Electronic Journal of Differential Equations, Vol. 2009(2009), No. 22, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL INCLUSIONS ON CLOSED MOVING CONSTRAINTS IN BANACH SPACES

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Abstract. In this paper, we prove the existence of solutions to a multivalued differential equation with moving constraints. We use a weak compactness type condition expressed in terms of a strong measure of noncompactness.

1. Introduction

In this paper we study the existence of solutions to the multivalued differential equation with moving constraints

$$\begin{gathered} \dot{x}(t) \in F\big(t, x(t)\big) \quad \text{a.e. on } I, \\ x(t) \in \Gamma(t) \quad \forall t \in [0,T], \\ x(0) = x_0 \in \Gamma(0). \end{gathered} \tag{1.1}$$

Where $F : [0,T] \times E \to P_{ck}(E)(P_{ck}$ is the family of nonempty convex compact subsets of $E)$ and $\Gamma : [0,T] \to P_f(E)$, ($P_f(E)$ is the family of closed subsets of E). Problem (1.1) has been studied by many authors; see for example [6, 2, 25, 8, 26] when F is lower semicontinuous, and [14, 6] when F is upper semicontinuous with Γ is independent of t. For Γ depending on t, we refer to [2, 20, 5]. In [16] we consider the differential inclusions $\dot{x}(t) \in A(t)x(t) + F(t, x(t))$, $x(0) = x_0$ where $\{A(t) : 0 \le t \le T\}$ is a family of densely defined closed linear operators generating a continuous evolution operator $S(t, s)$ and F is a multivalued function with closed convex values in Banach spaces. there, we show how that this results can be used in abstract control problems. Also in [17] we consider the Cauchy problem

$$\begin{gathered} \dot{x}(t) = f\big(t, x(t)\big), \quad t \in [0,T] \\ x(0) = x_0, \end{gathered}$$

where $f : [0,T] \times E \to E$ and E is a Banach space. In [11, 12], we study nonlinear differential equations. In [10] we study some differential inclusions with delay and their topological properties. Much work has been done in the study of topological properties of solution for differential inclusions; see [1, 3, 7, 21, 19, 15, 18].

In this paper we to prove the existence of solutions to (1.1) by using a measure of strong noncompactness, γ, (see the next section). Since the Kuratowksi measure

2000 *Mathematics Subject Classification.* 32F27, 32C35, 35N15.
Key words and phrases. Differential inclusions; moving constraints; existence solutions.
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Submitted December 1, 2008. Published January 27, 2009.

of noncompactness and the ball measure of noncomactness are measures of strong noncomactness and we can construct many measures such γ as in [9], in this paper Theorem 3.1 is a generalization of results for example Szufla [26] and Ibrahim-Gomaa [20]. In Theorem 3.2, the assumption on noncompactness is weaker than that of Benabdellah-Castaing and Ibrahim [5].

2. Preliminaries

Let E be a Banach space, E^* its topological dual space, E_w the Banach space E endowed with the weak topology, $B(0,1)$ unit ball of E, $I=[0,T]$, $(T>0)$, and λ be the Lebesgue measure on I . Consider B is the family of all bounded subsets of E and $C(I,E)$ is the space of all weakly continuous functions from I to E endowed with the topology of weak uniform convergence.

Definition 2.1. By a measure of strong noncompactness, γ, we will understand a function $\gamma : B \to \mathbb{R}^+$ such that, for all $U, V \in B$,

- (M1) $U \subset V \Longrightarrow \gamma(U) \leq \gamma(V)$,
- (M2) $\gamma(U \cup V) \leq \max(\gamma(U), \gamma(V))$,
- (M3) $\gamma(\overline{\text{conv}} U) = \gamma(U)$,
- (M4) $\gamma(U + V) \leq \gamma(U) + \gamma(V)$,
- (M5) $\gamma(cU) = |c|\gamma(U), \quad c \in \mathbb{R}$,
- (M6) $\gamma(U) = 0 \Longleftrightarrow U$ is relatively compact in E,
- (M7) $\gamma(U \cup \{x\}) = \gamma(U)$, $x \in E$.

Definition 2.2. For any nonempty bounded subset U of E the weak measure of noncompactness, β, and the Kuratowski's measure of noncompactness, α, is defined as:

$$\alpha(U) = \inf\{\varepsilon > 0 : U \text{ admits a finite number of sets with diameter less than } \varepsilon.\}$$

For the properties of β and α we refer the reader to [4, 13].

Definition 2.3. By a Kamke function we mean a function $w : I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that:

- (i) w satisfies the Caratheodry conditions,
- (ii) for all $t \in I$; $w(t, 0) = 0$,
- (iii) for any $c \in (0, b]$, $u \equiv 0$ is the only absolutely continuous function on $[0, c]$ which satisfies $\dot{u}(t) \leq w\big(t, u(t)\big)$ a.e. on $[0, c]$ and such that $u(0) = 0$.

