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OSCILLATION CRITERIA FOR FIRST-ORDER SYSTEMS OF LINEAR DIFFERENCE EQUATIONS

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Abstract. In this article, we obtain conditions for the oscillation of vector solutions to the first-order systems of linear difference equations

$$\begin{aligned} x(n+1) &= a(n)x + b(n)y \\ y(n+1) &= c(n)x + d(n)y \end{aligned}$$

and

$$\begin{aligned} x(n+1) &= a(n)x + b(n)y + f_1(n) \\ y(n+1) &= c(n)x + d(n)y + f_2(n) \end{aligned}$$

where $a(n), b(n), c(n), d(n)$ and $f_i(n), i = 1, 2$ are real valued functions defined for $n \geq 0$.

1. Introduction

Consider the system of k-equations of the form

$$X(n+1) = AX(n), \tag{1.1}$$

where $A = (a_{ij})_{k\times k}$ is a constant matrix. The characteristic equation of (1.1) is given by

$$\det(\lambda I - A) = 0;$$

that is,

$$\lambda^k + (-1)^k b_1 \lambda^{k-1} + \cdots + (-1)^k b_k = 0, \tag{1.2}$$

where $b_k = \det A$. If k is odd, then from the theory of algebraic equations (see e.g. [2]), it follows that (1.2) admits at least one real root λ_1 such that the sign of λ_1 is opposite to that of the last term, namely $(-1)^k b_k$. Hence we have the following result.

Theorem 1.1. *Let k be odd. If $\det A < 0$, then (1.1) admits at least one oscillatory solution; if $\det A > 0$, then (1.1) admits at least one nonoscillatory solution.*

Proof. When $\det A < 0$, we find a real root λ_1 of (1.2) such that $\lambda_1 < 0$ and $X(n) = (\lambda_1)^n C$, where $C = (C_1 C_2 \dots C_k)^T$ is a column vector of constants. Thus $X(n)$ is oscillatory. Similarly for $\det A > 0$. □

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Remark. If $\det A = 0$, then (1.1) admits a nonoscillatory solution. Indeed, $\det A = 0$, implies that $\lambda = 0$ is a solution of (1.2) and hence $X(n) = C$ is a solution of (1.1), where C is a non-zero constant vector. We note that $AC = 0$ always admits a nontrivial solution.

Theorem 1.2. *Let k be even. If $\det A < 0$, then (1.1) admits an oscillatory solution and a nonoscillatory solution.*

The proof is simple and can be obtained from the following Theorem in [2].

Theorem 1.3.
(I) *Every equation of an even degree, whose constant term is negative has at least two real roots one positive and the other negative.*
(II) *If the equation contains only even powers of x and the coefficients are all of the same sign, then the equation has no real root; that is, all roots are complex.*

Remarks. If the last term of an even degree equation is positive, no definite conclusion can be drawn regarding the roots of the equation. If $\det A > 0$, then no definite conclusion can be drawn regarding the oscillation of solutions of (1.1) when k is even.

Theorem 1.4. *Let k be even and A be such that $b_1 = b_3 = \dots = b_{k-1} = 0$, $b_2 > 0$, $b_4 > 0 \dots b_k > 0$. Then every component of the vector solution of (1.1) is oscillatory.*

The proof of the above theorem follows from the above Theorem 1.3(II).

The literature on study of system of difference equations does not conisder the case when k is even. Therefore the present work is devoted to study the system of equations

$$\begin{aligned} x(n+1) &= a(n)x + b(n)y \\ y(n+1) &= c(n)x + d(n)y \end{aligned} \tag{1.3}$$

and the corresponding nonhomogeneous system of equations

$$\begin{aligned} x(n+1) &= a(n)x + b(n)y + f_1(n) \\ y(n+1) &= c(n)x + d(n)y + f_2(n), \end{aligned} \tag{1.4}$$

where $a(n), b(n), c(n), d(n), f_1(n), f_2(n)$ are real-valued functions defined for $n \geq n_0 \geq 0$. One may think of systems (1.3) and (1.4) as being a discrete analoge of the differential systems

$$\begin{aligned} x'(t) &= a(t)x + b(t)y \\ y'(t) &= c(t)x + d(t)y \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} x'(t) &= a(t)x + b(t)y + f_1(t) \\ y'(t) &= c(t)x + d(t)y + f_2(t) \end{aligned} \tag{1.6}$$

