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EXISTENCE AND ANALYTICITY OF A PARABOLIC EVOLUTION OPERATOR FOR NONAUTONOMOUS LINEAR EQUATIONS IN BANACH SPACES

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Dedicated to Professor Daniel Henry

Abstract. We give conditions for the parabolic evolution operator to be analytic with respect to a coefficient operator. We also show that the solution of a homogeneous parabolic evolution equation is analytic with respect to the coefficient operator and to the initial data. We apply our results to example that can not be studied by the standard methods.

1. Introduction

Despite the great development in the theory of nonlinear parabolic equations, some gaps remain in the theory of nonautonomous linear parabolic equations. To formulate the question more precisely, consider two Banach Spaces X, Y with $Y \subset X$, densely, with continuous immersion and call by $Z_\alpha = (X, Y)_\alpha$, $0 \leq \alpha < 1$, an interpolation space between X and Y obtained by a suitable interpolation method $(\ ,\ )_\alpha$. For all $t \in J$, where J is an interval, let $R(t), S(t)$ be closed linear operators in X with constant domain Y such that there exist parabolic evolution operators T_R and T_S satisfying the equations:

$$\begin{gathered}\frac{dT_R}{dt}(t,s) + R(t)T_R(t,s) = 0, \quad T_R(s,s) = I \\ \frac{dT_S}{dt}(t,s) + S(t)T_S(t,s) = 0, \quad T_S(s,s) = I,\end{gathered}$$

where $(t,s) \in \{(t,s) : t, s \in J, t > s\}$ and I is the identity operator in X. We have estimates such as

$$\|T_R(t,s) - T_S(t,s)\|_{\mathcal{L}(Z_\alpha, Z_\beta)} \leq c(t-s)^{\beta-\alpha} \max_{t\in J}\{\|R(t) - S(t)\|_{\mathcal{L}(Y,X)}\},$$

using many types of interpolation methods, where $c > 0$, $\alpha \in (0,1]$ and $\beta \in [0,1)$. In particular, if $\alpha = \beta$, roughly speaking, we have a Lipschitz continuous dependence of the evolution operator in relation to the operator. In fact, this seems be the best available result for parabolic evolution operators in infinite dimension. Here, in a less general setting, we present better results.

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Let X be a Banach space and A a constant linear closed operator in X and T_P the parabolic evolution operator which is the solution of the equation

$$\frac{dT_P}{dt}(t,s) + P(t)AT_P(t,s) = 0, \quad T_P(s,s) = I,$$

where $P = P(t)$ is a time dependent operator called here a coefficient operator, such that P varies in an open set of the space of the functions which are continuous functions from J to $\mathcal{L}(Z_\mu)$, with $Z_\mu = (X,Y)_\mu$ for some $\mu \in (0,1)$, and Hölder continuous from J to $\mathcal{L}(X)$. We define the open condition, putting the usual hypothesis to obtain the existence of a parabolic evolution operator. As an additional hypothesis, we suppose that $P(t)$ is an isomorphism in X and in X_μ. Thus, we prove that the evolution operator with respect to the coefficient operator is analytic and the solution of the equation

$$\frac{du}{dt} + P(t)Au(t) = 0, \quad u(s) = \xi$$

is analytic with respect to P and ξ.

The main references are: [5], [6] and [10], for the theory of parabolic evolution operators; [1] and [7], for the application of those operators in the context of the interpolation spaces; [3], for a direction and motivation in a geometrical point of view.

Finally, we need to observe that the equations which are being considered have operators with constant domain. This restriction limits the applications of the results obtained here in concrete cases, for example, in diffusion equations with time-dependent linear boundary conditions.

