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EXISTENCE OF CONVEX AND NON CONVEX LOCAL SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we establish the existence theorems for a class of fractional differential inclusion of order $n-1 < \alpha \leq n$. The study holds in two cases, when the set-valued function has convex and non-convex values.

1. Introduction

We study the existence of solutions for a class of nonlinear differential inclusions of fractional order. The operators are taken in the Riemann-Liouville sense and the initial conditions are specified according to Caputo's suggestion, thus allowing for interpretation in a physically meaningful way. There are numerous books focused in this direction, that is concerning the linear and nonlinear problems involving different types of fractional derivatives as well as integral (see [21, 24, 25, 27, 28]). El-Sayed and Ibrahim [13, 14, 18] gave the concept of the definite integral of fractional order for set-valued function. As applications of this type of problem, it arises in the study of control systems, game theory and programing languages (see [2, 3, 20]).

The Riemann-Liouville fractional operators are defined as follows; see [24, 27]:

Definition 1.1. The fractional integral operator I^α of order $\alpha > 0$ of a continuous function $f(t)$ is given by

$$I_0^\alpha f(t) := I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

We can write $I_0^\alpha f(t) = f(t) * \psi_\alpha(t)$ where $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$ and $\psi_\alpha(t) \to \delta(t)$ (the delta function) as $\alpha \to 0$ (see [24, 27]).

Definition 1.2. The fractional derivatives D^α of order $n-1 < \alpha \leq n$ of the function $f(t)$ is given by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau.$$

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This paper concerns the fractional differential inclusion

$$\begin{gathered} D^{\alpha}(u-T_{n-1}[u])(t)\in F(t,u(t),\rho(t));\quad n-1<\alpha\leq n,\ t\in J:=[0,T],\\ u^{(k)}(0)=u_0^{(k)}\in\mathbb{R},\quad k=0,1,\dots,n-1 \end{gathered} \tag{1.1}$$

where $T_{n-1}[u]$ is the Taylor polynomial of order $(n-1)$ for u, centered at 0, $\rho : J\to\mathbb{R}$ is a continuous function and $F : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a set-valued function with nonempty values in $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

This paper is organized as follows: In Section 2, we will recall briefly some basic definitions and preliminary facts from set-valued analysis which will be used later. In Section 3, we shall establish the existence and uniqueness solution for the single-valued problem

$$\begin{gathered} D^{\alpha}(u-T_{n-1}[u])(t)=f(t,u(t),\rho(t));\quad n-1<\alpha\leq n,\ t\in J=[0,T],\\ u^{(k)}(0)=u_0^{(k)},\ k=0,1,\dots,n-1 \end{gathered} \tag{1.2}$$

by using the Schauder fixed point theorem (see [6]) and the Banach fixed point theorem (see [29]) respectively. In Section 4, we shall study the existence of solution for the set-valued problem (1.1) when F has a convex as well as non-convex values via the single-valued problem as well as fixed point theorems of the set-valued function. In the first case (convex) a fixed point theorem due to Martelli [23] is used. A fixed point theorem for contraction set-valued functions due to Covitz and Nadler [9] is applied in the second one (non-convex).

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts from set-valued analysis which are used throughout this paper. For further background and details pertaining to this section we refer the reader to [4, 7, 16, 17, 19, 26, 30].

$\mathcal{B}:=C[J,\mathbb{R}]$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|=\sup\{|u(t)|:t\in J\}$$

for each $u\in\mathcal{B}$. $\mathcal{L}:=L^1[J,\mathbb{R}]$ denotes the Banach space of measurable functions $u:J\to\mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1}=\int_0^T|u(t)|dt,$$