respectively, where a, b, c, d, f_1, f_2 are in $C(\mathbb{R}, \mathbb{R})$. If $a(n) \equiv a$, $b(n) \equiv b$, $c(n) \equiv c$ and $d(n) \equiv d$, then the characteristic equation of (1.3) is

$$\lambda^2 - (a+c)\lambda + (ad - bc) = 0. \tag{1.7}$$

We note that this equation is the same for both the systems (1.3) and (1.5). Hence the oscillatory behaviour of solutions of these systems are comparable. Clearly, the components of the vector solution of (1.5) are oscillatory only if (1.7) has complex roots. Otherwise, it is nonoscillatory. On the other hand, the behaviour of the components of the vector solution of (1.3) is given below.

Proposition 1.5. *Let λ_1 and λ_2 be the roots of* (1.7). *If any one of the following three conditions*

(1) $(a-d)^2+4bc<0$,
(2) $(a-d)^2+4bc>0$ *but* $(a+d)\pm[(a-d)^2+4bc]^{\frac{1}{2}}<0$,
(3) $(a-d)^2+4bc=0$ *and* $(a+d)<0$

hold, then every component of the vector solution of (1.3) *is oscillatory. Otherwise, there exists a nonoscillatory solution to* (1.3).

The proof is simple and hence it is omitted.

The object of this work is to establish the sufficient conditions for the oscillation of all solutions of the systems (1.3) and (1.4). Proposition 1.4 which demonstrate the difference in the behaviour of the solutions of the systems (1.3)-(1.4) and (1.5)-(1.6) motivate us to study further for the oscillatory behaviour of solutions of (1.3)-(1.4). Furthermore, an attempt is made here to apply some of the results of [6] for the oscillatory behaviour of solutions of the systems (1.3) and (1.4).

A close observation reveals that, all most all works in difference equations / system of equations are the discrete analogue of the differential equations / system of equations see for e.g. [1, 3, 4] and the references cited therein. Agarwal and Grace [1] have discussed the oscillatory behavour of solutions of the system of equations of the form

$$(-1)^{m+1}\Delta^m y_i(n)+\sum_{j=1}^{N} q_{ij}y_j(n-\tau_{jj})=0,\quad m\ge 1,\ i=1,2,\dots,N$$

which is the discrete analogue of the functional differential equations

$$\frac{d^m}{dt^m}y_i(t)+\sum_{j=1}^{N} q_{ij}y_j(t-\tau_{jj})=0,\quad m\ge 1, i=1,2,\dots,N,$$

where q_{ij} and τ_{jj} are real numbers and $\tau_{jj}>0$. It seems that the results in [1] are the discrete analog results of the continuous case. We note that, in this work an investigation is made to study the system of equations (1.3)/(1.4) without following any results of the continuous case.

By a solution of (1.3)/(1.4) we mean a real valued vector function $X(n)$ for $n=0,1,2\dots$ which satisfies (1.3)/(1.4). We say that the solution $X(n)=[x(n),y(n)]^T$ oscillates componentwise or simply oscillates if each component oscillates. Otherwise, the solution $X(n)$ is called non-oscillatory. Therefore a solution of (1.3)/(1.4) is non-oscillatory if it has a component which is eventually positive or eventually negative.

We need the following two results from [6] for our use in the sequel.

Theorem 1.6. *If $a_n>0$, $b_n>0$ and*

$$a_n\le\frac{b_{n+1}}{a_{n+1}}+\frac{b_n}{a_{n-1}}$$

for large n, then $y_{n+2}-a_ny_{n+1}+b_ny_n=0$ is oscillatory.

Theorem 1.7. *Let $0\le a_n\le 1$ and $c_n\ge 0$. Let $f_n=g_{n+2}-g_{n+1}$, where for each $n\ge 1$, there exists $m>n$ such that $g_ng_m<0$. If*

$$\sum_{n=1}^{\infty}[(1-a_n)g^+_{n+1}+C_ng^+_n]=\infty,\quad \sum_{n=1}^{\infty}[(1-a_n)g^-_{n+1}+C_ng^-_n]=\infty,$$

then all solutions of

$$y_{n+2} - a_n y_{n+1} + c_n y_n = f_n$$

oscillate, where $g_n^+ = \max\{g_n, 0\}$ *and* $g_n^- = \max\{-g_n, 0\}$.