Lemma 2.4 ([22, 4])**.** *If $\gamma : B \to \mathbb{R}^+$ satisfies conditions* (M2), (M4), (M6)*, then, for any nonempty $U \in B$,*

$$\gamma(U) \leq \gamma(B(0,1))\alpha(U)$$

Lemma 2.5 ([24, 23])**.** *If γ is a measure of weak (strong) noncompactness and $A \subset C(I,E)$ be a family of strongly equicontinuous functions, then $\gamma(A(I)) = \sup\{\gamma(A(t)) : t \in I\}$.*

3. Main Results

Theorem 3.1. *Let $\Gamma : I \to P_f(E)$ be a set-valued function such that its graph, G, is left closed and $F : I \times E \to P_{ck}(E)$ be a scalarly measurable set-valued function such that for any $t \in I$, $F(t,.)$ is upper semicontinuous on E. Suppose that F satisfies the following conditions:*

(A1) *For each* $\varepsilon > 0$, *there exists a closed subset* I_ε *of* I *with* $\lambda(I - I_\varepsilon) < \varepsilon$ *such that for any nonempty bounded subset* Z *of* E, *one has*

$$\gamma(F(J \times Z)) \leq \sup_{t \in J} w(t, \gamma(Z))$$

for any compact subset J *of* I_ε;

(A2) *there is* $\mu \in L^1(I, \mathbb{R}^+)$, *such that* $\|F(t,x)\| < \mu(t)(1 + \|x\|)$, *for all* $(t,x) \in G$;

(A3) *for each* $(t,x) \in ([0,T[\times E) \cap G$ *and* $\varepsilon > 0$ *there is* $(t_\varepsilon, x_\varepsilon) \in G$ *such that* $0 < t_\varepsilon - t < \varepsilon$ *and that*

$$x_\varepsilon - x \in \int_t^{t_\varepsilon} F(s,x)ds + (t_\varepsilon - t)\varepsilon B(0,1).$$

Then, for any $x_0 \in \Gamma(0)$, *there is a solution for* (1.1).

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $]0,1]$ with $\varepsilon_n = 0$. By [5, Proposition 6.1], there exist $m > 1$, a sequence $(\theta_n)_{n\in\mathbb{N}}$ of right continuous functions $\theta_n : I \to I$ such that $\theta_n(0) = 0$, $\theta_n(T) = T$ and $\theta_n(t) \in [t - \varepsilon_n, t]$, and a sequence $(x_n)_{n\in\mathbb{N}}$ from I to E with

(i) for all $t \in I$, $x_n(t) = x_0 + \int_0^t \dot{x}_n(s)\,ds$, where $\dot{x}_n \in L^1(I,E)$;
(ii) for all $t \in I, x_n(\theta_n(t)) \in \Gamma(\theta_n(t))$;
(iii) $\dot{x}_n(t) \in F(t, x_n(\theta_n(t)) + \varepsilon_n B(0,1)$ a.e on I;
(iv) $\|\dot{x}_n(t)\| \leq m\mu(t) + 1$, a.e on I.

From (iv) the sequence (x_n) is equicontinuous in $C(I,E)$. For each $t \in I$, set

$$A(t) = \{x_n(t) : n \in \mathbb{N}\} \quad \text{and} \quad \rho(t) = \gamma(A(t)).$$

We claim that $(x_n)_{n\in\mathbb{N}}$ is relatively compact in the space $C(I,E)$. So we will show that $\rho \equiv 0$. Since for each $(t,\tau) \in I \times I$, we have

$$\gamma\{(x_n)(\tau) : n \in \mathbb{N}\} \leq \gamma\{(x_n)(t) : n \in \mathbb{N}\} + \gamma\{(x_n)(\tau) - (x_n)(t) : n \in \mathbb{N}\}$$

and

$$\gamma\{(x_n)(t) : n \in \mathbb{N}\} \leq \gamma\{(x_n)(\tau) : n \in \mathbb{N}\} + \gamma\{(x_n)(t) - (x_n)(\tau) : n \in \mathbb{N}\},$$

then, from Lemma 2.4,

$$|\rho(\tau) - \rho(t)| \leq \gamma(B(0,1))\alpha(\{x_n(t) - x_n(\tau) : n \in \mathbb{N}\}),$$

which implies

$$|\rho(\tau) - \rho(t)| \leq 2\gamma(B(0,1))|\int_t^\tau (m\mu(s) + 1)\,ds|.$$

It follows that ρ is an absolutely continuous and hence differentiable a.e. on I. Let $\varepsilon > 0$. Since $\varepsilon_n \to 0$ as $n \to \infty$, then we can find $n_0 \in \mathbb{N}$ such that $T\varepsilon_n < \frac{\varepsilon}{\gamma(B(0,1))}$, for all $n \geq n_0$. Now let $(t,\tau) \in I \times I$ with $t \leq \tau$. In view of Condition (iii) and properties of γ ((M4), (M7)), we have

$$\begin{aligned}
\rho(\tau)-\rho(t) &\le \gamma(\int_t^\tau \dot{x}_n(s)\,ds : n\in N)\\
&\le \gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,\theta_n(s))ds)+\gamma(\{\varepsilon_n B(0,1)(\tau-t):n\in\mathbb{N}\})\\
&= \gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,\theta_n(s))ds)+\gamma(\{\varepsilon_n B(0,1)(\tau-t):n\ge n_0\})\\
&\le \gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,\theta_n(s))ds)+\frac{\varepsilon}{\gamma(B(0,1))}\gamma(B(0,1))\\
&\le \gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,\theta_n(s))ds)+\varepsilon.
\end{aligned}$$