for $u\in\mathcal{L}$. Let $(X,|\cdot|)$ be a normed space, $\mathcal{P}_{cl}(X)=\{Y\in\mathcal{P}(X):Y\text{ is closed}\}$, $\mathcal{P}_b(X)=\{Y\in\mathcal{P}(X):Y\text{ is bounded}\}$, $\mathcal{P}_{cp}(X)=\{Y\in\mathcal{P}(X):Y\text{ is compact}\}$, $\mathcal{P}_c(X)=\{Y\in\mathcal{P}(X):Y\text{ is convex}\}$, $\mathcal{P}_{cl,c}(X)=\{Y\in\mathcal{P}(X):Y\text{ is closed and convex}\}$, $\mathcal{P}_{cp,c}(X)=\{Y\in\mathcal{P}(X):Y\text{ is compact and convex}\}$. A set-valued function $F : X\to\mathcal{P}(X)$ is called **convex (closed)** valued if $F(x)$ is convex (closed) for all $x\in X$. F is called **bounded** valued on bounded set B if $F(B)=\bigcup_{x\in B}F(x)$ is bounded in X for all $B\in\mathcal{P}_b(X)$ i.e. $\sup_{x\in B}\{\sup\{|u|:u\in F(x)\}\}<\infty$. F is called **upper semi-continuous (u.s.c)** on X if for each $x_0\in X$ the set $F(x_0)$ is nonempty closed subset of X and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0)\subseteq N$. In other wards F is u.s.c if the set $F^{-1}(A)=\{x\in X:Fx\subset A\}$ is open in X for every open set A in X. F is called **lower semi-continuous (l.s.c)** on X if A is any open subset of X then $F^{-1}(A)=\{x\in X:Fx\cap A\neq\emptyset\}$ is open in X. F is called **continuous** if it is lower as well as upper semi-continuous on X. F is called **compact** if for every M bounded subset of X, $F(M)$ is relatively compact. Finally F is called **completely continuous** if it is upper semi-continuous and compact on X. The following definitions are used in the sequel.

Definition 2.1. A mapping $p : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

(i) $t \to p(t, u)$ is measurable for each $u \in \mathbb{R}$,
(ii) $u \to p(t, u)$ is continuous a.e. for $t \in J$.

A Carathéodory function $p(t, u)$ is called $L^1(J, \mathbb{R})$-Carathéodory if

(iii) for each number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that $|p(t, u)| \le h_r(t)$ a.e. $t \in J$ for all $u \in \mathbb{R}$ with $|u| \le r$.

A Carathéodory function $p(t, u)$ is called $L^1_X(J, \mathbb{R})$-Carathéodory if

(iv) there exists a function $h \in L^1(J, \mathbb{R})$ such that $|p(t, u)| \le h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ where h is called the bounded function of p.

Definition 2.2. A set-valued function $F : J \to \mathcal{P}(\mathbb{R})$ is said to be **measurable** if for any $x \in X$, the function $t \mapsto d(x, F(t)) = inf\{|x - u| : u \in F(t)\}$ is measurable.

Definition 2.3. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto F(t, x)$ is u.s.c. for almost $t \in J$.

Definition 2.4. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if

(i) F is Carathéodory and
(ii) For each $r > 0$, there exists $h_r \in L^1(J, \mathbb{R})$ such that $\|F(t, u)\| = \sup\{|f| : f \in F(t, u)\} \le h_r(t)$ for all $|u| \le r$ and for a.e. $t \in J$.

Definition 2.5 ([11])**.** A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called L^1_X-Carathéodory if there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$\|F(t, u)\| = \sup\{|f| : f \in F(t, u)\} \le h(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of F on $J \times \mathbb{R}$.

Let $A, B \in \mathcal{P}_{cl}(X)$, let $a \in A$ and let

$$D(a, B) = \inf\{\|a - b\| : b \in B\} \quad \text{and} \quad \rho(A, B) = \sup\{D(a, B) : a \in A\}.$$

The function $H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl,b}(X) \to \mathbb{R}^+$ defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

is a metric and is called Hausdorff metric on X. Moreover $(\mathcal{P}_{cl,b}(X), H)$ is a metric space and $(\mathcal{P}_{cl}(X), H)$ is a complete metric space (see [22]). It is clear that

$$H(0, C) = sup\{\|c\| : c \in C; C \in \mathcal{P}_b(X)\}.$$

Definition 2.6. A set-valued function $F : \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called
(i) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H(F(x), F(y)) \le \gamma\|x - y\|, \quad \text{for each } x, y \in X$$

the constant γ is called a Lipschitz constant.
(ii) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.7. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called
(i) $\gamma(t)$-Lipschitz if there exists $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$H(F(t,x), F(t,y)) \leq \gamma(t)\|x-y\|, \quad \text{for each } x, y \in X.$$

(ii) a contraction if it is $\gamma(t)$-Lipschitz with $\|\gamma\| < 1$.

The following remark and lemmas are used in the sequel.

Remark 2.8 ([5])**.** Let $M \subset X$. If $F : M \to \mathcal{P}(X)$ is closed and $F(M)$ is relatively compact then F is u.s.c. on M. And if $F : X \to \mathcal{P}(X)$ is closed and compact operator then F is u.s.c.on X.

