Electronic Journal of Differential Equations, Vol. 2009(2009), No. 30, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

AN OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

JAGMOHAN TYAGI, VENKATARAMANARAO RAGHAVENDRA

Abstract. We establish an oscillation criteria of the second-order nonlinear damped differential equation with functional arguments

$$x''(t) + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t), x[h(t)]) = 0, \quad t \in [t_0, \infty).$$

1. Introduction

Over the previous three decades, many studies have dealt with the oscillation theory for functional differential equations. For an excellent bibliography and later developments of this theory, we refer the books by Agarwal, Bohner and Wan-Tong Li [1], Erbe, Kong and Zhang [3] and research articles [4, 9, 2, 8, 10, 11, 12]. In this note, we consider the second-order nonlinear differential equations with functional arguments of the type

$$x''(t) + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t), x[h(t)]) = 0, \quad t \in [t_0, \infty), \tag{1.1}$$

where $p, q \in C([t_0, \infty), \mathbb{R}^+)$, $f \in C([t_0, \infty) \times \mathbb{R}^2, \mathbb{R}^+)$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g(y_1, y_2) > 0$ if $y_i > 0$; $g(y_1, y_2) < 0$ if $y_i < 0$, for all $i = 1, 2$ and $h \in C^1([t_0, \infty), \mathbb{R})$.

We consider only nontrivial solutions of (1.1) which are defined for all $t \geq t_0 \geq 0$. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; i.e., for any $T > t_0$, there exists a $t \geq T$ such that $x(t) = 0$, otherwise the solution is said to be non-oscillatory. When either $p(t) = 0$ or $f = 0$, the oscillatory behavior of (1.1) is investigated by many investigators, (see, e.g., Bradley [2], Travis [10], Yeh [11]). For the convenience of the reader, we give a brief introduction about the earlier developments. In 1970, Bradley considered the equation

$$(r(t)x'(t))' + q(t)g(x(t), x[h(t)]) = 0, \quad [t_0, \infty), \tag{1.2}$$

where $r(t) > 0$, $q(t) \geq 0$ and

(i) $h(t) \to \infty$ as $t \to \infty$.
(ii) If y_1, y_2 are of the same sign, then $g(y_1, y_2)$ has that sign.
(iii) $g(y_1, y_2)$ is bounded away from zero when y_1, y_2 are.

Bradley stated the following theorem.

2000 *Mathematics Subject Classification.* 34K15, 34C10.
Key words and phrases. Nonlinear; functional differential equations; oscillation.

Submitted July 19, 2008. Published February 9, 2009.

Theorem 1.1 ([2]). *If $q(t) \geq 0$, $r(t) > 0$, $\int_{t_0}^{\infty} q(t)dt = \infty$, $\int_{t_0}^{\infty} \frac{1}{r(t)}dt = \infty$, and conditions* (i)–(iii) *hold, then any solution of* (1.2) *that exists on a ray* $[t_0, \infty)$ *is oscillatory.*

In 1972, Travis considered the equation

$$x''(t) + q(t)g(x(t), x[h(t)]) = 0, \quad [t_0, \infty), \tag{1.3}$$

where q, g, h are continuous functions. If y_1 and y_2 are of one sign, then $g(y_1, y_2)$ has that sign. To avoid the assumption that h is differentiable, he introduced a differentiable minorant $j(t)$ and gave the following result

Theorem 1.2 ([10]). *If*

(i) $j(t) \leq h(t)$ *and* $0 < \alpha \leq j(t) \leq 1$.

(ii) *There exists* $M > 0$ *such that* $y_1 \geq M$ *implies*

$$\liminf_{|y_2| \to \infty} |\frac{g(y_1, y_2)}{y_2}| \geq \epsilon > 0.$$

(iii) $q(t) \geq 0$ *and* $\limsup_{x \to \infty} x \int_x^{\infty} q(t)dt = \infty$,

then all solutions of (1.3) *existing on* (t_0, ∞) *are oscillatory.*

In 1980, Yeh also considered (1.3), where q, g, h are continuous functions and if y_1 and y_2 are of one sign, then $g(y_1, y_2)$ has that sign. He gave a new integral criterion for the oscillation of (1.3). He used the n-th primitive

$$A_n(t) = \frac{1}{n!} \int_{t_0}^{t} (t-u)^{n-1} q(u)du$$

of the coefficient $q(t)$. He established the following result.