Thus,

$$\rho(\tau)-\rho(t)\le \gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,\theta_n(s))ds). \tag{3.1}$$

Since ρ is continuous and w is Caratheodory we can find a closed subset I_ε of I, $\delta>0$, $\eta>0$ ($\eta<\delta$) and for $s_1,s_2\in I_\varepsilon$; $r_1,r_2\in[0,2T]$ such that if $|s_1-s_2|<\delta$, $|r_1-r_2|<\delta$, then $|w(s_1,r_1)-w(s_2,r_2)|<\varepsilon$ and if $|s_1-s_2|<\eta$, then $|\rho(s_1)-\rho(s_2)|<\frac{\delta}{2}$. Consider the following partition, of $[t,\tau]$, $t=t_0<t_1<\dots<t_m=\tau$ such that $t_i-t_{i-1}<\eta$ for $i=1,\dots,n$. From Condition (A1) we can find a closed subset J_ε of I such that $\lambda(I-J_\varepsilon)<\varepsilon$ and that for any compact subset K of J_ε and any bounded subset Z of E, $\gamma(f(K\times Z))\le \sup_{s\in K} w(s,\gamma(Z))$. Let $T_i=J_\varepsilon\cap[t_{i-1},t_i]\cap I_\varepsilon$, $P=\cup_{i=1}^m T_i=[t,\tau]\cap J_\varepsilon\cap I_\varepsilon$, $Q=[t,\tau]-P$ and $A_i=\{x_n(\theta_n(t)):n\in\mathbb{N},t\in T_i\}$, $i=1,\dots,m$. In view of the mean value theorem, properties of γ ((M3), (M5)) and Condition (A1), this implies

$$\begin{aligned}
\gamma(\cup_{n\in\mathbb{N}}\int_P F(s,x_n(\theta_n(s)))ds) &\le \gamma(\sum_{i=1}^m \cup_{n\in\mathbb{N}}\int_{T_i}F(s,x_n(\theta_n(s)))ds)\\
&\le \gamma(\sum_{i=1}^m \lambda(T_i)(\overline{\mathrm{conv}}F(T_i\times A_i)))\\
&\le \sum_{i=1}^m \lambda(T_i)\gamma(\overline{\mathrm{conv}}F(T_i\times A_i))\\
&\le \sum_{i=1}^m \lambda(T_i)\sup_{s_i\in T_i} w(s_i,\gamma(A_i)).
\end{aligned}$$

Now we have

$$\begin{aligned}
\gamma(A_i) &= \gamma(\{x_n(\theta_n(s)):n\in\mathbb{N},s\in T_i\})\\
&\le \gamma(\{x_n(s):n\in\mathbb{N},s\in T_i\})+\gamma(\{x_n(\theta_n(s))-x_n(s):n\in\mathbb{N},s\in T_i\})\\
&\le \gamma(\{x_n(s):n\in\mathbb{N},s\in T_i\})+\gamma(\{\int_s^{\theta_n(s)}\dot{x}_n(r)dr:n\in\mathbb{N},s\in T_i\}).
\end{aligned}$$

From Lemma 2.4 we know that

$$\begin{aligned}
&\gamma\Big(\Big\{\int_s^{\theta_n(s)}\dot{x}_n(r)dr : n\in\mathbb{N}, s\in T_i\Big\}\Big)\\
&\le \gamma(B(0,1))\alpha\Big(\Big\{\int_s^{\theta_n(s)}\dot{x}_n(r)dr : n\in\mathbb{N}, s\in T_i\Big\}\Big).
\end{aligned}$$

Also $\lim_{n\to\infty}|\theta_n(s)-s|=0$. So, $\gamma(A_i)=\gamma(\{x_n(s):n\in\mathbb{N},s\in T_i\})+\frac{\delta}{2}$. Applying Lemma 2.5, we get $\gamma(A_i)=\sup_{\xi_i\in T_i}\rho(\xi_i)+\frac{\delta}{2}$. Since w and ρ are continuous on the closed subsets of T_i, then

$$\begin{aligned}
\gamma\Big(\cup_{n\in\mathbb{N}}\int_P F(s,x_n(\theta_n(s)))ds\Big) &\le \sum_{i=1}^m \lambda(T_i)\sup_{s_i\in T_i} w\big(\big(s_i,\sup_{\xi_i\in T_i}\rho(\xi_i)+\frac{\delta}{2}\big)\\
&\le \sum_{i=1}^m \lambda(T_i) w\big(q_i,\rho(\xi_i)+\frac{\delta}{2}\big),
\end{aligned}$$

where q_i and ξ_i are elements of T_i. Moreover, for all $s\in T_i$, we have

$$|\rho(s)-\rho(\xi_i)+\frac{\delta}{2}|\le|\rho(s)-\rho(\xi_i)|+\frac{\delta}{2}<\frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