Lemma 2.9 ([23])**.** *Let $T : X \to \mathcal{P}_{c,cp}(X)$ be a completely continuous set-valued function. If*

$$\varepsilon = \{u \in X : \lambda u \in Tu, \ \text{for some } \lambda > 1\}$$

is a bounded set, then T has a fixed point.

Lemma 2.10 ([9])**.** *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then G has a fixed point.*

3. Single-valued problem

In this section we prove that the fractional differential equation (1.2) has a solution $u(t)$ on J. By using some classical results from the fractional calculus, the following result held (see [28]).

Lemma 3.1. *If the function f is continuous, then the initial value problem* (1.2) *is equivalent to the nonlinear Volterra integral equation of the second kind,*

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau), \rho(\tau)) d\tau, \tag{3.1}$$

where $n - 1 < \alpha \leq n$, $t \in J = [0, T]$. In other words, every solution of the Volterra equation (3.1) *is also a solution of the initial value problem* (1.2) *and vice versa.*

Diethelm and Ford [10] proved the existence of solutions for (3.1) in the case $0 < \alpha < 1$. Let us formulate the following assumption:

- (H1) The function f is L^1_X-Carathéodory with bounded function $h \in L^1(J \times \mathbb{R}, \mathbb{R}^+)$; i.e., $|f(t,u,\rho)| \leq h(t,\rho)$ a.e $t \in J$ for all $u \in \mathbb{R}$ such that $\|h\|_{L^1} < \infty$.

Theorem 3.2. *Let the assumption* (H1) *hold. Then the fractional differential equation* (1.2) *has at least one solution $u(t)$ on J.*

Proof. Define an operator P by

$$(Pu)(t) := \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau), \rho(\tau)) d\tau \tag{3.2}$$

then by the assumption of the theorem and the properties of fractional calculus we obtain

$$\begin{aligned}|(Pu)(t)| &\le \sum_{k=0}^{n-1}\frac{t^k}{k!}|u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau),\rho(\tau))|d\tau\\ &\le \sum_{k=0}^{n-1}\frac{t^k}{k!}|u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau,\rho)d\tau\\ &\le \sum_{k=0}^{n-1}\frac{T^k}{k!}|u^{(k)}(0)| + \frac{\|h\|_{L^1}T^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Hence

$$\|Pu\| \le \sum_{k=0}^{n-1}\frac{T^k}{k!}|u^{(k)}(0)| + \frac{\|h\|_{L^1}T^\alpha}{\Gamma(\alpha+1)}.$$

Set $r := \sum_{k=0}^{n-1}\frac{T^k}{k!}|u^{(k)}(0)| + \frac{\|h\|_{L^1}T^\alpha}{\Gamma(\alpha+1)}$ that is $P: B_r \to B_r$. Then P maps B_r into itself. In fact, P maps the convex closure of $P[B_r]$ into itself. Since f is bounded on B_r, thus $P[B_r]$ is equicontinuous and the Schauder fixed point theorem shows that P has at least one fixed point $u \in \mathcal{B} = C[J,\mathbb{R}]$ such that $Pu = u$, which is corresponding to the solution of (1.2). □

For the uniqueness of solutions, we introduce the following assumption:

(H2) The function f satisfies that there exists a function $\ell(t) \in L^1(J,\mathbb{R}^+)$ with, $\|\ell\|_{L^1} < \infty$, such that for each $u, v \in C[J,\mathbb{R}]$ we have
$$|f(t,u,\rho) - f(t,v,\rho)| \le \ell(t)\|u-v\|.$$

Theorem 3.3. *Let* (H2) *hold. If* $\|\ell\|_{L^1}T^\alpha/\Gamma(\alpha+1) < 1$*, then the fractional differential equation* (1.2) *has a unique solution* $u(t)$ *on* J.

Proof. Using the operator P defined in (1.2), we have

$$\begin{aligned}|(Pu)(t)-(Pv)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau),\rho(\tau)) - f(\tau,v(\tau),\rho(\tau))|d\tau\\ &\le \frac{\|u-v\|_\infty}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\ell(\tau)d\tau\\ &\le \frac{\|\ell\|_{L^1}T^\alpha}{\Gamma(\alpha+1)}\|u-v\|_\infty.\end{aligned}$$

Hence P is a contraction mapping. Then in virtue of the Banach fixed point theorem, P has a unique fixed point which is corresponding to the solution of equation (1.2). □