2. Oscillation for System (1.3)

Consider the system of equations (1.3) as

$$X(n+1) = A(n)X,$$

where $X(n) = [x(n), y(n)]^T$ and

$$A(n) = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}.$$

We assume that $a(n), b(n), c(n), d(n)$ are real valued functions defined for $n \geq n_0 > 0$. Let $b(n) \neq 0$ for all $n \geq n_0$. Then it follows from (1.3) that

$$y(n) = \frac{x(n+1)}{b(n)} - \frac{a(n)}{b(n)} x(n);$$

that is,

$$y(n+1) = \frac{x(n+2)}{b(n+1)} - \frac{a(n+1)}{b(n+1)} x(n+1).$$

Hence

$$c(n)x(n) + d(n)y(n) = \frac{x(n+2)}{b(n+1)} - \frac{a(n+1)}{b(n+1)} x(n+1);$$

that is,

$$x(n+2) - P_1(n)x(n+1) + Q_1(n)x(n) = 0, \tag{2.1}$$

where we define

$$P_1(n) = a(n+1) + \frac{d(n)b(n+1)}{b(n)},$$
$$Q_1(n) = \frac{b(n+1)}{b(n)}[a(n)d(n) - b(n)c(n)]$$

for all $n \geq n_0$. Similarly, if $c(n) \neq 0$ for all $n \geq n_0$, then

$$y(n+2) - P_2(n)y(n+1) + Q_2(n)y(n) = 0, \tag{2.2}$$

where we define

$$P_2(n) = d(n+1) + \frac{a(n)d(n)}{c(n)},$$
$$Q_2(n) = \frac{c(n+1)}{c(n)}[a(n)d(n) - b(n)c(n)]$$

Theorem 2.1. *Let* $P_i(n) > 0$, $Q_i(n) > 0$, $i = 1, 2$ *be such that*

$$P_i(n) \leq \frac{Q_i(n+1)}{P_i(n+1)} + \frac{Q_i(n)}{P_i(n-1)} \tag{2.3}$$

for all large n*, then every solution* $X(n)$ *of* (1.3) *oscillates.*

Proof. Suppose, on the contrary, that $X(n)$ is a nonoscillatory solution of (1.3). Then there exists $n_0 > 0$ such that at least one component of $X(n)$ is nonoscillatory for $n \geq n_0$. Let $x(n)$ be the nonoscillatory component of $X(n)$ such that $x(n)$ is eventually positive for $n \geq n_0$. Then applying Theorem 1.5, we have a contradiction to (2.3). Similarly, one can proceed for $y(n)$, if we assume that $y(n)$ is a nonoscillatory component of $X(n)$ for $n \geq n_0$. Hence the proof is complete. □

Remark. If (2.3) holds for either $i = 1$ or $i = 2$, then there could be a possibility for the existence of nonoscillatory solution. However, we are not sure about the fact. We note that (2.1) and (2.2) are non self-adjoint difference equations. Hence the above possibility may be true.

Remark. If $P_i(n) = p_i$ and $Q_i(n) = q_i, i = 1, 2$ then (2.3) becomes $p_i^2 \leq 2q_i$, $i = 1, 2$. Hence the inequalities $p_1^2 \leq 2q_1$ and $p_2^2 \leq 2q_2$ reduce to $(a+d)^2 \leq 2(ad-bc)$. Thus we have the following corollary.