This implies $|w(s,\rho(s))-w(q_i,\rho(\xi_i)+\frac{\delta}{2})|<\varepsilon$ for all $s\in T_i$. Consequently, $\lambda(T_i)w(q_i,\rho(\xi_i)+\frac{\delta}{2})\le\int_{T_i}w(s,\rho(s))\,ds+\varepsilon\lambda(T_i)$. So,

$$\begin{aligned}
\gamma\Big(\cup_{n\in\mathbb{N}}\int_P F(s,x_n(\theta_n(s)))ds\Big) &\le \sum_{i=1}^m\Big(\int_{T_i}w(s,\rho(s))ds+\varepsilon\lambda(T_i)\Big)\\
&=\int_P w(s,\rho(s))ds+\varepsilon\lambda(P)\\
&\le\int_t^\tau w(s,\rho(s))ds+\varepsilon(\tau-t).
\end{aligned}$$

On the other hand,

$$\gamma(\cup_{n\in\mathbb{N}}\int_Q F(s,x_n(\theta_n(s)))ds)\le 2m\gamma(B(0,1))\int_Q\mu(s)(1+\|x_n(\theta_n(s))\|)ds.$$

As $\lambda(Q)<2\varepsilon$ and since ε is arbitrary, then

$$\gamma(\cup_{n\in\mathbb{N}}\int_t^\tau F(s,x_n(\theta_n(s)))ds)\le\int_t^\tau w(s,\rho(s))ds, \tag{3.2}$$

Thus, from two relations (3.1), (3.2),

$$\dot\rho(t)\le w(s,\rho(s))\quad \text{a.e. on}I.$$

$\rho(0)=0$ and w is a Kamke function, then ρ is identically equal to zero. It follows that (x_n) is relatively compact in $C(I,E)$. Since, for all $t\in I$,

$$\gamma(\{x_n(\theta_n(t)):n\in\mathbb{N}\})\le\gamma(\{x_n(t):n\in\mathbb{N}\})+\gamma(\int_{\theta_n(t)}^t(m\mu(s)+1)ds)B(0,1)$$

and since $\lim_{n\to\infty}|\theta_n(t)-t|=0$, the set $\tilde{A}(t):=\{x_n(\theta_n(t)):n\in\mathbb{N}\}$ is relatively compact in E. By our assumption $F(t,.)$ is upper semicontinuous, it follows that $F(t,\overline{A}(t))$ is compact for all $t\in I$, Furthermore, we have

$$\dot{x}_n(t)\in F(t,\overline{A(t)})+\varepsilon_nB(0,1),\quad\forall n\in\mathbb{N},\forall t\in I.$$

Since $\dot{x}_n$ is uniformly integrable, by [5, Theorem 5.4], the sequence $\dot{x}_n$ is relatively $\sigma(L^1(I,E), L^\infty(I,E))$ compact. Therefore there are $x_{nk} \in C(I,E), g \in L^1(I,E)$ and a subsequence (x_{nk}) of (x_n) such that (x_{nk}) converges to x in $C(I,E)$ and $(\dot{x_{nk}})$ converges to g in $L^1(I,E)$ for $\sigma(L^1(I,E), L^\infty(I,E))$, with

$$x(t) = x_0 + \int_0^t g(s)ds, \forall t \in I.$$

Thus $g = \dot{x}$. Clearly for all $t \in I$, $\lim_{n\to\infty} x_{nk}(\theta_n(t)) = x(t)$, and $x(t) \in \Gamma(t)$, for all $t \in I$. Finally, in virtue [5, Theorem 5.6, Remark 6.3] and the property (iii) we obtain

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. on} I.$$

□

Theorem 3.2. *Let $\Gamma : I \to P_f(E)$ be a set-valued function with closed graph, G, and $F : G \to P_{ck}(E)$ be a set-valued function such that for any $t \in I$, $F(t,.)$ is upper semicontinuous on E. Assume that F satisfies the following conditions:*

(A1') *For each $\varepsilon > 0$, there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that for any nonempty bounded subset Z of E, one has*

$$\gamma(F(G \cap (I \times Z))) \le \sup_{t\in I} w(t, \gamma(Z)),$$

for any compact subset J of I_ε;

(A2') *there is a positive number c such that*

$$\|F(t,x)\| < c(1 + \|x\|), \forall (t,x) \in G;$$

(A3') *for each $(t,x) \in ([0,T[\times E) \cap G$ and for any $\varepsilon > 0$ there is $(t_\varepsilon, x_\varepsilon) \in G$ such that $0 < t_\varepsilon - t < \varepsilon$ and*

$$\frac{x_\varepsilon - x}{t_\varepsilon - t} \in F(t,x) + \varepsilon\overline{B(0,1)}.$$

Then, for each $x_0 \in \Gamma(0)$, there is a solution of (1.1).