Notation: For the readers convenience, we introduce here the basic notation. When necessary, additional notation will be given. We refer to X, Y, Z, and so on, as complex Banach spaces, J as a real interval and $\mathcal{L}(X,Y)$ as the Banach space of all linear bounded operators from X to Y. If $X = Y$, we use $\mathcal{L}(X)$. For a linear operator T, we use $\rho(T)$ and $\sigma(T)$ as the resolvent and spectral set of T and we denote $\operatorname{Re}\sigma(T) > c$ as the subset $\{\lambda \in \sigma(T) | \operatorname{Re}(\lambda) > c\}$. Also, we denote as $C(J,Z)$ the Banach space of all bounded continuous functions u defined in J with values in Z with the norm given by

$$\max_{t\in J} \|u(t)\|_Z \,.$$

Moreover, for $\epsilon \in (0,1]$, $C^\epsilon(J,Z)$ denotes the Banach space of all Hölder continuous functions whose the norm of the space is the finite number

$$\sup_{t\in J} \|u(t)\|_Z + \sup_{t,s\in J, t\neq s} \frac{\|u(t)-u(s)\|_Z}{|t-s|^\epsilon}.$$

For the Banach space $X \cap Y$, we use the norm $\|w\|_{X\cap Y} = \min\{\|w\|_X, \|w\|_Y\}$ with $w \in X \cap Y$. Finally, the symbol Δ is the set $\{(t,s),\ t > s, t, s \in J\}$.

2. Analytic Semigroups and Interpolation Spaces

On using the interpolation spaces theory, we adopt a non-direct method. It consists on considering only the necessary properties to reach an estimate which is related to the analytic semigroups used to obtain the stated regularity results. In the following, only the first definition is unusual in classical books about interpolation space theory. Indeed, in such books, these properties are obtained as a consequence of an explicit definition for each interpolation method. The other two definitions are standard and included here for the sake of the understanding.

Notation: In the following, for any Banach space, for convenience, we denote $(X,Y)_0 := X$ and $(X,Y)_1 := Y$.

Definition 2.1. We say that an interpolation method $(,)_\alpha$ has the property 1 if for any two Banach spaces X, Y with $Y \subset X$, continuously, it is true that:

(i) Each $(X,Y)_\alpha$, $0 < \alpha < 1$, is a Banach space;
(ii) $(X,X)_\theta = X$, $(Y,Y)_\theta = Y$ with equivalent norms for each $\theta \in (0,1)$;
(iii) $(X,Y)_\alpha \subset (X,Y)_\beta$, continuously, if $\alpha \geq \beta$ with $\alpha, \beta \in [0,1]$.

Definition 2.2. We say that the interpolation method $(,)_\alpha$ has the reiteration property if for any two Banach spaces Y, Z, we have:

$$((Y,Z)_\alpha, (Y,Z)_\beta)_\theta = (Y,Z)_{(1-\theta)\alpha+\theta\beta}$$

with equivalent norms form each $\alpha, \beta \in [0,1]$ and $\theta \in (0,1)$.

Definition 2.3. We say that an interpolation method $(,)_\alpha$ has the interpolation property if, for all Banach spaces Z_1, Z_2, W_1, W_2 such that $W_1 \subset Z_1$, $W_2 \subset Z_2$, continuously, and for all $T \in \mathcal{L}(Z_1, Z_2) \cap \mathcal{L}(W_1, W_2)$, we have that, for each $\theta \in (0,1)$,

$$\|T\|_{\mathcal{L}((Z_1,W_1)_\theta,(Z_2,W_2)_\theta)} \leq c_0 \|T\|^{1-\theta}_{\mathcal{L}(Z_1,Z_2)} \|T\|^{\theta}_{\mathcal{L}(W_1,W_2)}$$

where $c_0 > 0$ does not depend on T.

Definition 2.4. Let X, Y be Banach spaces such that $Y \subset X$, continuously. We say that $(X,Y)_\alpha$ is an interpolation space between X and Y if the method $(,)_\alpha$ has the property 1, the interpolation and the reiteration properties.

In the following, if X_0, X_1 are two Banach spaces with $X_1 \subset X_0$, continuously, we denote as X_θ the interpolation space $(X_0, X_1)_\theta$ for $\theta \in (0,1)$. Calderon [12] and Hans Triebel [13, sections 1.9.3, theorem A and remark 1] give us that the complex interpolation method is an interpolation method as defined above. By Lunardi [7], the same is true for the real interpolation method.