Corollary 2.2. *If $A(n) \equiv A$ and $(\operatorname{tr} A)^2 \leq 2 \det A$, then (1.3) is oscillatory.*

Example. Consider

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \tag{2.4}$$

Indeed, $\operatorname{tr} A = 2$ and $\det A = 3$. $\lambda_1 = 1+i\sqrt{2}$ and $\lambda_2 = 1-i\sqrt{2}$ are two characteristic roots of the coefficient matrix A. Clearly,

$$\begin{aligned} x(n) &= \lambda_1^n \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= (1+i\sqrt{2})^n \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= 3^{n/2}(\cos n\theta + i \sin n\theta) \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3^{n/2}(\cos n\theta + i \sin n\theta \\ -3^{n/2}(\sin n\theta - i \cos n\theta) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} y(n) &= \lambda_2^n \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= (1-i\sqrt{2})^n \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= 3^{n/2}(\cos n\theta - i \sin n\theta) \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3^{n/2}(\cos n\theta + i \sin n\theta \\ 3^{n/2}(\sin n\theta - i \cos n\theta) \end{bmatrix}, \end{aligned}$$

where $\theta = \tan^{-1}(\sqrt{2})$. By Corollary 2.2, the system (2.4) is oscillatory.

If we define $a(n) = \frac{r(n)}{r(n+1)}$ and $d(n) = \frac{t(n)}{t(n+1)}$, then $r(n+1) = \frac{r(n)}{a(n)}$ and $t(n+1) = \frac{t(n)}{d(n)}$ and hence solving the two relations we get

$$r(n) = \frac{r(0)}{\prod_{i=0}^{n-1} a(i)}, \quad t(n) = \frac{d(0)}{\prod_{j=0}^{n-1} d(j)},$$

where $r(0)$ and $d(0)$ are non-zero constants if $a(n) \neq 0 \neq d(n)$ for $n \geq n_0 > 0$. From (1.3) it follows that

$$r(n+1)x(n+1) - r(n)x(n) = b(n)r(n+1)y(n);$$

that is,

$$\Delta(r(n)x(n)) = b(n)r(n+1)y(n).$$

Consequently,

$$\sum_{s=0}^{n-1} \Delta[r(s)x(s)] = \sum_{s=0}^{n-1} b(s)r(s+1)y(s);$$

that is,

$$\begin{aligned} x(n) &= \frac{r(0)x(0)}{r(n)} + \frac{1}{r(n)} \sum_{s=0}^{n-1} b(s)r(s+1)y(s) \\ &= \prod_{i=0}^{n-1} a(i) \Big[x(0) + \sum_{s=0}^{n-1} \frac{b(s)y(s)}{\prod_{i=0}^{s} a(i)} \Big]. \end{aligned}$$

Similarly,

$$y(n) = \prod_{j=0}^{n-1} d(j) \Big[y(0) + \sum_{s=0}^{n-1} \frac{c(s)x(s)}{\prod_{j=0}^{s} d(j)} \Big].$$

Hence or otherwise the following theorem holds

Theorem 2.3. *Let $A(n)$ be a real valued coefficient matrix such that $a(n) \neq 0 \neq d(n)$ for $n \geq n_0 > 0$. Then* (1.3) *is either oscillatory or nonoscillatory.*

Theorem 2.4. *Suppose that $a(n) = 0 = d(n)$ and $c(n) \neq 0 \neq b(n)$ for all $n \geq n_0 > 0$. If $\liminf_{n\to\infty} b(n) = \alpha \neq 0$ and $\liminf_{n\to\infty} c(n) = \beta \neq 0$ such that $\alpha\beta < 0$, then* (1.3) *is oscillatory.*

Proof. Let $X(n)$ be a nonoscillatory solution of (1.3) for $n \geq n_0$. Let $x(n)$ be a component of $X(n)$ such that $x(n)$ is eventually positive for $n \geq n_0$. Clearly, from (1.3) we obtain that, $x(n)$ is a solution of

$$z(n+2) - b(n+1)c(n)z(n) = 0. \tag{2.5}$$

Without any loss of generality, we may assume that $z(n) > 0$ for $n \geq n_0$. Equation (2.5) can be written as

$$\frac{z(n+2)}{z(n+1)} \frac{z(n+1)}{z(n)} = b(n+1)c(n)$$

for $n \geq n_0$. If we denote $u(n) = \frac{z(n+1)}{z(n)} > 0$ for $n \geq n_1$, then the above equation yields

$$\begin{aligned} \liminf_{n\to\infty}[u(n+1)u(n)] &= \liminf_{n\to\infty}[b(n+1)c(n)] \\ &= [\liminf_{n\to\infty} b(n+1)][\liminf_{n\to\infty} c(n)] = \alpha\beta. \end{aligned} \tag{2.6}$$