Theorem 1.3 ([11]). *Let conditions* (i)–(ii) *of Theorem 1.2 hold. Let $q(t) \geq 0$ and*

$$\limsup_{t \to \infty} t^{1-n} A_n(t) = \infty,$$

where $A_n(t)$ is the n-th primitive of $q(t)$ for some $n > 2$, then all solutions of (1.3) *are oscillatory.*

The oscillatory behavior of a class of second-order functional equations which have the potential, $q(t) = t, t^2, \dots$, are discussed in Theorems 1.1–1.3. It is worth mentioning that the oscillatory behavior of the equations which have the potential, like $q(t) = e^{-t} + \frac{2}{t^2}$, $t > 0$, cannot be discussed by the above approaches.

Koplatadze et al. [8] gave some oscillation theorems for second-order linear delay differential equations. Recently, Zayed and El-Moneam [12] gave some oscillation criteria for second-order nonlinear functional differential equations with linear damping. They point out that the oscillation of some nonlinear functional differential equations is studied by comparison with related to some linear equations.

All the above cited results do not include a nonlinear damping term. The main result is proved in section 2 which includes a nonlinear damping term. Our approach is not only different from other approaches but also it deals with nonlinear functional equations with nonlinear damping and more general potentials.

Komkov [7] considered the equation

$$(a(t)x'(t))' + q(t)x(t) = 0, \tag{1.4}$$

where $a, q \in C([t_0, \infty), \mathbb{R})$ and $a(t) > 0$. He proved the following result.

Theorem 1.4. *Suppose there exist a C^1 function $u(t)$ defined on $[t_1, t_2]$ and a function $G(u)$ such that $G(u(t))$ is not constant on $[t_1, t_2]$, $G(u(t_1)) = G(u(t_2)) = 0$, $g(u) = G'(u)$ is continuous,*

$$\int_{t_1}^{t_2} [a(t)(u'(t))^2 - q(t)G(u(t))]dt < 0,$$

and $(g(u(t)))^2 \le 4G(u(t))$ for $t \in [t_1, t_2]$. Then every solution of (1.4) must vanish on $[t_1, t_2]$.

For a proof of the above theorem, we refer the reader to [7]. Also, this result is used by Graef and Spikes [6] for getting the sufficient conditions for nonoscillation of a second-order nonlinear differential equations.

We need the following hypotheses for further studies.

- (H1) $g(y_1, y_2)$ is a continuously differentiable function with respect to y_1 and y_2. Also suppose there exist $k > 0$ such that $\frac{\partial}{\partial y_i} g(y_1, y_2) \ge k/2 > 0$, for $y_i \ne 0$ for $i = 1, 2$.
- (H2) There exist a C^1 function u defined on $[t_0, \infty)$, a C^1 function F on $\mathbb{R}$, and a continuous function G on $\mathbb{R}$ such that $F'(u) = \sqrt{k}G(u)$, $F(u) \ge \frac{(G(u))^2}{4}$.
- (H3) $\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} [(u'(s))^2 - q(s)F(u(s))]ds < 0$.
- (H4) $h \in C^1([t_0, \infty), \mathbb{R})$ such that $h(t) \to \infty$ as $t \to \infty$, $h'(t) \ge 1$, $h(t) \le t$ for all large t.

Remark 1.5. Hypotheses (H2) is more general than the condition used by [7, Theorem 1.4]. If we restrict $F(u(t_1)) = F(u(t_2)) = 0$, $F(u(t))$ is not constant on $[t_1, t_2], k = 1$ and $[t_1, t_2] \subseteq [t_0, \infty)$ in (H2), then (H2) implies a condition used in Theorem 1.4. Similarly, (H3) can be viewed as a more general condition than an integral inequality used in Theorem 1.4.

Remark 1.6. Let $\tau \in C^1([t_0, \infty), \mathbb{R}^+)$ such that $\tau(t) \to 0$ as $t \to \infty$ and $\tau'(t) \le 0$. Let $h(t) = t - \tau(t)$. Then $h(t)$ satisfies the hypothesis (H_4).