Proof. Let $A_\varepsilon([0,\tau])$ $(\varepsilon > 0, \tau \in I)$ be the set of all points (x,θ) where $\theta : [0,\tau] \to [0,\tau]$ is an increasing right continuous function with $\theta(0) = 0, \theta(\tau) = \tau$ and for all $t \in]0,\tau[$, $\theta(t) \in [t - \varepsilon, t]$ and $x : [0,\tau] \to E$ is such that:

(i) for all $t \in [0,\tau]$, $x(t) = x_0 + \int_0^t \dot{x}(s)\, ds$, where $\dot{x} \in L^1(I,E)$;
(ii) for all $t \in [0,\tau]$, $x\big(\theta(t)\big) \in \Gamma(\theta(t))$;
(iii) for all $t \in [0,\tau]$, $\dot{x}(t) \in F\big(t, x(\theta(t)\big) + \varepsilon\overline{B(0,1)}$, a.e.

Let $\varepsilon \in]0,1]$ and $(\theta,x) \in A_\varepsilon([0,\tau])$. Then by (A2') and the fact that, for all $t \in [0,\tau]$, $\theta(t) \in [t-\varepsilon, t]$, we have

$$\begin{aligned}\|x(\theta(t))\| &\le \|x_0\| + \int_0^{\theta(t)} \|\dot{x}(s)\|\, ds \le \|x_0\| + \int_0^t \|\dot{x}(s)\|\, ds \\ &\le \|x_0\| + \varepsilon T + \int_0^t c(1 + \|x(\theta((s))\|)\, ds.\end{aligned}$$

By Gronwall's lemma, we obtain $\|x(\theta(t))\| \le (\|x_0\| + T)e^{cT}$ which gives us

$$\|x(\theta(t))\| + 1 \le (1 + \|x_0\| + T)e^{cT}.$$

Consequently we get for all $t \in [0,\tau]$,

$$F(t, x(\theta(t))) \subseteq pc\overline{B(0,1)} \tag{3.3}$$

where $p = (1 + \|x_0\| + T)e^{cT}$. Let $A_\varepsilon = \bigcup_{\tau\in I} A_\varepsilon([0,\tau])$. Obviously $A_\varepsilon \neq \emptyset$. Partially order A_ε such that for any $(\theta_i, x_i) \in A_\varepsilon([0,\tau_i]) \subseteq A_\varepsilon$ $(i = 1,2)$ $(\theta_1, x_1) \leq (\theta_2, x_2) \Longleftrightarrow \tau_1 \leq \tau_2, \theta_1 = \theta_2|_{[0,\tau_1]}$ and $x_1 = x_2|_{[0,\tau_2]}$. Let C be a subset of A_ε such that each two elements of it are comparable that is there exists a subset $\mathbb{N}' \subseteq \mathbb{N}$ such that $C = \{(\theta_j, x_j) : j \in \mathbb{N}'\} \subseteq A_\varepsilon$ and each $(\theta_n, x_n), (\theta_m, x_n) \in C$ we have $(\theta_n, x_n) \leq (\theta_m, x_m)$ or $(\theta_m, x_m) \leq (\theta_n, x_n)$. Now we prove that C has an upper bound. Let $\tau = \sup_{j\in\mathbb{N}'} \tau_j$. Also let $\theta : [0,\tau] \to [0,\tau]$ is such that, for each $j \in \mathbb{N}', \theta|_{[0,\tau_j]} = \theta_j$ and $x : [0,\tau[\to E$ with $x|_{[0,\tau_j]} = x_j$, for each $j \in \mathbb{N}'$. Let $\{\tau_{k_n}\}$ be increasing sequence in $\mathbb{N}'$ such that $\tau = \sup_{n\in\mathbb{N}} \tau_{k_n}$ and for any $n, m \in \mathbb{N}$, $m < n$ we have $\dot{x}_{k_n} = \dot{x}_{k_m}$ a.e. on $[0,\tau_{k_n}]$. Now we can define $\dot{x} : [0,\tau[\to E$ by, for any $n \in \mathbb{N}$, $\dot{x}(t) = \dot{x}_{k_n}(t)$ a.e. on $[0,\tau_{k_n}]$. From (3.3) $\dot{x}$ is measurable and $\|\dot{x}(t)\| \leq pc + \varepsilon \leq pc + 1$. We claim that $x, \dot{x}$ can be extend to $[0,\tau]$. Now for all $t \in [0,\tau[$, $x(t) = x_0 + \int_{]0,t]} \dot{x}(s)\,ds$, for all $t \in [0,\tau[$, $\dot{x}(\theta(t)) \in \Gamma(\theta(t))$ and $\dot{x}(t) \in F(t, \dot{x}(\theta(t))) + \varepsilon\overline{B(0,1)}$ a.e. on $[0,\tau[$. If $x'(t) = x_0 + \int_{]0,t[} \dot{x}(s)\,ds$ for all $t \in [0,\tau]$ then, for $(t,t') \in [0,\tau[\times[0,\tau[$, we have $\|x'(t) - x'(t')\| \leq \int_{[t,t'[}(ps+1)\,ds$. Then $x^* := \lim_{t\to\tau^{-0}}(x_0 + \int_{]0,t[} \dot{x}(s)\,ds) = \lim_{n\to\infty}(x_0 + \int_{]0,\tau_{k_n}[} \dot{x}(s)\,ds)$ exists. Since $x'(\tau_{k_n}) \in \Gamma(\tau_{k_n})$ and G is left closed, then $(\tau, x^*) \in G$ and hence the result. Let $x^* = x(\tau)$ and $\dot{x}(\tau) = 0$. Then $x(\tau) = x_0 + \int_{]0,\tau]} \dot{x}(s)\,ds$, $x^* = x(\tau) \in \Gamma(\tau)$ and $\dot{x}(t) \in F(t, x(\theta(t))) + \varepsilon\overline{B(0,1)}$ a.e. on $[0,\tau]$. Consequently we can extend (θ, x) to $[0,\tau]$ such that (θ, x) belongs to $A_\varepsilon([0,\tau])$ and it is an upper bound for C. By Zorn's lemma $(A_\varepsilon, \leq)$ has a maximal element $(\theta_\varepsilon, x_\varepsilon) \in A_\varepsilon([0,\tau_\varepsilon])$. We shall prove that $\tau_\varepsilon = T$. Let $\tau_\varepsilon < T$. If $\delta_\varepsilon > 0$ such that $\delta_\varepsilon < \inf(\varepsilon, T - \tau_\varepsilon)$. Then by (A3') there exists $(\hat{t}, \hat{x}) \in G$ such that $0 < \hat{t} - \tau_\varepsilon \leq \delta_\varepsilon$ and