Since $\alpha\beta \neq 0$, then $\liminf_{n\to\infty}[u(n)u(n+1)]$ exists. Let $\lambda = \liminf_{n\to\infty} u(n)$. From (2.6), it follows that $f(\lambda) = \lambda^2 - \alpha\beta = 0$. It is easy to see that $f(\lambda)$ attains minimum at $\lambda = 0$. Consequently, $\min f(\lambda) \leq f(\lambda)$ implies that $\alpha\beta \geq 0$, a contradiction. Hence (2.5) is oscillatory. Similarly, we can show that $y(n)$ is a solution of

$$w(n+2) - b(n)c(n+1)w(n) = 0, \tag{2.7}$$

and (2.7) is oscillatory. This completes the proof. □

Example Consider the system of equations

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & -2+(-1)^n \\ 2+(-1)^n & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \quad n \geq 0. \tag{2.8}$$

Indeed,

$$y(n+2) + (5 - 4(-1)^n)y(n) = 0, \quad n \geq 0 \tag{2.9}$$

and $\alpha = -3$, $\beta = 1$, $\alpha\beta = -3 < 0$. From Theorem 2.4, it follows that (2.8) is oscillatory. We note that

$$y(n) = y(0)(-1)^{n/2} \prod_{i=0}^{n-2} [5 - 4(-1)^i]$$

is one of the solution of (2.9), where $(n/2)$ is an odd positive integer.

We conclude this section with the following result.

Theorem 2.5. *Let $X(n_0) \in R \times R$ for $n_0 \in Z^+$. If $\det A(n) \neq 0$, then (1.3) is oscillatory if and only if every component of the matrix $\prod_{i=n_0}^{n-1} A(i)$ is oscillatory, where*

$$\prod_{i=n_0}^{n-1} A(i) = \begin{cases} A(n-1)A(n-2)\dots A(n_0) & n > n_0 \\ I & n = n_0. \end{cases}$$

The proof of the above theorem follows from the proof of the [3, Theorem 3.3] and hence it is omitted.

Remark, If (1.3) is an autonomous system, then $\prod_{i=n_0}^{n-1} A(i) = A^{n-n_0}$ and Theorem 2.5 holds for A^{n-n_0} for all $n > n_0$.

3. Oscillation for System (1.4)

This section presents sufficient conditions for the oscillation of all solutions of the system of equations (1.4). If we assume that $b(n) \neq 0$ for all $n \geq n_0$, then

$$y(n) = \frac{x(n+1)}{b(n)} - \frac{a(n)}{b(n)} x(n) - \frac{f_1(n)}{b(n)};$$

that is,

$$y(n+1) = \frac{x(n+2)}{b(n+1)} - \frac{a(n+1)}{b(n+1)} x(n+1) - \frac{f_1(n+1)}{b(n+1)}.$$

Consequently,

$$c(n)x(n) + d(n)y(n) + f_2(n) = y(n+1)$$

implies that

$$x(n+2) - P_1(n)x(n+1) + Q_1(n)x(n) = G_1(n), \tag{3.1}$$

where $G_1(n) = f_2(n) + \frac{f_1(n+1)}{b(n+1)}$, for $n \geq n_0$ and $P_1(n), Q_1(n)$ are same as in (2.1). Similarly, if we assume that $c(n) \neq 0$ for all $n \geq n_0$, then we find

$$y(n+2) - P_1(n)y(n+1) + Q_2(n)y(n) = G_2(n), \tag{3.2}$$

where $P_2(n)$ and $Q_2(n)$ are same as in (2.2) and $G_2(n) = f_1(n) + \frac{f_2(n+1)}{c(n+1)}$. We note that $G_i(n)$ could be oscillatory or could be nonoscillatory for $i = 1, 2$.