Next, we consider sectorial operators, i.e., operators which generate analytic semigroups (for definition of sectorial operator and analytic semigroup see [3]). It is well known a sectorial operator A generates an analytic semigroup e^{-tA}. But here, we have to obtain estimates for the bounded operators e^{-tA} between interpolation spaces uniformly with respect to A, thus we need to consider a slight modification on the definition of sectorial operator.

Definition 2.5. Let X, Y be Banach spaces such that $Y \subset X$, continuous and densely. We define a family of sectorial operators in X with domain Y as any set $\mathcal{S}$ of closed linear operators in X such that:

(i) $D(S) = Y$ with uniformly equivalents norms for all $S \in \mathcal{S}$;
(ii) There exists ω and $\theta \in (0, \pi/2)$ such that the subset $S_{\omega,\theta} = \{\lambda \in \mathbb{C} | \arg(\lambda - \omega) > \pi/2 - \theta \text{ or } \lambda = \omega\}$ is in the resolvent set of each S, $S \in \mathcal{S}$, and $(|\lambda| + 1)\|(\lambda - S)^{-1}\|_{\mathcal{L}(X)}$ is uniformly bounded for all $\lambda \in S_{\omega,\theta}$ and $S \in \mathcal{S}$.

Proposition 2.6. *Let $\mathcal{S}$ be a family of sectorial operators in X_0 with domain X_1. If all $S \in \mathcal{S}$ have $\operatorname{Re}\sigma(S) > \omega$, for a constant ω, then there exist $c, c' > 0$ such that*

(i) $\|e^{-tS}\|_{L(X_\alpha, X_\beta)} \leq c(1 + t^{-1})^{\beta-\alpha} e^{-\omega t}$ *for* $t > 0$ *and* $0 \leq \alpha \leq \beta \leq 1$*;*
(ii) $\|Se^{-tS}\|_{\mathcal{L}(X_\alpha, X_\beta)} \leq c'(1 + t^{-1})^{1+\beta-\alpha} e^{-\omega t}$ *for* $t > 0$ *and* $0 \leq \alpha \leq \beta \leq 1$*.*