$$\frac{\hat{x} - x_\varepsilon}{\hat{t} - \tau_\varepsilon} \in F(\tau_\varepsilon, x_\varepsilon(\tau_\varepsilon)) + \varepsilon\overline{B(0,1)}.$$

Let $\hat{y} \in F(\tau_\varepsilon, x_\varepsilon(\tau_\varepsilon)) + \varepsilon\overline{B(0,1)}$ such that $\hat{x} - x_\varepsilon(\tau_\varepsilon) = (\hat{t} - \tau_\varepsilon)\hat{y}$. If $\hat{\theta} : [0,\hat{t}] \to [0,\hat{t}]$ and $\tilde{x} : [0,\hat{t}] \to E$ are defined as:

$$\hat{\theta}(t) = \begin{cases} \theta_\varepsilon & \text{if } t \in [0,\tau_\varepsilon] \\ \tau_\varepsilon & \text{if } t \in]\tau_\varepsilon, \hat{t}] \\ \hat{t} & \text{if } t = \hat{t}, \end{cases} \qquad \tilde{x}(t) = \begin{cases} x_\varepsilon & \text{if } t \in [0,\tau_\varepsilon] \\ \hat{x} & \text{if } t \in [\tau_\varepsilon, \hat{t}] \end{cases}$$

Then it is easy to check that [5, p. 10.25] $(\hat{\theta}, \tilde{x}) \in A_\varepsilon([0,\hat{t}])$ and $(\theta_\varepsilon, x_\varepsilon) < (\hat{\theta}, \tilde{x})$. This contradicts the fact that $(\theta_\varepsilon, x_\varepsilon)$ is maximal. Now there exist $p > 1$, (from (3.3)) a sequence $(\theta_n)_{n\in\mathbb{N}}$ of right continuous functions $(\theta)_n : I \to I$ such that $\theta_n(0) = 0$, $\theta_n(T) = T$ and $\theta_n(t) \in [t - \varepsilon_n, t]$, if we have decreasing sequence (ε_n) such that $0 < \varepsilon_n \leq 1$ $\varepsilon_n \to 0$ as $n \to \infty$ and $T\varepsilon_n < \frac{\varepsilon}{\gamma(B(0,1))}$, for all $n \geq n_0$ we can define a sequence (x_n) of approximated solutions as the follows:

$\forall t \in I, \quad x_n(t) = x_0 + \int_0^t \dot{x}_n(s)\,ds$, where $\dot{x}_n \in L^1(I,E)$. $(\theta_n(t), x_n(\theta_n(t))) \in G$. $\dot{x}_n(t) \in F\big(t, x_n(\theta_n(t)\big) + \varepsilon_n B(0,1)$, a.e on I. $\|\dot{x}_n(t)\| \leq pc + 1$, a.e on I.