Theorem 3.1. *Let $P_i(n) < 0, Q_i(n) > 0$ for $n \geq n_0$ and $i = 1, 2$. Assume that $G_i(n)$ changes sign. In addition, there exists $g_i(n)$ which changes sign such that $G_i(n) = g_i(n+2) - g_i(n+1), i = 1, 2$. If*

$$\sum_{n=0}^{\infty}[Q_i(n)g_i^+(n) - P_i(n)g_i^+(n+1)] = \infty, \tag{3.3}$$

$$\sum_{n=0}^{\infty}[Q_i(n)g_i^-(n) - P_i(n)g_i^-(n+1)] = \infty \tag{3.4}$$

hold, then (1.4) *is oscillatory, where*

$$g_i^+(n) = \max\{g_i(n), 0\} and g_i^-(n) = \max\{0, -g_i(n)\}$$

Proof. Suppose on the contrary that $X(n) = [x(n), y(n)]^T$ is a nonoscillatory solution of (1.4). Then there exists $n_0 > 0$ such that at least one component of $X(n)$ is nonoscillatory for $n \geq n_0$. Let $x(n)$ be the nonoscillatory component of $X(n)$ such that $x(n) > 0$ for $n \geq n_0$. Consequently, $x_1(n)$ and $x_2(n)$ are two solutions of (3.1). Applying Theorem 1.6, we obtain a contradiction to our hypothesis (3.3). A contradiction can be obtained to (3.4) if we assume that $x(n) < 0$ eventually for $n \geq n_0$. Similar observations can be dealt with the solution $y(n)$ if we assume that $y(n)$ is a nonoscillatory component of $X(n)$ for $n \geq n_0$. Hence or otherwise the proof of the theorem is complete. □

Theorem 3.2. *Let $0 \leq P_i(n) < 1$, $Q_i(n) > 0$ and $G_i(n)$ changes sign for $i = 1, 2$. Assume that there exists $g_i(n)$ which changes sign such that $G_i(n) = g_i(n+2) - g_i(n+1)$, $i = 1, 2$. If*

$$\sum_{n=0}^{\infty}[Q_i(n)g_i^+(n) + (1 - P_i(n))g_i^+(n+1)] = \infty,$$

$$\sum_{n=0}^{\infty}[Q_i(n)g_i^-(n) + (1 - P_i(n))g_i^-(n+1)] = \infty$$

hold, then (1.4) *is oscillatory, where $g_i^+(n)$ and $g_i^-(n)$ are same as in Theorem 3.1.*

The proof of the above theorem follows from the Theorem 3.1 and Theorem 1.6 and hence it is omitted.

Theorem 3.3. *Let $P_i(n) < 0$ and $Q_i(n) > 0$ for all $n \geq n_0$ and $i = 1, 2$. Assume that $G_i(n)$ is nonoscillatory for all large n. Furthermore, assume that there exists $g_i(n)$ such that $G_i(n) = g_i(n+2) - g_i(n+1)$ and $0 < \lim_{n\to\infty} |g_i(n)| < \infty$. If*

$$\sum_{n=0}^{\infty}[Q_i(n)g_i(n) - P_i(n)g_i(n+1)] = +\infty, \tag{3.5}$$

$$\sum_{n=0}^{\infty}[Q_i(n) - P_i(n)] = +\infty \tag{3.6}$$

hold, then (1.4) *is oscillatory.*

Proof. Suppose on the contrary that $X(n) = [x(n), y(n)]^T$ is a nonoscillatory solution of (1.4). Proceeding as in the proof of the Theorem 3.1, we may assume that $x(n)$ and $y(n)$ are nonosillatory solutions of (3.1) and (3.2) respectively. Assume that there exists $n_0 > 0$ such that $x(n) > 0$ for $n \geq n_0$. Then from (3.1), it follows that

$$\Delta[x(n+1) - g_1(n+1)] = [P_1(n) - 1]x(n+1) - Q_1(n)x(n) \leq 0 \tag{3.7}$$

but not identically zero for $n \geq n_0$. Ultimately, $(x(n+1)-g_i(n+1))$ is nonincreasing on $[n_0, \infty)$. We consider two cases viz. $g_1(n) > 0$ and $g_1(n) < 0$ for $n \geq n_0$. Suppose the former holds. If $(x(n+1) - g_1(n+1)) > 0$ for $n \geq n_1 > n_0$, then $\lim_{n\to\infty}(x(n+1) - g_1(n+1))$ exists and hence (3.7) becomes

$$\sum_{n=n_1}^{\infty} [Q_1(n)g_1(n) - (P_1(n))g_1(n+1)] < \infty,$$

a contradiction to (3.5). Thus $(x(n+1) - g_1(n+1)) < 0$ for $n \geq n_1$. Consequently, $x(n) < 0$ for large n, a contradiction. Let the latter hold. Ultimately, $x(n+1) - g_1(n+1) > 0$ for $n \geq n_1$. It is easy to verify that $0 < \lim_{n\to\infty} x(n+1) < \infty$. Let $\lim_{n\to\infty} x(n) = \ell$, $\ell \in (0, \infty)$. For every $\epsilon > 0$, there exists $n^* > 0$ such that $x(n+1) > \ell - \epsilon > 0$ for $n \geq n^*$.