Proof. By [3, Theorem 1.3.4], there exist $c_1, c_2 > 0$ such that
$$\|e^{-tS}\|_{\mathcal{L}(X_0)} \le c_1 e^{-\omega t}, \quad \|Se^{-tS}\|_{\mathcal{L}(X_0)} \le c_2 t^{-1} e^{-\omega t}$$
for all $S \in \mathcal{S}$. Call m_1 and m_2 two positive numbers such that:
$$m_1 \|y\|_{D(S)} \le \|y\|_{X_1} \le m_2 \|y\|_{D(S)}$$
for all $S \in \mathcal{S}$ in which we have denoted $\|y\|_{D(S)} = \|y\|_{X_0} + \|Sy\|_{X_0}$. So
$$\|e^{-tS}\|_{\mathcal{L}(X_0,X_1)} \le c_3(1+t^{-1})e^{-\omega t}$$
where $c_3 = m_2 \max\{c_1, c_2\}$. Suppose $y \in X_1$, $t > 0$. Since $Se^{-tS}y = e^{-tS}Sy$, we have that
$$\|e^{-tS}\|_{\mathcal{L}(X_1)} \le \frac{c_1 m_2}{m_1} e^{-\omega t}.$$
For $\alpha \in (0,1)$, $X_\alpha = (X_0, X_1)_\alpha$ and $X_1 = (X_1, X_1)_\alpha$ with equivalent norms, calling $c_4 > 0$ for $\|y\|_{X_1} \le c_4 \|y\|_{(X_1,X_1)_\alpha}$, we have that:
$$\|e^{-tS}\|_{\mathcal{L}(X_\alpha,X_1)} \le c_4 \|e^{-tS}\|_{\mathcal{L}((X_0,X_1)_\alpha,(X_1,X_1)_\alpha)}.$$
But, by the interpolation property,
$$\|e^{-tS}\|_{\mathcal{L}((X_0,X_1)_\alpha,(X_1,X_1)_\alpha)} \le c_0 \|e^{-tS}\|^{1-\alpha}_{\mathcal{L}(X_0,X_1)} \|e^{-tS}\|^{\alpha}_{\mathcal{L}(X_1,X_1)}$$
so
$$\|e^{-tS}\|_{\mathcal{L}(X_\alpha,X_1)} \le c_0 (c_1 m_2/m_1)^\alpha c_3^{1-\alpha} c_4 (1+t^{-1})^{1-\alpha} e^{-\omega t}.$$
Now, by the reiteration property, for $\alpha \in [0,1)$, $\beta \in (0,1)$, $X_\alpha = (X_0, X_{\alpha/\beta})_\beta$, so taking c_5 such that $\|w\|_{(X_0,X_{\alpha/\beta})_\beta} \le c_5 \|w\|_{X_\alpha}$ for all $w \in X_\alpha$, we have that:
$$\|e^{-tS}\|_{\mathcal{L}(X_\alpha,X_\beta)} \le c_5 \|e^{-tS}\|_{\mathcal{L}((X_0,X_{\alpha/\beta})_\beta,(X_0,X_1)_\beta)}.$$
Thus the interpolation property gives
$$\|e^{-tS}\|_{\mathcal{L}(X_\alpha,X_\beta)} \le c_0 c_5 \|e^{-tS}\|^{1-\beta}_{\mathcal{L}(X_0)} \|e^{-tS}\|^{\beta}_{\mathcal{L}(X_{\alpha/\beta},X_1)},$$
or, if $\alpha \le \beta$,
$$\|e^{-tS}\|_{\mathcal{L}(X_\alpha,X_\beta)} \le c(1+t^{-1})^{\beta-\alpha} e^{-\omega t}$$
where $c = c_0^{1+\beta} c_1^{1-\beta+\alpha} c_3^{\beta-\alpha} c_4^{\beta} c_5 (m_2/m_1)^\alpha$. Finally, since, for $t > 0$, $Se^{-tS} = e^{-tS/2} S e^{-tS/2}$, we have:
$$\|Se^{-tS}\|_{\mathcal{L}(X_\alpha,X_\beta)} \le \|e^{-tS/2}\|_{\mathcal{L}(X_0,X_\beta)} \|S\|_{\mathcal{L}(X_1,X_0)} \|e^{-tS/2}\|_{\mathcal{L}(X_\alpha,X_1)}$$
or
$$\|Se^{-tS}\|_{\mathcal{L}(X_\alpha,X_\beta)} \le c'(1+t^{-1})^{\beta-\alpha+1} e^{-\omega t},$$
where $c' = 2^{\beta-\alpha+1} c_0^{2+\beta} c_1^{1-\beta+\alpha} c_3^{1+\beta-\alpha} c_4^{1+\beta} m_2^\alpha / m_1^{1+\alpha}$. □

3. Topology

We start this section with a preliminary result on linear operators in a way we have not seen in classical references such as [2] or [4].

Proposition 3.1. *Let A be a linear closed operator, densely defined in a Banach space X, and let Y be the domain of A with the graph norm (or only that Y is a Banach space, continuously immersed in X, such that $D(A) \subset Y$, continuously). Then*

(a) *The normed space $D(A+H)$ with the graph norm satisfies $D(A) \subset D(A+H)$, continuously, for any $H \in \mathcal{L}(Y,X)$ and uniformly in a bounded subset of $\mathcal{L}(Y,X)$;*