4. Set-valued problem

In this section we study the existence results for the differential inclusion (1.1) when the right hand side is convex as well as non-convex valued. The study will be taken in view of the single-valued problem (Theorems 3.2, 3.3) as well as fixed point theorems of set-valued function. The definite integral for the set-valued function F of order α defines as follows:

$$I^\alpha F(t,u(t),\rho(t)) = \{\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u(\tau),\rho(\tau))d\tau : f(t,u,\rho) \in S_F(u)\},$$

where

$$S_F(u) = \{f \in L^1(J, \mathbb{R}) : f(t) \in F(t, u(t), \rho(t)) \text{ a.e. } t \in J\}$$

denotes the set of selections of F. Let us introduce the following assumption

(H3) The set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl,c}(\mathbb{R})$ is L^1_X-Carathéodory with a growth function $h \in L^1(J \times \mathbb{R}, \mathbb{R}^+)$; i.e., $\|F(t, u, \rho)\| \leq h(t, \rho)$ a.e $t \in J$ for all $u \in \mathbb{R}$ such that $\|h\|_{L^1}$.

Theorem 4.1. *Let* (H3) *hold. If F is lower semi-continuous (l.s.c). Then the differential inclusion* (1.1) *has at least one solution $u(t)$ on J.*

Proof. This proof depends on the (single-valued problem). Inclusion (1.1) can reduce to the integral inclusion

$$u(t) \in \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau, u(\tau), \rho(\tau)) d\tau, \tag{4.1}$$

where $n - 1 < \alpha \leq n$, $t \in J = [0, T]$. For each $u(t)$ in $\mathbb{R}$, the set $S_F(u)$ is nonempty since by (H3), F has a non-empty measurable selection (see [8]). Thus there exists a function $f(t) \in F$ where f is a L^1_X-Carathéodory function with a bounded function $h \in L^1(J \times \mathbb{R}, \mathbb{R}^+)$ such that $\|f\| \leq \|h\|$ a.e $t \in J$ for all $u \in \mathbb{R}$. Hence the assumptions of Theorem 3.2 are satisfied then the inclusion (4.1) has a solution and consequently (1.1). □

We define the partial ordering $\leq$ in $W^{n,1}(J, \mathbb{R})$, the Sobolev class of functions $u : J \to \mathbb{R}$ for which $u^{(n-1)}$are absolutely continuous and $u^{(n)} \in L^1(J, \mathbb{R})$ as follows: Let $u, v \in W^{n,1}(J, \mathbb{R})$ then define

$$u \leq v \Leftrightarrow u(t) \leq v(t), \quad \text{for all } t \in J.$$

If $a, b \in W^{n,1}(J, \mathbb{R})$ and $a \leq b$ then we define an order interval $[a, b] \in W^{n,1}(J, \mathbb{R})$ by

$$[a, b] := \{u \in W^{n,1}(J, \mathbb{R}) : a \leq u \leq b\}.$$

Definition 4.2 ([1])**.** A function $\underline{u}$ is called a lower solution of (1.1) if there exists an $L^1(J, \mathbb{R})$ function $f_1(t)$ in $F(t, \underline{u}(t), \rho(t))$ a.e. $t \in J$. such that $\underline{u}^{(n)}(t) \leq f_1(t), a.e. t \in J$ and $\underline{u}^{(k)}(0) \leq \underline{u}_0^{(k)}$, $k = 0, 1, \dots, n-1$. Similarly a function $\overline{u}$ is called an upper solution of the problem (1.1) if there exists an $L^1(J, \mathbb{R})$ function $f_2(t)$ in $F(t, \overline{u}(t), \rho(t))$, a.e. $t \in J$ such that $\overline{u}^{(n)}(t) \geq f_2(t)$, a.e. $t \in J$ and $\overline{u}^{(k)}(0) \geq \overline{u}_0^{(k)}$, $k = 0, 1, \dots, n-1$.

(H4) The initial value problem (1.1) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ with $\underline{u} \leq \overline{u}$.