Lemma 1.7. *Let $p(t) \ge 0$ and $q(t)$ be continuous non-negative and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \ge t_0$ and assume that*

- (i) *$f(t, x, y) \le |y|^\alpha$, $-\infty < x, y < \infty$, $t \ge t_0$ and some constant $\alpha \ge 0$.*
- (ii) *$\left(1 + \int_{t_0}^{t} p(s)ds\right)^{-1/\alpha} \notin L(t_0, \infty)$, if $\alpha > 0$,*

$$\int_{t_0}^{\infty} \exp\Big(\int_{t_0}^{s} -p(\tau)d\tau\Big)ds = \infty, \quad \text{if } \alpha = 0.$$

If $x(t)$ is a non-oscillatory solution of (1.1), then $x(t)x'(t) > 0$ for all large t.

For a proof of above lemma, we refer the reader to [5, p. 1083].

This paper is organized as follows. Section 2 deals with the main result. In Section 3, we construct some examples for the illustration of this result.

2. Main results

The main result of the paper is as follows.

Theorem 2.1. *Let the conditions (i)–(ii) of Lemma 1.7 hold. Let $p(t) \ge 0$, $q(t)$ be non-negative and not eventually zero on $[t_0, \infty)$. Then under the hypotheses (H1)–(H4), (1.1) is oscillatory.*

Proof. Suppose on the contrary, (1.1) has a non-oscillatory solution $x(t)$. Then, there exist some $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$, for all $t \geq t_1$.

Case 1. $x(t) > 0$, for all $t \geq t_1$. For large t, we have, $x(t) > 0$, $x[h(t)] > 0$, for all $t \geq T$, where T is sufficiently large. By Lemma 1.7, we have $x'(t) > 0$, for all $t \geq T$. From (1.1), $x''(t) < 0$, for all $t \geq T$. Now we note that the following identity is valid on $[T, \infty)$,

$$
\begin{aligned}
&(u'(t))^2 - q(t)F(u(t)) \\
&= (u'(t))^2 - q(t)F(u(t)) + \Big(\frac{x'(t)F(u(t))}{g(x(t), x[h(t)])}\Big)' \\
&\quad + \frac{x'(t)\frac{\partial}{\partial x[h(t)]}g(x(t), x[h(t)])x'[h(t)]h'(t)F(u(t))}{(g(x(t), x[h(t)])^2} \\
&\quad + \frac{x'(t)\frac{\partial}{\partial x(t)}g(x(t), x[h(t)])x'(t)F(u(t))}{(g(x(t), x[h(t)]))^2} - \Big(\frac{x'(t)F'(u(t))u'(t)}{g(x(t), x[h(t)])}\Big) - \frac{x''(t)F(u(t))}{g(x(t), x[h(t)])} \\
&= (u'(t))^2 + p(t)f(t, x(t), x'(t))x'(t)\frac{F(u(t))}{g(x(t), x[h(t)])} \\
&\quad - \frac{F(u(t))}{g(x(t), x[h(t)])}[x''(t) + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t), x[h(t)])] \\
&\quad + \Big(\frac{x'(t)F(u(t))}{g(x(t), x[h(t)])}\Big)' + \frac{x'(t)\frac{\partial}{\partial x[h(t)]}g(x(t), x[h(t)])x'[h(t)]h'(t)F(u(t))}{(g(x(t), x[h(t)])^2} \\
&\quad + \frac{x'(t)\frac{\partial}{\partial x(t)}g(x(t), x[h(t)])x'(t)F(u(t))}{(g(x(t), x[h(t)]))^2} - \Big(\frac{x'(t)F'(u(t))u'(t)}{g(x(t), x[h(t)])}\Big).
\end{aligned} \tag{2.1}
$$

Since x' is a decreasing function for large t, so, $x'[h(t)] \geq x'(t)$, for $t \geq T$ and using the hypotheses (H1) and (H2) in (2.1), we get