(b) *If $\rho(A)$ is not void, we have that $D(A+H) \subset D(A)$, continuously for all $H \in \mathcal{L}(Y,X)$, such that $\|H\|_{\mathcal{L}(Y,X)} \le \|(\omega - A)^{-1}\|^{-1}_{\mathcal{L}(X,Y)}$ for any $\omega \in \rho(A)$, and with uniformly continuous immersion for all H, with $\|H\|_{\mathcal{L}(Y,X)} \le l\|(\omega - A)^{-1}\|^{-1}_{\mathcal{L}(X,Y)}$ and $0 < l < 1$. In any case, $\omega \in \rho(A+H)$.*

Proof. Let $m_1 > 0$ be such that $m_1\|y\|_Y \le \|y\|_{D(A)}$. So

$$\|y\|_{D(A+H)} = \|y\|_X + \|(A+H)y\|_X \le (1 + \frac{\|H\|_{\mathcal{L}(Y,X)}}{m_1})\|y\|_{D(A)}$$

which proves item (a).

The proof of (b) is more delicate. Take $\omega \in \rho(A)$. Firstly, we recall that $\omega \in \rho(A+H)$ if $\|H\|_{\mathcal{L}(Y,X)}$ is sufficiently small. In fact:

$$\omega - (A+H) = (I - H(\omega - A)^{-1})(\omega - A)\,,$$

so if $\|H\|_{\mathcal{L}(Y,X)}\|(\omega - A)^{-1}\|_{\mathcal{L}(X,Y)} = h < 1$, h depending on H or $\|H\|_{\mathcal{L}(Y,X)}\|(\omega - A)^{-1}\|_{\mathcal{L}(X,Y)} = l$ with $0 < l < 1$, l a constant, that is true and

$$\|(\omega - (A+H))^{-1}\|_{\mathcal{L}(X)} \le \frac{1}{1-l_1}\|(\omega - A)^{-1}\|_{\mathcal{L}(X)}\,,$$

where $l_1 = h$ or $l_1 = l$. We also observe that

$$\|(\omega - A)^{-1}\|_{\mathcal{L}(X,Y)} \le \frac{1}{m_1}(1 + (|\omega| + 1)\|(\omega - A)^{-1}\|_{\mathcal{L}(X)})\,.$$

Then, writing $A = A + H - \omega + \omega - H$, we have that

$$A = (I + (\omega - H)(A + H - \omega)^{-1})(A + H - \omega)\,.$$

As a necessary step, we estimate $A(A+H-\omega)^{-1}$, following from the identity:

$$A(A+H-\omega)^{-1} = A(A-\omega)^{-1}(I + H(A-\omega)^{-1})^{-1}\,.$$

Thus,

$$\|A(A+H-\omega)^{-1}\|_{\mathcal{L}(X)} \le \frac{1 + |\omega|\|(A-\omega)^{-1}\|_{\mathcal{L}(X)}}{1-l_1}$$

which implies

$$\|A(A+H-\omega)^{-1}\|_{\mathcal{L}(X,Y)} \le \frac{1}{m_1(1-l_1)}(1 + (|\omega|+1)\|(A-\omega)^{-1}\|_{\mathcal{L}(X)})\,.$$

Finally, for any $y \in Y$,

$$\begin{aligned}\|Ay\| \le (1 + \frac{\omega}{1-l_1}\|(\omega - A)^{-1}\|_{\mathcal{L}(X)} \\ + \frac{\|H\|_{\mathcal{L}(Y,X)}}{m_1(1-l_1)}(1 + (1+|\omega|)\|(A-\omega)^{-1}\|_{\mathcal{L}(X)}))(\|(A+H)y\| + |\omega|\|y\|)\,.\end{aligned}$$