Hence summing (3.7) from n_2 to ∞, we get

$$\sum_{n=n_2}^{\infty} [Q_1(n)P_1(n)] < \infty, \quad n_2 > \max\{n_1, n^*\},$$

a contradiction to our assumption (3.6). Same type of reasoning can be made if we assume $x(n) < 0$ for $n \geq n_0$. A similar type of observation can be formulated when $y(n)$ is a non-oscillatory component of (1.4) for $n \geq n_0$. This completes the proof. □

Remark. Without any loss of generality, we may assume that $g_i(n) > 0$ for $i = 1, 2$.

Theorem 3.4. *Let $0 \leq P_i(n) < 1$ and $Q_i(n) > 0$ for large n. Assume that all the conditions of Theorem 3.3 hold except* (3.5) *and* (3.6). *If*

$$\sum_{n=0}^{\infty}[Q_i(n)g_i(n) + (1 - P_i(n))g_i(n+1)] = \infty,$$

$$\sum_{n=0}^{\infty}[Q_i(n) + (1 - P_i(n))] = \infty,$$

hold, then (1.4) *is oscillatory.*

The proof of the above theorem follows from the proof of the Theorem 3.3.

Example Consider

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} (-1)^n \\ (-1)^n \end{bmatrix}, \quad n \geq 0.$$

Clearly, $P_1(n) = 0 = P_2(n)$, $Q_1(n) = \frac{1}{2} = Q_2(n)$, $G_1(n) = 2(-1)^n$ and $G_2(n) = (-1)^{n+1}$. Indeed, $x(n)$ and $y(n)$ are two solutions of

$$z(n+2) + \frac{1}{2}z(n) = 2(-1)^n, \tag{3.8}$$

$$w(n+2) + \frac{1}{2}w(n) = (-1)^{n+1} \tag{3.9}$$

respectively. If we choose $g_1(n) = (-1)^n$ and $g_2(n) = \frac{1}{2}(-1)^{n+1}$, then $G_1(n) = 2(-1)^n$ and $G_2(n) = (-1)^{n+1}$ for all $n \geq 0$.

It follows that all the conditions of Theorem 3.2 are satisfied and hence the given system of equations is oscillatory. In particular, $x(n) = \frac{4}{3}(-1)^n$ is a solution of (3.8) and $y(n) = \frac{2}{3}(-1)^{n+1}$ is a solution of (3.9).

Example. Consider

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} 1-2(-1)^n \\ 1-2(-1)^n \end{bmatrix}, \quad n \geq 0.$$

where $P_1(n) = -3 = P_2(n)$, $Q_1(n) = 1 = Q_2(n)$, $G_1(n) = 2 = G_2(n)$. Clearly, $x(n)$ and $y(n)$ are solutions of

$$z(n+2) + 3z(n+1) + z(n) = 2, \tag{3.10}$$

$$w(n+2) + 3w(n+1) + w(n) = 2 \tag{3.11}$$

respectively. If we choose $g_1(n) = 2(n-1) = g_2(n)$, then $G_1(n) = 2 = G_2(n)$ and hence (3.5) and (3.6) hold good. But we can not apply the Theorem 3.3 due to the fact that

$$\liminf_{n\to\infty} g_1(n) = \limsup_{n\to\infty} g_1(n) = \infty.$$

Then $x(n) = \frac{2}{5} + \big(\frac{3+\sqrt{5}}{2}\big)^n(-1)^n$ is a solution of (3.10) and $y(n) = \frac{2}{5} + \big(\frac{3+\sqrt{5}}{2}\big)^n(-1)^n$ is a solution of (3.11). We note that the given system of equations is oscillatory.

Remark. In view of the above example, it seems that some additional condition is necessary to prove the Theorem 3.3 when $\lim_{n\to\infty} |g_i(n)| = \infty$.