Theorem 4.3 (Convex case)**.** *Let* (H3)-(H4) *hold. Then the differential inclusion* (1.1) *has at least one solution $u(t)$ such that*

$$\underline{u}(t) \leq u(t) \leq \overline{u}(t), \quad \text{for all } t \in J.$$

Proof. Now we shall show that the assumptions of Lemma 2.9 are satisfied in a suitable Banach space. Consider the problem

$$\begin{gathered} D^\alpha(u - T_{n-1}[u])(t) \in F(t, Au(t), \rho(t)), \quad n-1 < \alpha \leq n,\ t \in J := [0, T], \\ u^{(k)}(0) = u_0^{(k)} \in \mathbb{R}, \quad n = 0, 1, \dots, n-1 \end{gathered} \tag{4.2}$$

where $A : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is the truncation operator defined by

$$(Au)(t) = \begin{cases} \underline{u}(t) & \text{if } u(t) < \underline{u}(t); \\ u(t) & \text{if } \underline{u}(t) \le u(t) \le \overline{u}(t); \\ \overline{u}(t) & \text{if } \overline{u}(t) < u(t). \end{cases}$$

The problem of the existence of a solution to (1.1) reduce to finding a solution to the integral inclusion

$$u(t) \in \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau, Au(\tau), \rho(t)) d\tau, \tag{4.3}$$

where $n-1 < \alpha \le n$, $t \in J = [0,T]$. We study (4.3) in the space of all continuous real functions on J endow with a supremun norm. Define a set-valued function operator $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ by

$$Nu = \{u \in C(J,\mathbb{R}) : u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, f \in \overline{S}_F(Au)\} \tag{4.4}$$

where

$$\overline{S}_F(Au) = \{f \in S_F(Au) : f(t) \ge \underline{u}(t) \text{ a.e. } t \in J_1 \text{ and } f(t) \le \overline{u}(t) \text{ a.e. } t \in J_2\}$$

and

$$\begin{aligned} J_1 &= \{t \in J : u(t) < \underline{u}(t) \le \overline{u}(t)\}, \\ J_2 &= \{t \in J : \underline{u}(t) \le \overline{u}(t) < u(t)\}, \\ J_3 &= \{t \in J : \underline{u}(t) \le u(t) \le \overline{u}(t)\}. \end{aligned}$$

We shall show that the set-valued operator N satisfies all the conditions of Lemma 2.9. Firstly, since F is measurable (H3), then it has a nonempty closed selection set $S_F(u)$ (see [8]) consequently $\overline{S}_F(u)$. The proof holds in several steps.

Step 1: $N(u)$ is convex subset of $C(J,\mathbb{R})$. Let $u_1, u_2 \in N(u)$. Then there exist $f_1, f_2 \in \overline{S}_F(u)$ satisfy

$$u_i(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_i(\tau) d\tau, \quad i = 1, 2.$$

Since $F(t,u)$ has convex values, then for $0 \le \delta \le 1$ we obtain

$$\begin{aligned} [\delta f_1 + (1-\delta) f_2](t) &= \delta[\sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_1(\tau) d\tau] \\ &\quad + (1-\delta)[\sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_2(\tau) d\tau] \\ &= \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [\delta f_1 + (1-\delta) f_2](\tau) d\tau. \end{aligned}$$

Therefore, $[\delta f_1 + (1-\delta) f_2] \in Nu$ and consequently N has a convex values in $C(J,\mathbb{R})$.

Step 2: $N(u)$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. Let B be bounded set in $C(J,\mathbb{R})$. Then there exists a real number $r > 0$ such that $\|u\| \le r$, for all

$u \in B$. Now for each $u \in N$ there exists $f \in \overline{S}_F(u)$ such that

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Then for each $t \in J$,

$$\begin{aligned}
|u(t)| &\leq \sum_{k=0}^{n-1} \frac{t^k}{k!} |u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau)| d\tau \\
&\leq \sum_{k=0}^{n-1} \frac{t^k}{k!} |u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \\
&\leq \sum_{k=0}^{n-1} \frac{T^k}{k!} |u^{(k)}(0)| + \frac{\|h\|_{L^1} T^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

Implies that $N(B)$ is bounded such that $\|u(t)\|_C \leq \sum_{k=0}^{n-1} \frac{T^k}{k!} |u^{(k)}(0)| + \frac{\|h\|_{L^1} T^\alpha}{\Gamma(\alpha+1)} := r$.

Step 3: $N(u)$ maps bounded sets into equicontinuous sets in $C(J, \mathbb{R})$. From above we have for any $t_1, t_2 \in J$ such that $|t_1 - t_2| \leq \delta,\ \delta > 0$