$$
\begin{aligned}
&(u'(t))^2 - q(t)F(u(t)) \\
&\geq (u'(t))^2 + p(t)f(t, x(t), x'(t))x'(t)\frac{F(u(t))}{g(x(t), x[h(t)])} \\
&\quad - \frac{F(u(t))}{g(x(t), x[h(t)])}[x''(t) + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t), x[h(t)])] \\
&\quad + \Big(\frac{x'(t)F(u(t))}{g(x(t), x[h(t)])}\Big)' - \Big(\frac{x'(t)\sqrt{k}G(u(t))u'(t)}{g(x(t), x[h(t)])}\Big) + \frac{k(x'(t))^2(G(u(t)))^2}{4(g(x(t), x[h(t)]))^2} \\
&\geq p(t)f(t, x(t), x'(t))x'(t)\frac{F(u(t))}{g(x(t), x[h(t)])} \\
&\quad - \frac{F(u(t))}{g(x(t), x[h(t)])}[x''(t) + p(t)f(t, x(t), x'(t))x'(t) + q(t)g(x(t), x[h(t)])] \\
&\quad + \Big(\frac{x'(t)F(u(t))}{g(x(t), x[h(t)])}\Big)' + \Big[u'(t) - \frac{x'(t)\sqrt{k}G(u(t))}{2g(x(t), x[h(t)])}\Big]^2.
\end{aligned}
$$

Therefore,

$$
(u'(t))^2 - q(t)F(u(t)) \geq \Big(\frac{x'(t)F(u(t))}{g(x(t), x[h(t)])}\Big)'.
$$

An integration over $[T,\infty)$ yields

$$\begin{aligned}\int_T^t [(u'(s))^2 - q(s)F(u(s))]ds &\geq \int_T^t \Big(\frac{x'(s)F(u(s))}{g(x(s),x[h(s)])}\Big)' ds \\ &= \Big(\frac{x'(t)F(u(t))}{g(x(t),x[h(t)])}\Big) - \Big(\frac{x'(T)F(u(T))}{g(x(T),x[h(T)])}\Big).\end{aligned}$$

So,

$$\frac{1}{t}\int_T^t [(u'(s))^2 - q(s)F(u(s))]ds \geq -\frac{1}{t}\frac{x'(T)F(u(T))}{g(x(T),x[h(T)])} \to 0 \quad \text{as } t\to\infty,$$

which contradicts to (H3).

Case 2. $x(t) < 0$, for all $t \geq t_1$. For large t, we have, $x(t) < 0$, $x[h(t)] < 0$, for all $t \geq T$, where T is sufficiently large. By the Lemma 1.7, we have $x'(t) < 0$, for all $t \geq T$. From (1.1), $x''(t) > 0$, for all $t \geq T$; i.e., x' is an increasing function for sufficiently large t. This implies that $x'[h(t)] \leq x'(t) < 0$ for sufficiently large t. Now the rest of the proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem. $\square$

Remark 2.2. The oscillatory behavior of (1.1) with $p(t) = 0$ has been investigated by Bradley [2], Travis [10], Yeh [11] by different conditions.

Remark 2.3. Let

- (H1') $g(y_1, y_2)$ is a continuously differentiable function with respect to y_1, y_2. Suppose there exists $k > 0$ such that $\frac{\partial}{\partial y_i} g(y_1, y_2) \geq k > 0$, for $y_i \neq 0$, for $i = 1, 2$.
- (H2') There exist a C^1 function u defined on $[t_0,\infty)$, a C^1 function F on $\mathbb{R}$, and a continuous function G on $\mathbb{R}$ such that $F'(u) = \sqrt{k}G(u)$ and suppose that there exists $\alpha > 0$ such that $F(u) \geq \frac{(G(u))^2}{4\alpha}$.
- (H4') $h \in C^1([t_0,\infty),\mathbb{R})$ such that $h(t) \to \infty$ as $t \to \infty, h'(t) \geq \alpha$, $h(t) \leq t$ for all large t.

Let (H1), (H2) and (H4) in Theorem 2.1 be replaced by (H1'), (H2') and (H4'), respectively. Then (1.1) is oscillatory

We also consider the equation

$$x''(t) + p(t)f(t,x(t),x'(t))x'(t) + q(t)g(x[h_1(t)], x[h_2(t)],\dots,x[h_n(t)]) = 0, \quad (2.2)$$

where $p, q \in C([t_0,\infty),\mathbb{R}^+)$, $f \in C([t_0,\infty)\times\mathbb{R}^2,\mathbb{R}^+)$, $g \in C(\mathbb{R}^n,\mathbb{R})$, $g(y_1,\dots,y_n) > 0$ if $y_i > 0$, $g(y_1,\dots,y_n) < 0$ if $y_i < 0$ and $h_i \in C^1([t_0,\infty),\mathbb{R})$, for all $i = 1,2,3,\dots,n$.