By the same arguments used in the proof of Theorem 3.1 we can prove that the sequence (x_n) converges to an absolutely continuous function x which is a solution for problem (1.1). □

4. Conclusion

Let us remark that, if we replace γ in (A1') by α, the condition

(A4) For each $\varepsilon > 0$, there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that for almost all $t \in I_\varepsilon$ and for any nonempty bounded subset Z of E, one has

$$\inf_{\delta>0} \alpha(F(G \cap (([t-\delta, t] \cap I) \times Z))) \le w(t, \alpha(Z))$$

implies Condition (A1') and the converse is not true. Indeed Let $\varepsilon > 0$. Since w is Caratheodory function, we can find a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ such that w is continuous on I_ε and Condition (A4) holds on I_ε. Let Z be a nonempty bounded subset of E. It follows from (A4) that, for any $\tau > 0$ and any $t \in I_\varepsilon$, there exists a $\delta_{\tau,t}$ such that $\alpha(F(G \cap (([t - \delta_{\tau,t}, t] \cap I) \times Z))) \le w(t, \alpha(Z)) + \tau$. Let τ be arbitrary but fixed, J be a compact subset of I_ε. The collection $\{(t - \frac{\delta_t}{2}, t + \frac{\delta_t}{2}) : t \in J\}$ is an open cover for J. By compactness of J, there exist $t'_1, t'_2 \dots, t'_n$ such that $J \subseteq \cup_{i=1}^n (t'_i - \frac{\delta_{t'_i}}{2}, t'_i + \frac{\delta_{t'_i}}{2}) \subseteq \cup_{i=1}^n [t'_i - \frac{\delta_{t'_i}}{2}, t'_i + \frac{\delta_{t'_i}}{2}]$. Now if $J_i = J \cap [t'_i - \frac{\delta_{t'_i}}{2}, t'_i + \frac{\delta_{t'_i}}{2}]$ and $t_i = \max J_i, 1 \le i \le n$, then there exist $t_1, t_2 \dots t_n \in J$ such that $J_i \subseteq [t_i - \delta_{t_i}, t_i]$ and $J \subseteq \cup_{i=1}^n [t_i - \delta_{t_i}, t_i]$. This implies that,

$$\begin{aligned}
\alpha(F(G \cap (J \times Z))) &\le \alpha(\cup_{i=1}^n F(G \cap (([t_i - \delta_{t_i}, t_i] \cap I) \times Z))) \\
&\le \max_{1 \le i \le n} \alpha(F(G \cap (([t_i - \delta_{t_i}, t_i] \cap I) \times Z))) \\
&\le \max_{1 \le i \le n} w(t_i, \alpha(Z)) + \tau \le \max_{t \in J} w(t, \alpha(Z)) + \tau
\end{aligned}$$

Since τ is arbitrary, Condition (A1') holds. To show that the converse is not true we give an example. Let $f : [0,1] \times B(0,1) \to E$ be the single valued function defined by $f(t,x) = k(t)x$, where $k : [0,1] \to \mathbb{R}$,

$$k(t) = \begin{cases} 1 & \text{if } t \text{ is irrational} \\ 1/t^2 & \text{if } t \text{ is rational} \end{cases}$$

Let also $w(t,s) = k(t)s$, for all $(t,s) \in I \times \mathbb{R}^+$. Clearly, w is a Kamke function. Let $\varepsilon > 0$ and choose a closed subset I_ε of I such that $\lambda(I - I_\varepsilon) < \varepsilon$ and k is continuous on I_ε. Then for any compact subset J of I_ε and any bounded subset Z of E,

$$\begin{aligned}
\alpha(f(G \cap (J \times Z))) \le \alpha(f(J \times Z)) &= \alpha\big(\cup_{t \in J, x \in Z} f\{(t,x)\}\big) \\
&= \alpha\big(\cup_{t \in J} k(t)Z\big) = \sup_{t \in J} k(t)\alpha(Z) \\
&= \sup_{t \in J} w\big(t, \alpha(Z)\big).
\end{aligned}$$

Then Condition (A1') holds as the measure γ replaced by the measure α. But for each $t \in (0,1)$ and each nonempty subset Z of E we have $\alpha\big(f([t-\delta, t] \times Z)\big) = \alpha\big(\cup_{s \in [t-\delta,t]} k(s)Z\big) = \alpha(Z) \cdot \big(\sup_{s \in [t-\delta,t]} k(s)\big) = \frac{\alpha(Z)}{(t-\delta)^2}$. Thus, $\inf_{\delta>0} \alpha(F((([t - \delta, t] \cap I) \times Z)) = \frac{\alpha(Z)}{t^2}$. So if t is irrational then $\inf_{\delta>0} \alpha(F((([t-\delta, t] \cap I) \times Z)) = \frac{\alpha(Z)}{t^2} > \alpha(Z) = k(t)\alpha(Z) = w(t, \alpha(Z))$. Then (A4) does not hold and consequently Theorem 3.2 is a generalization of the following theorem.