Calling the first factor l_2, we obtain that

$$\|Ay\| \le l_2(\|(A+H)y\| + |\omega|\|y\|)\,,$$

or

$$\|y\|_{D(A)} \le \max\{l_2, l_2|\omega| + 1\}\|y\|_{D(A+H)}\,,$$

which concludes the proof. $\square$

Proposition 3.2. *Let $\mathcal{S}$ be a family of sectorial operators in X with domain Y. Then there exists an open set V in $\mathcal{L}(Y,X)$ which contains $\mathcal{S}$ and is a family of sectorial operators in X with domain Y. Moreover, V can be taken, for a fixed $r>0$, as*

$$V = \cup_{A\in\mathcal{S}} B(A,r)$$

in which $B(A,r)$ in $\mathcal{L}(Y,X)$ is the ball of center A and radius r. The value of r can be chosen as any $r < m_1/(M+1)$ in which m_1 is the immersion constant of $D(A)\subset Y$ and M is such that $(|\lambda|+1)\|(\lambda-S)^{-1}\|_{\mathcal{L}(X)} \le M$ for all $S\in\mathcal{S}$ and $\lambda\in S_{\omega,\theta}$ for those ω and θ which define the family $\mathcal{S}$.

Proof. Take $m_1>0$ such that $m_1\|y\|_Y \le \|y\|_{D(A)}$ for all $y\in Y$ and $A\in\mathcal{S}$, and $M>0$ such that $(|\lambda|+1)\|(\lambda-S)^{-1}\|_{\mathcal{L}(X)} \le M$ for all $S\in\mathcal{S}$ and $\lambda\in S_{\omega,\theta}$ for some ω and $\theta\in(0,\pi/2]$. Since, for all $A\in\mathcal{S}$,

$$\|(\omega-A)^{-1}\|_{\mathcal{L}(X,Y)} \le \frac{1}{m_1}(1+(|\omega|+1)\|(\omega-A)^{-1}\|_{\mathcal{L}(X)})$$

so

$$\|(\omega-A)^{-1}\|_{\mathcal{L}(X,Y)} \le \frac{M+1}{m_1}$$

and, by Proposition 3.1, for a fix $r>0$, $r<\frac{m_1}{M+1}$, the first condition of sectorial family operators is true for $\cup_{A\in\mathcal{S}}B(A,r)$. Proceeding as in the proof of the last proposition, we obtain that

$$\|(\lambda-(A+H))^{-1}\|_{\mathcal{L}(X)} \le \frac{\|(\lambda-A)\|_{\mathcal{L}(X)}}{1-r\frac{M+1}{m_1}} < 1\,,$$

if $\|H\|_{\mathcal{L}(Y,X)} < \frac{m_1}{M+1}$. So the condition (ii) of the sectorial family definition is true for the subset $\cup_{A\in\mathcal{S}}B(A,r)$, with the same parameters θ and ω of $\mathcal{S}$. $\square$

Notation: Now, we design an open set which contains the coefficient operators $P(t)$ for which not only there is a parabolic evolution operator T_P which satisfies the equation

$$\frac{dT_P}{dt}(t,s) + P(t)AT_P(t,s) = 0\,,\ t>s\,, T_P(s,s) = I$$

$t,s\in J$, but also such that it can be conveniently estimated.

Proposition 3.3. *Let $\epsilon\in(0,1]$, $\mu\in(0,1)$ and A be a linear closed operator densely defined in X_0 with domain X_1. If W is the subset of any $P\in C^\epsilon(J,\mathcal{L}(X_0))\cap C(J,\mathcal{L}(X_\mu))$ which satisfies the following conditions:*

(i) *$\{P(t)A, t\in J\}$ is a sectorial family in X_0 with domain X_1;*
(ii) *$P(t): X_0\to X_0$ and $P(t): X_\mu\to X_\mu$ are isomorphisms for all $t\in J$;*
(iii) *$\|P^{-1}(t)\|_{\mathcal{L}(X_0)}$ and $\|P^{-1}(t)\|_{\mathcal{L}(X_\mu)}$ are both uniformly bounded for all $t\in J$.*

Then W is an open set in $C^\epsilon(J,\mathcal{L}(X_0))\cap C(J,\mathcal{L}(X_\mu))$. Moreover, given a set $V_0\in W$ such that the conditions (i), (ii), (iii) are satisfied uniformly for all $P\in V_0$, then there exists an open set $V\supset V_0$ in $C^\epsilon(J,\mathcal{L}(X_0))\cap C(J,\mathcal{L}(X_\mu))$ such that the conditions are satisfied uniformly for all $P\in V$. Indeed, the subset V can be taken as $V=\cup_{P\in V_0}B(P,r)$ for a fix $r>0$.