For the study of (2.2), we consider the following hypotheses:

- (H1") $g(y_1, y_2,\dots,y_n)$ is a continuously differentiable function with respect to $y_1, y_2,\dots,y_n$. Suppose there exists $k > 0$ such that $\frac{\partial}{\partial y_i} g(y_1,y_2,\dots,y_n) \geq k/n > 0$, for $y_i \neq 0$, for $i = 1,2,3,\dots,n$.
- (H4") $h_i \in C^1([t_0,\infty),\mathbb{R})$ such that $h_i(t) \to \infty$ as $t \to \infty$, $h_i'(t) \geq 1$, $h_i(t) \leq t$ for all large t for $i = 1,2,3,\dots,n$.

Theorem 2.4. *Let* (H1), (H4) *in Theorem 2.1 be replaced by* (H1"), (H4")*, respectively. Then* (2.2) *is oscillatory.*

Proof. For sufficiently large T, on $[T,\infty)$, the following identity plays the role of identity (2.1):

$$
\begin{aligned}
&(u'(t))^2 - q(t)F(u(t)) + \frac{F(u(t))}{g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])} \\
&\times \big[x''(t) + p(t)f(t,x(t),x'(t))x'(t) + q(t)g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])\big] \\
&= \Big(\frac{x'(t)F(u(t))}{g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])}\Big)' \\
&\quad + \frac{x'(t)\frac{\partial}{\partial x[h_1(t)]} g\big(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)]\big)x'[h_1(t)]h_1'(t)F(u(t))}{\big(g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])\big)^2} + \dots \\
&\quad + \frac{x'(t)\frac{\partial}{\partial x[h_n(t)])} g\big(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)]\big)x'[h_n(t)]h_n'(t)F(u(t))}{\big(g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])\big)^2} \\
&\quad - \Big(\frac{x'(t)F'(u(t))u'(t)}{g(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)])}\Big) + (u'(t))^2 \\
&\quad + p(t)f(t,x(t),x'(t))x'(t)\frac{F(u(t))}{g\big(x[h_1(t)], x[h_2(t)], \dots, x[h_n(t)]\big)}.
\end{aligned}
$$

The rest of the proof of Theorem 2.4 is similar to the proof of Theorem 2.1, in view of hypotheses (H1") and (H4"). So, we omit the proof. $\square$

3. Examples

Finally, we give some examples to illustrate our results.

Example 3.1. Consider the differential equation

$$
x''(t) + (x'(t))^3 + x(t) + (x(t))^{2n+1} + x[t - \frac{1}{t+1}] + \big(x[t - \frac{1}{t+1}]\big)^{2m+1} = 0, \quad (3.1)
$$

for $t > 0$, $n, m \in \mathbb{N}$. This equation can be viewed as (1.1) with $p(t) = 1$, $f(t,x,y) = y^2$, $q(t) = 1$, $g(y_1, y_2) = y_1 + y_1^{2n+1} + y_2 + y_2^{2m+1}$, $h(t) = t - \frac{1}{t+1}$. With the choice of $k = 1$, $F(u) = u^2$, $u(t) = t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied; therefore, (3.1) is oscillatory.

Example 3.2. Consider the differential equation

$$
x''(t) + x'(t) + \big(e^{-t} + \frac{2}{t^2}\big)\big(x(t) + x(\frac{t}{2}) + x(\frac{t}{2})^3\big) = 0, \quad t > 0. \quad (3.2)
$$

This equation can be viewed as (1.1) with $p(t) = 1$, $f(t,x,y) = 1$, $q(t) = e^{-t} + \frac{2}{t^2}$, $g(y_1, y_2) = y_1 + y_2 + y_2^3$, $h(t) = \frac{t}{2}$. With the choice of $k = 1$, $F(u) = u^2$, $u(t) = t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied; therefore (3.2) is oscillatory.