Theorem 4.1 (Benabdellah-Castaing and Ibrahim [5])**.** *Let F and Γ be as in Theorem 3.2 except F satisfies Condition* (A4) *instead of* (A1')*. Then, for any $x_0 \in \Gamma(0)$, there is a solution for* (1.1).

References

[1] Anguraj, A.,; Balachandran K.; *On the solution sets of differential inclusion in Banach spaces.* Tam., 23(1):59-65, 1992.

[2] Aubin, J. P.; Cellina, A.; *Differential inclusions set-valued maps and viability theory.* Springer-Verlage, Berlin, 1984.

[3] Avgerinos, E. P.; Papageorgiou, N. S.; *On the solution set of maximal monotone differential inclusions in* $\mathbb{R}^m$. Math. Japonica, 38(1):91-110,1993.

[4] Banasm, J; Goebel, K.; *Measure of noncompactness in Banach spaces, Lecture notes in pure and applied mathematics.* Marcel Dekker Inc v 60, New York; 1980.

[5] Benabdellah, H.; Castaing, C.; Ibrahim, A.-G.; *Bv solution of multivalued differential equations on closed moving sets in Banach spaces.* Geometry in nonlinear control and differential inclusions, Banach Center Publications, Polish Acadamy of Science, Warswzaws, 32:53-81, 1995.

[6] Castaing, C.; Valadier, M.; *Convex analysis and measurable multifunctions*, Lecture notes in mathematics v 580. Springer Verlage; 1977.

[7] Cellina, A.; Ornelas, A.; *Convexity and the closure of the solution set to differential inclusions.* Bollettino U.M.I., (7)4-B:255-263, 1993.

[8] Cichon, M.; *Multivalued differential equations with noncontinuous righthand side.* Fasciculi Mathematici, 23:43-50, 1991.

[9] Cichon, M.; *On measure of weak noncompactness.* Publications, Math. Debrecen, 45:93-102, 1994.

[10] Gomaa, A. M.; *Existence and topological properties of solution sets for differential inclusions with delay*, Annales Societatis Mathematicae Polonae, Comm. Math., 48(1): 45-57, 2008.

[11] Gomaa, A. M.; *Existence of weak and strong solutions of nonlinear differential equations with delay in Banach space*, to appear in Journal of Physics: Conference Series Second Intenational Symposium on Nonlinear Dynamics, 27-30 Oct, 2007, Shanghai, China.

[12] Gomaa, A. M.; *On theorems for weak solutions of nonlinear equations with and without delay in Banach spaces*, Annales Societatis Mathematicae Polonae, Comm. Math., XLVII(2): 179-191, 2007.

[13] Deimling, K.; *Ordinary Differential Equations in Banach Spaces.* LNM 596, Berlin-Heidelberg, New- York; 1992.

[14] Deimling, K.; *Multivalued differential equations.* Walter de Gruyter, Berlin, New York; 1992.

[15] Gomaa, A. M.; *On the solution sets of three-point boundary value problems for nonconvex differential inclusions.* J. Egypt. Math. Soc. 12(2):97-107, 2004.

[16] Gomaa, A. M.; *On the topological properties of the solution set of differential inclusions and abstract control problems.* J. Frac. Calcu., 24:87-99, 2003.

[17] Gomaa, A. M.; *Weak and strong solutions for differential equations in Banach spaces.* Chaos, Solitons & Fractals, 18(4):687-692, 2003.

[18] Ibrahim, A. G., Gomaa, A. M.; *Existence theorems for a functional multivalued three-point boundary value problem of second order.* J. Egypt. Math. Soc., 8(2):155-168, 2000.

[19] Ibrahim, A. G., Gomaa, A. M.; *Extremal solutions of classes of multivalued differential equations.* Appl. Math. Comput.,136 no. 2-3:297-314, 2003.

[20] Ibrahim, A.G.; Gomaa, A. M.; *Strong and weak solutions for differential inclusions with moving constraints in Banach spaces.* PU. M. A., 8(1):53-65, 1997.

[21] Ibrahim, A.G.; Gomaa, A. M.; *Topological properties of the solution sets of some differential inclusions.* PU. M. A., 10(2):197-223, 1999.

[22] Janicka, R.; *Zastosowanie miar niezwartosciw teorii rownan rozniczkowych i calkowych.* Ph. D thesis, Lublin, 1978.

[23] Kubiaczyk, I.; *On the existence of solutions of differential equations in Banach spaces.* Bull. Polish Acad. Sci. Math., 33: 607-614, 1985.

[24] Mitchell, A. R.; Smith, C. H.; *An existence theorem for weak solutions of differential equations in Banach spaces.* Nonlinear Equations in Abstract Spaces, 387-404, 1978.

[25] Papageorgiou, N.S.; *Weak solutions of differential equation in Banach spaces.* Bull Austral Math Soc. 33:407-418, 1986.

[26] Szufla, S.; *On existence of solutions of differential equations in Banach spaces.* Bull Acad Polon Sci Math 1982;30:507-514.

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