Proof. Take $P\in W$. Proposition 3.2 states the existence an open set V' in $\mathcal{L}(X_1,X_0)$ such that $V'\supset\{P(t)A, t\in J\}$ which can be chosen as

$$V' = \cup_{t\in J}B(P(t)A, r_1),$$

for a fix $r_1 > 0$, denoting $B(P(t)A, r_1)$ as the open ball in $\mathcal{L}(X_1, X_0)$ with center $P(t)A$ and radius r_1. Let $B(P, r_1/\|A\|_{\mathcal{L}(X_1,X_0)})$ be the ball in $C(J, \mathcal{L}(X_0))$ with center P and radius $r_1/\|A\|_{\mathcal{L}(X_1,X_0)}$. So, if $Q \in B(P, r_1/\|A\|_{\mathcal{L}(X_1,X_0)})$ then

$$\|(Q(t) - P(t))A\|_{\mathcal{L}(X_1,X_0)} < r_1$$

yielding that the set $\{Q(t)A | Q \in B(P, r_1/\|A\|_{\mathcal{L}(X_1,X_0)}), t \in J\}$ is a sectorial family in X_0 with domain X_1. Take now M_1 such that $\|P^{-1}(t)\|_{\mathcal{L}(X_0)} \le M_1$, for all $t \in J$. The Identity Perturbation Theorem gives that $\|Q^{-1}(t)\|_{\mathcal{L}(X_0)}$ is uniformly bounded if $Q \in B(P, r_2)$, such that $B(P, r_2)$ is the ball in $C(J, \mathcal{L}(X_0))$ of center P and radius r_2, with $r_2 < 1/M_1$. In fact,

$$\|Q^{-1}(t)\|_{\mathcal{L}(X_0)} \le \frac{M_1}{1 - r_2 M_1}\,.$$

By the same argument, $\|Q^{-1}(t)\|_{\mathcal{L}(X_\mu)} \le M_2/(1 - r_3 M_2)$ if $Q \in B(P, r_3) \subset C(J, \mathcal{L}(X_\mu))$, where $r_3 < 1/M_2$. Since $C^\epsilon(J, \mathcal{L}(X_0)) \subset C(J, \mathcal{L}(X_0))$, continuously, (the immersion constant can be taken as 1), if $r = \min\{r_1, r_1/\|A\|_{\mathcal{L}(X_1,X_0)}, r_2, r_3\}$, then the ball $B(P, r)$ of $C^\epsilon(J, \mathcal{L}(X_0)) \cap C(J, \mathcal{L}(X_\mu))$ is in W. Finally, for any $P \in V_0$, following the above argument, it can be taken r_1, r_2, r_3 independent of P, so $\cup_{P \in V_0} B(P, r)$ has the enunciated properties. $\square$

4. Estimates for the parabolic evolution operator

The basic properties of the parabolic evolution operators in many contexts may be obtained from the classical works of Sobolevskii [10], Kato [4] and [5], Tanabe [11], Pazy [9] or from the recent of Amann [1] or Lunardi [7]. Here, before going to the estimates, we give a definition and a condition for its existence. So, let X and Y be Banach spaces such that $Y \subset X$ continuous and densely and suppose that $S(t)$, $t \in J$, where J is an interval, is a closed linear operator in X with domain Y and, for each $t \in J$, it generates an analytical semigroup $e^{-rS(t)}$, $r \ge 0$. Thus, we define the parabolic evolution operator for the equation $x'(t) + S(t)x(t) = 0$, $t \in J$, as the operator $T(t, s)$ which has the following properties:

- (i) for all $t, s \in J$, $\mathcal{L}(X) \ni T(t, s)$ is differentiable with respect to t, $t \in J$, in $\mathcal{L}(X)$ and $T(t, s) \in Y$ if $t > s$, $t \in J$;
- (ii) $\frac{\partial T(t,s)}{\partial t} + S(t)T(t, s) = 0$, $t \in J$, $t > s$, and $T(s, s) = I$.