$$\begin{aligned}
|u(t_1) - u(t_2)| &= \Big| \sum_{k=0}^{n-1} \frac{t_1^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} f(\tau) d\tau \\
&\quad - \sum_{k=0}^{n-1} \frac{t_2^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-\tau)^{\alpha-1} f(\tau) d\tau \Big| \\
&\leq 2 \sum_{k=0}^{n-1} \frac{T^k}{k!} |u^{(k)}(0)| + [\frac{\|h\|_{L^1}}{\Gamma(\alpha+1)}](t_1^\alpha - t_2^\alpha + 2(t_1 - t_2)^\alpha)) \\
&\leq 2 \sum_{k=0}^{n-1} \frac{T^k}{k!} |u^{(k)}(0)| + [\frac{2\|h\|_{L^1}}{\Gamma(\alpha+1)}] |(t_1 - t_2)|^\alpha \\
&\leq 2[\sum_{k=0}^{n-1} \frac{T^k}{k!} |u^{(k)}(0)| + \frac{\delta^\alpha \|h\|_{L^1}}{\Gamma(\alpha+1)}]
\end{aligned}$$

which is independent of u hence $N(B)$ is equicontinuous set.

Step 4: $N(u)$ is u.s.c. As an application of the Arzela-Ascoli theorem yields that $N(B)$ is relatively compact set. Thus N is compact operator, hence in view of Remark 2.8, we have that N is u.s.c.

Step 5: Finally we show that the set

$$\varepsilon = \{u \in C(J, \mathbb{R}) : \lambda u \in Nu \text{ for some } \lambda > 1\}$$

is bounded. Let $u \in \varepsilon$. Then there exists a $f \in \overline{S}_F(u)$ such that

$$\begin{aligned}|u(t)| &\le \lambda^{-1}\sum_{k=0}^{n-1}\frac{t^k}{k!}|u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau)|d\tau\\ &\le \sum_{k=0}^{n-1}\frac{t^k}{k!}|u^{(k)}(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau\\ &\le \sum_{k=0}^{n-1}\frac{T^k}{k!}|u^{(k)}(0)| + \frac{\|h\|_{L^1}T^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Hence ε is bounded set. As a consequence of Lemma 2.9, we deduce that N has a fixed point which is a solution for A. Next we show that u is a solution for the problem (1.1). First we show that $u \in [\underline{u}, \overline{u}]$. Suppose not, then either $\underline{u} \not\le u$ or $u \not\le \overline{u}$ on $\overline{J} \subset J$. If $\underline{u} \not\le u$ then for $t_1 < t_2$ we have $\underline{u}(t) > u(t)$ for all t in $(t_1, t_2) \subset J$. Since $\underline{u}$ is the lower solution of the problem then for $f \in \overline{S}_F(u)$ yields

$$\begin{aligned}u(t) &= \sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau\\ &\ge \sum_{k=0}^{n-1}\frac{t^k}{k!}\underline{u}^{(k)}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\underline{u}(\tau)d\tau = \underline{u}(t)\end{aligned}$$

for all $t \in (t_1, t_2)$. This is a a contradiction. Similarly for $u \not\le \overline{u}$ yields a contradiction. Hence $\underline{u}(t) \le u(t) \le \overline{u}(t)$, for all $t \in J$. As a result, problem (1.1) has a solution $u \in [\underline{u}, \overline{u}]$. $\square$

Example 4.4. Let $J = [0, 1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider $\alpha = 1/2$ and

$$F(t,u,\rho) = \begin{cases} p(t,\rho), & \text{if } u < 1;\\ [p(t,\rho)\exp(-u^2(t)), p(t,\rho)], & \text{if } u \ge 1.\end{cases}$$

in problem (1.1), subject to the condition $u(0) = 1$. It is clear that $F(t, u, \rho)$ is L^1_X-Carathéodory with a growth function $p \in L^1(J \times \mathbb{R}, \mathbb{R})$ such that $\|F(t, u, \rho)\| \le p(t, \rho)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Thus we have

$$\overline{u}(t) = 1 + \frac{1}{\Gamma(1/2)}\int_0^t (t-\tau)^{-0.5}p(\tau)d\tau \quad \text{and} \quad \underline{u}(t) = 1.$$

In view of Theorem 4.3, the problem has a convex solution $u \in [\underline{u}, \overline{u}]$.

To study the existence for the problem (1.1) in non-convex case by using Theorem 3.3 (the existence of the single valued problem (1.2)) and Lemma 2.10 (the fixed point theorem for set valued functions), we introduce the following assumptions.

- (H5) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$, $(t, .) \mapsto F(t, u, \rho)$ is measurable for each $u \in \mathbb{R}$.
- (H6) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is $\ell(t)$-Lipschitz; i.e., $H(F(t, u, \rho), F(t, v, \rho)) \le \ell(t)\|u - v\|$.

