}_n\|)\|\bar{u}_n\|]^p}\int_0^T F(t,\bar{u}_n)dt\Big] \to -\infty \quad \text{as } \|\bar{u}_n\| \to \infty,
\end{aligned}
\tag{3.10}
$$

which also contradicts the boundedness of $\varphi(u_n)$. Then

$$
\{\|\bar{u}_n\|\} \text{ is also bounded.} \tag{3.11}
$$

From (3.9) and (3.11), we have $\{\|u_n\|\}$ is bounded, thus φ satisfies the (PS) condition.

Since $W_T^{1,p} = \mathbb{R}^{\mathbb{N}} \oplus \tilde{W}_T^{1,p}$, where $\tilde{W}_T^{1,p} := \{u \in W_T^{1,p} | \int_0^T u(t)dt = 0\}$. Next, we shall prove that

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty \text{ in } \tilde{W}_T^{1,p}. \tag{3.12}$$

In fact, since $\tilde{\omega}(t) \to 0$ as $t \to \infty$, then there exists $A > 0$ such that $\tilde{\omega}(A) \leq \frac{1}{2pC_{29}}$. In a similar way to (3.7), we have

$$\begin{aligned}
&\Big|\int_0^T [F(t,u(t)) - F(t,A)]dt\Big| \\
&\leq C_{17}\|\tilde{u}\|[\tilde{\omega}(A)A]^{p-1} + C_{17}\|\tilde{u}\|[\tilde{\omega}(A)\|\tilde{u}\|]^{p-1} + C_{18}\|\tilde{u}\| \\
&\leq C_{27}\|\tilde{u}\| + C_{17}\tilde{\omega}(A)\|\tilde{u}\|^p \\
&\leq C_{28}\|\dot{u}\|_{L^p} + C_{29}\tilde{\omega}(A)\|\dot{u}\|_{L^p}^p \\
&\leq C_{28}\|\dot{u}\|_{L^p} + \frac{1}{2p}\|\dot{u}\|_{L^p}^p,
\end{aligned}$$

which implies

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,A)]dt - \int_0^T F(t,A)dt \\
&\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p - \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_{28}\|\dot{u}\|_{L^p} - \int_0^T F(t,A)dt
\end{aligned} \tag{3.13}$$

for all $u \in \tilde{W}_T^{1,p}$. By Wirtinger's inequality, one has

$$\|u\| \to \infty \Leftrightarrow \|\dot{u}\|_{L^p} \to \infty \quad \text{on } \tilde{W}_T^{1,p}.$$

Hence, (3.12) follows from (3.13). On the other hand, by (H2),

$$\varphi(u) \to -\infty \quad \text{as } |u| \to \infty \quad \text{in } \mathbb{R}^{\mathbb{N}}. \tag{3.14}$$

Combine (3.12) and (3.14), applying saddle point theorem, then problem (1.1) has at least one solution in $W_T^{1,p}$. □

Proof of Theorem 1.3. We commence by showing that φ satisfies (PS) condition. Let $\{u_n\}$ be a sequence in $W_T^{1,p}$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. If $\{u_n\}$ is unbounded, without loss of generality, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from Lemma 2.3 (b) that

$$\begin{aligned}
\Big|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt\Big| &\leq \|\tilde{u}_n\|_\infty[\|f\|_{L^2}[\tilde{\omega}(\|u_n\|)\|u_n\|]^{p-1} + \|G\|_{L^1}] \\
&\leq C_{30}[\tilde{\omega}(\|u_n\|)\|u_n\|]^{p-1}\|\tilde{u}_n\| + C_{31}\|\tilde{u}_n\|.
\end{aligned} \tag{3.15}$$

Hence, we have

$$\begin{aligned}
\|\tilde{u}_n\| &\geq (\varphi'(u_n), \tilde{u}_n) \\
&= \int_0^T |\dot{u}_n(t)|^p dt + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt \\
&\geq C_{32}\|\tilde{u}_n\|^p - C_{30}[\tilde{\omega}(\|u_n\|)\|u_n\|]^{p-1}\|\tilde{u}_n\| - C_{31}\|\tilde{u}_n\|,
\end{aligned}$$

which implies

$$\limsup_{\|u_n\|\to\infty} \frac{\|\tilde{u}_n\|}{\tilde{\omega}(\|u_n\|)\|u_n\|} < +\infty. \tag{3.16}$$

Therefore, by Lemma 2.3 (d), one has

$$\frac{1}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p}\int_0^T F(t,u_n(t))dt \to -\infty \quad \text{as } \|u_n\| \to \infty. \tag{3.17}$$

However, by the boundedness of $\varphi(u_n)$ and (3.16), we get

$$\begin{aligned}
\Big|\frac{1}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p}\int_0^T F(t,u_n(t))dt\Big| &= \Big|\frac{\varphi(u_n)}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p} - \frac{\frac{1}{p}\int_0^T |\dot{u}_n(t)|^p dt}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p}\Big| \\
&\le \frac{\frac{1}{p}\int_0^T |\dot{u}_n(t)|^p dt}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p} \\
&\le \frac{C_{33}\|\tilde{u}_n\|^p}{[\tilde{\omega}(\|u_n\|)\|u_n\|]^p} < +\infty,
\end{aligned} \tag{3.18}$$

which contradicts (3.17). So, φ satisfies the (PS) condition.

As in the proof of Theorem 1.1, we can obtain

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty \text{ in } \tilde{W}_T^{1,p}.$$

Furthermore, from (2.10), it is easy to see that

$$\varphi(u) \to -\infty \quad \text{as } |u| \to \infty \quad \text{in } \mathbb{R}^{\mathbb{N}}.$$

Thus, using the saddle point theorem, problem (1.1) has at least one solution in $W_T^{1,p}$. □

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