Let $a(n) = 0 = d(n)$ for all $n \geq n_0 \geq 0$. Then the system of equations (1.4) becomes

$$x(n+1) = b(n)y(n) + f_1(n),$$
$$y(n+1) = c(n)x(n) + f_2(n)$$

Solving the above two equations, it follows that $x(n)$ and $y(n)$ are solutions of

$$z(n+2) - c(n)b(n+1)z(n) = E_1(n), \tag{3.12}$$

$$w(n+2) - c(n+1)b(n)w(n) = E_2(n) \tag{3.13}$$

respectively, where $E_1(n) = f_1(n+1) + f_2(n)b(n+1)$, $E_2(n) = f_2(n+1) + f_1(n)c(n+1)$ and we assume that $\det A(n) \neq 0$ for all $n \geq n_0 \geq 0$.

Theorem 3.5. *Assume that $c(n)b(n+1) < 0$ for all large n. If there exists $e_i(n)$, $i = 1, 2$ which changes sign such that $E_i(n) = \Delta e_i(n+1)$ and*

$$\sum_{n=0}^{\infty} [c(n)b(n+1)e_1^+(n) - e_1^+(n+1)] = -\infty,$$

$$\sum_{n=0}^{\infty} [b(n)c(n+1)e_2^+(n) - e_2^+(n+1)] = -\infty$$

where $e_i^+(n) = \max\{e_i(n), 0\}$, $e_i^-(n) = \max\{-e_i(n), 0\}$, then (1.4) is oscillatory.

It is easy to verify that, (3.12) and (3.13) can be written as

$$\Delta[z(n+1) - e_1(n+1)] = c(n)b(n+1)z(n) - z(n+1), \tag{3.14}$$

$$\Delta[w(n+1) - e_2(n+1)] = c(n+1)b(n)w(n) - w(n+1) \tag{3.15}$$

respectively. To prove this theorem it is sufficient to prove that (3.14) and (3.15) are oscillatory. Moreover, the proof of the theorem can be done as in Theorems 3.1 and 1.6.

Remark. $E_i(n)$ could be nonoscillatory also. If $e_i(n)$ is nonoscillatory such that $E_i(n) = \Delta e_i(n+1)$, then a result corresponding to the Theorem 3.3 can be formulated under the conditions $0 < \lim_{n\to\infty} |e_i(n)| < \infty$ and $c(n)b(n+1) < 0$ for all large n.

Concluding Remarks. In this work, specific results regarding the oscillatory behaviour of vector solutions of the systems (1.3) and (1.4) have been established under the criteria $\det A(n) \neq 0$ subject to the fundamental matrix $\Phi(n)(\det \Phi(n) \neq 0)$. Indeed, the discrete analog of a second order differential equation is not necessarily a self adjoint difference equation. Since the work in [6] based on the oscillatory behaviour of solutions of a non-self adjoint difference equation and the author has followed the work of [6], then it follows that the present work is not the analog work of continuous case. Hence the results developed here may initiate further study for the system of equations (1.3)/(1.4).

Existence of nonoscillatory vector solution of (1.3)/(1.4) is not discussed in this work. However, the same can be followed from [3] and [4].

It is interesting to apply this work to study the system of equations

$$X(n+1) = A(n)h(X(n))$$

and

$$X(n+1) = A(n)h(X(n)) + F(n),$$

where $h \in C(R, R)$.

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References

[1] R. P. Agarwal, S. R. Grace; *The oscillation of systems of difference equations*, Appl. Math. Lett. 13 (2000), 1-7.

[2] W. S. Burnside, A. W. Panton; *The Theory of Equations*, S. Chand and Company Ltd., New Delhi, 1979.

[3] S. N. Elaydi; *An Introduction to Difference Equations*, Springer - Verlag, New York, 1996.

[4] I. Gyori, G. Ladas; *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[5] W. G. Kelley, A. C. Peterson; *Difference Equations: An Introduction with Applications*, Academic Press, INC, New York, 1991.

[6] N. Parhi, A. K. Tripathy; *Oscillatory behaviour of second order difference equations*, Commu. Appl. Nonlin. Anal. 6(1999), 79 - 100.

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