Theorem 4.5 (Non-convex case)**.** *Let* (H5-H6) *hold. If* $\|\ell\|_{L^1}T^\alpha/\Gamma(\alpha + 1) < 1$, *then the differential inclusion* (1.1) *has at least one solution* $u(t)$ *on* J.

Proof. For each $u(t)$ in $\mathbb{R}$, F has a nonempty measurable selection (H5) then the set $S_F(u)$ is nonempty (see [8]). Then there exists a function $f(t) \in F$ such that f is $\ell(t) - Lipschitz$. Thus by the assumption (H6), we deduce that the conditions of Theorem 3.3 hold, which implies that the inclusion (1.1) has a solution. Hence the proof is complete in view of the single-valued problem. □

Theorem 4.6 (Non-convex case)**.** *Let* (H4-H6) *hold. If* $\|\ell\|_{L^1}T^\alpha/\Gamma(\alpha+1) < 1$, *then the differential inclusion* (1.1) *has at least one solution* $u(t)$ *on* J *such that* $\underline{u}(t) \le u(t) \le \overline{u}(t)$, *for all* $t \in J$.

Proof. Define the operator N as in (4.4) then the proof is done in two steps.
Step 1: $N(u) \in \mathcal{P}_{cl}(\mathcal{B})$ for each $u \in \mathcal{B} := C(J, \mathbb{R})$. Let $\{u_m\}_{m\ge 0} \in N(u)$ such that $u_m \to \widetilde{u}$ in $\mathcal{B}$. Then $\widetilde{u} \in \mathcal{B}$ and there exists $f_m \in S_F(u)$ such that for $t \in J$

$$u_m(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_m(\tau)d\tau.$$

Using the fact that F has closed values, we get that f_m converges to f in $L^1(J, \mathbb{R})$ and hence $f \in S_F(u)$. Then for each $t \in J$,

$$u_m(t) \to \widetilde{u}(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

So $\widetilde{u} \in N(u)$.
Step 2: There exists $\gamma < 1$ such that

$$H(N(u), N(v)) \le \gamma \|u - v\|_{\mathcal{B}}, \quad \text{for each } u, v \in \mathcal{B}.$$

Let $u, v \in \Omega$. Then by (H6) there exists $f \in F$ satisfies

$$|f(t, u, \rho) - f(t, v, \rho)| \le \ell(t)\|u - v\|_{\mathcal{B}}$$

then for $h_1(t) \in N(u)$ where

$$h_1(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau), \rho(\tau))d\tau.$$

And for $h_2(t) \in N(v)$ where

$$h_2(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, v(\tau), \rho(\tau))d\tau$$

we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau, u(\tau), \rho(\tau)) - f(\tau, v(\tau), \rho(\tau))| d\tau \\ &\le \frac{\|u - v\|_{\mathcal{B}}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \ell(\tau) d\tau \\ &\le \frac{\|\ell\|_{L^1} T^\alpha}{\Gamma(\alpha+1)} \|u - v\|_{\mathcal{B}}. \end{aligned}$$

Let

$$\gamma := [\frac{\|\ell\|_{L^1} T^\alpha}{\Gamma(\alpha+1)}].$$

It follows that

$$H(N(u),N(v)) \le \gamma\|u-v\|_{\mathcal{B}}, \quad \text{for each } u,v\in\mathcal{B},$$

where $\gamma<1$. Implies that N is a contraction set-valued mapping. Then in view of Lemma **??**, N has a fixed point which is corresponding to a solution of inclusion (1.1). The same conclusion holds in Theorem 4.3, we obtain that problem (1.1) has a solution $u\in[\underline{u},\overline{u}]$. □

Example 4.7. Let $J=[0,1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider

$$F(t,u,\rho)=\begin{cases}[0,ul(t)], & \text{if } 1\le u\le 2;\\ 1, & \text{if } u>2.\end{cases}$$

in problem (1.1), subject to the condition $u(0)=1$. It is clear that F is γ-Lipschitzean continuous and bounded function on $J\times\mathbb{R}$ with bound 1. Thus we have

$$\overline{u}(t)=1+\frac{2}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}l(\tau)d\tau \quad \text{and} \quad \underline{u}(t)=1$$

where $\gamma:=2\|l\|/\Gamma(\alpha+1)<1$. In view of Theorem 4.6, the problem has a non-convex solution $u\in[\underline{u},\overline{u}]$.