Remark 3.3. Consider the differential equation

$$
x''(t) + x'(t) + \big(e^{-t} + \frac{4}{t^2}\big)\big(x(t) + x(\frac{t}{2}) + x(\frac{t}{2})^3\big) = 0, \quad t > 0. \quad (3.3)
$$

This equation can be viewed as (1.1) with $p(t) = 1$, $f(t,x,y) = 1$, $q(t) = e^{-t} + \frac{4}{t^2}$, $g(y_1, y_2) = y_1 + y_2 + y_2^3$, $h(t) = \frac{t}{2}$. In view of Remark 2.3, with the choice of $k = 1$, $F(u) = \frac{u^2}{2}$, $u(t) = t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied; therefore (3.3) is oscillatory.

Remark 3.4. By Theorem 2.1, the equation

$$x''(t) + \big(e^{-t} + \frac{2}{t^2}\big)\big(x(t) + x(\frac{t}{2}) + x(\frac{t}{2})^3\big) = 0, \quad t > 0 \tag{3.4}$$

is oscillatory, whereas none of the known criteria [2, 10, 11] can obtain this result.

Example 3.5. Consider the differential equation

$$x''(t) + \frac{1}{t}x'(t) + x(t) + (x(t))^3 + x[t - e^{-t}] + \big(x[t - e^{-t}]\big)^5 = 0, \quad t > 0. \tag{3.5}$$

This equation can be viewed as (1.1) with $p(t) = \frac{1}{t}$, $f(t, x, y) = 1$, $q(t) = 1$, $g(y_1, y_2) = y_1 + y_1^3 + y_2 + y_2^5$, $h(t) = t - e^{-t}$. With the choice of $k = 1$, $F(u) = u^2$, $u(t) = t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied; therefore, (3.5) is oscillatory.

Acknowledgments. Authors thank Prof. Julio G. Dix and the anonymous referee for their constructive suggestions. The first author would like to thank the National Board for Higher Mathematics (NBHM), DAE, Govt. of India for providing him a financial support under the grant no. 40/1/2008–R&D–II/3230.

References

[1] R. P. Agarwal, M. Bohner, Wan-Tong Li; *Nonoscillation and oscillation: theory for functional differential equations*, Marcel Dekker, New York, 2004.

[2] John S. Bradley; *Oscillation theorems for a second-order delay equation*, J. Diff. Eqs., **8** (1970), pp. 397–403.

[3] L. H. Erbe, Q. Kong, B. G. Zhang; *Oscillation theory for functional differential equations*, Dekker, New York, 1995.

[4] S. R. Grace and B. S. Lalli; *Oscillation theorems for damped differential equations of even order with deviatng arguments*, SIAM J. Math. Anal., **15**(2) 1984, pp. 308–316.

[5] S. R. Grace, B. S. Lalli and C. C. Yeh; *Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term*, SIAM J. Math. Anal., **15** (1984), pp. 1082–1093.

[6] J. R. Graef and Paul W. Spikes; *Sufficient conditions for nonoscillation of a second-order nonlinear differential equation*, Proc. Amer. Math. Soc., **50** (1975), pp. 289–292.

[7] V. Komkov; *A generalization of Leighton's variational theorem*, Applicable Analysis **1** (1972), 377–383.

[8] R. G. Koplatadze, G. Kvinikadze and I. P. Stavroulakis; *Oscillation of second-order linear delay differential equations*, Funct. Diff. Equ., **7** (2000), pp. 121–145.

[9] I. P. Stavroulakis; *Oscillation criteria for functional differential equations*, Electronic Journal of Differential Equations, Conference **12** (2005), pp. 171–180.

[10] C. C. Travis; *Oscilllation theorems for second-order differential equations with functional arguments*, Proc. Amer. Math. Soc., **31** (1972), pp. 199–202.

[11] C. C. Yeh; *An oscilllation criterion for second-order nonlinear differential equations with functional arguments*, J. Math. Anal. Appl., **76** (1980), pp. 72–76.

[12] E. M. E. Zayed and M. A. El–Moneam; *Some oscillation criteria for second-order nonlinear functional ordinary differential equations*, Acta Mathematica Scientia, **27B(3)** (2007), pp. 602–610.

Jagmohan Tyagi
TIFR Centre For Applicable Mathematics, Post Bag No. 6503, Sharda Nagar, Chikkabommasandra, Bangalore - 560065, Karnataka, India
E-mail address: jtyagi1@gmail.com

Venkataramanarao Raghavendra
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur - 208016, India
E-mail address: vrag@iitk.ac.in