5. Extremal solutions

In this section, we establish the existence of extremal solutions to (1.1) on ordered Banach spaces. The cone $K=\{u\in C(J,\mathbb{R}): u(t)\ge 0, \forall\, t\in J\}$ defines an order relation, $\le$ in $C(J,\mathbb{R})$ by $u\le v \Leftrightarrow u(t)\le v(t)$, for all $t\in J$. It is clear that K is normal in $C(J,\mathbb{R})$ (see [15]). Let $S_1,S_2\in\mathcal{P}(X)$. Then by $S_1\le S_2$ we mean $s_1\le s_2$ for all $s_1\in S_1$ and $s_2\in S_2$. Thus if $S_1\le S_1$ then it follows that S_1 is a singleton set.

We need to the following definitions and result due to Dhage.

Definition 5.1. Let X be an ordered Banach space. A mapping $T:X\to\mathcal{P}(X)$ is called **isotone increasing** if $x,y\in X$ with $x<y$, then we have that $T(x)\le T(y)$.

Definition 5.2. A solution $u_M(t)$ of (1.1) is said to be **maximal solution** if for every solution $u(t)$ of (1.1), we have $u(t)\le u_M(t)$ for all $t\in J$. A solution $u_m(t)$ of (1.1) is said to be **minimal solution** if $u_m(t)\le u(t)$ for all $t\in J$ where $u(t)$ is any solution of (1.1).

Lemma 5.3 ([12])**.** *Let $[\underline{u},\overline{u}]$ be an order interval in a Banach space and let $T:[\underline{u},\overline{u}]\to\mathcal{P}([\underline{u},\overline{u}])$ be a completely continuous and isotone increasing set-valued. Further if the cone K in X is normal, then T has a least u_* and a greatest fixed point v^* in $[\underline{u},\overline{u}]$. Moreover, the sequences $\{u_n\}$ and $\{v_n\}$ defined by $u_{n+1}\in Tu_n$, $u_0=\underline{u}$ and $v_{n+1}\in Tv_n$, $v_0=\overline{u}$, converge to u_* and v^* respectively.*

Let us consider the following assumptions:

- (H7) The set-valued function $F:J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is Carathéodory.
- (H8) $F(t,u(t))$ is nondecreasing in u a.e. $t\in J$; i.e., if $u<v$ then $F(t,u)\le F(t,v)$ a.e. $t\in J$.

Theorem 5.4. *Assume* (H4), (H7), (H8) *hold. Then* (1.1) *has a minimal and a maximal solution on J.*

Proof. Define an operator $H : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ as follows

$$Hu = \big\{u \in C(J,\mathbb{R}) : u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, f \in S_F(u)\big\}. \tag{5.1}$$

We show that H satisfies the conditions of Lemma 5.3. Firstly, proceeding as in Theorem 4.3, is proved that H is completely continuous set-valued operator on $[\underline{u}, \overline{u}]$. Finally, we show that H is isotone increasing on $C(J,\mathbb{R})$. Let $u, v \in C(J,\mathbb{R})$ be such that $u < v$. Let $\underline{u} \in Hu$ be arbitrary. Then there is a function $f_1 \in S_F(u)$ such that

$$\underline{u}(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_1(\tau) d\tau.$$

Since F is nondecreasing in u we obtain that $S_F(u) \le S_F(v)$. As a result for any $f_2 \in S_F(v)$ we have

$$\underline{u}(t) \le \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_2(\tau) d\tau = \overline{u}$$

for all $t \in J$ and $\overline{u} \in Hv$. This shows that the set-valued operator H is isotone increasing on $C(J,\mathbb{R})$. And in particular in $[\underline{u}, \overline{u}]$. Since $\underline{u}$ and $\overline{u}$ are lower and upper solutions of the problem (1.1) on J we have

$$\underline{u}(t) \le \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

for all $f \in S_F(\underline{u})$ and so $\underline{u} \le H\underline{u}$. Similarly $\overline{u} \ge H\overline{u}$. Hence we have

$$\underline{u} \le H\underline{u} \le H\overline{u} \le \overline{u}.$$

Since H satisfies all the conditions of Lemma 5.3, yields that H has a least and greatest fixed point $[\underline{u}, \overline{u}]$. This implies that problem (1.1) has a minimal and maximal solution on J. □

Conclusion. We remark that when $\alpha = n$ in problem (1.1), we obtain the existence of solution of the n-th order differential inclusions studied in [12]. Again problem (1.1) has special cases that have been discussed in [1]. Further, this work holds for any kind of fractional operators: Caputo's, Erdelyi-Kober, Weyl-Riesz, etc.

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