Proposition 4.1. *Let $\{S(t), t \in J\}$ be a family of sectorial operators in X with domain Y and suppose that $S \in C^\epsilon(J, \mathcal{L}(Y, X))$ for some $\epsilon \in (0, 1]$. Then there is a unique parabolic evolution operator for the equation $x'(t) + S(t)x(t) = 0$, $t \in J$.*

For a proof of the above proposition, see [1, 4, 9, 10, 11].

Next, we present a type of singular Gronwall inequality. In fact, this is the kernel of the estimates and so we try to obtain a clear form for the constants.

Proposition 4.2. *Suppose $\beta \in (0, 1]$ and $x > 0$. So, for any $\delta > 0$, we have the estimate*

$$\sum_{i=1}^{\infty} \frac{x^{i-1}}{\Gamma(\beta i)} \le c_1 e^{(1+\delta)x^{1/\beta}}$$

in which, as follows from Amann [1, Section 3.2], *$c_1 = c_1(\beta, \delta)$ can be taken as*

$$c_1 = \max_{y \in [0,1]} \Big(\frac{e^{\frac{1+\delta}{\beta} y^{1/\beta}} - 1}{y}\Big)^\beta (2\pi)^{\frac{\beta-1}{2}} e^{\frac{\beta}{12}} \beta^{\frac{1}{12}} \sum_{i=1}^{\infty} \frac{i^{\frac{1+\beta}{2(1-\beta)}}}{(1+\delta)^{\frac{\beta}{1-\beta} i}}$$

if $\beta \in (0,1)$ *and as* $c_1 = 1$ *(including* $\delta = 0$*) if* $\beta = 1$.

Now suppose $b \geq 0$, $a(t)$ is a locally integrable non-negative function on $0 \leq t < T$ (some $T \leq \infty$)) and suppose $u(t)$ is non-negative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds$$

on this interval; then

$$u(t) \leq a(t) + bc_1\Gamma(\beta) \int_0^t (t-s)^{\beta-1} \exp[(1+\delta)(b\Gamma(\beta))^{1/\beta}(t-s)] a(s) ds$$

on $0 \leq t < T$.

Proof. By Henry [3, Lemma 7.1.1]

$$u(t) \leq a(t) + \int_0^t \sum_{i=1}^{\infty} \frac{(b\Gamma(\beta))^i (t-s)^{i\beta-1}}{\Gamma(i\beta)} a(s) ds$$

and so the conclusion is immediate. □

Notation. In the following, consider a family of sectorial operators of W, which is denoted as $\mathcal{S}$, such that for all $P \in \mathcal{S}$, we call a_0, b_0, $a_{\alpha,\beta}$, $b_{\alpha,\beta}$, $\overline{b}$ and $\overline{a}$ and $\overline{a}_\mu$ constants such that:

(1) $\|P(t)Ae^{-(t-s)P(t)A}\|_{\mathcal{L}(X_0)} \leq a_0(t-s)^{-1}e^{-\omega(t-s)}$;
(2) $\|e^{-(t-s)P(t)A}\|_{\mathcal{L}(X_0)} \leq b_0 e^{-\omega(t-s)}$;
(3) $\|Ae^{-(t-s)P(t)A}\|_{\mathcal{L}(X_\alpha,X_\beta)} \leq a_{\alpha,\beta}(1+(t-s)^{-1})^{1+\beta-\alpha}e^{-\omega(t-s)}$;
(4) $\|e^{-(t-s)P(t)A}\|_{\mathcal{L}(X_\alpha,X_\beta)} \leq b_{\alpha,\beta}(1+(t-s)^{-1})^{\beta-\alpha}e^{-\omega(t-s)}$;
(5) $\|P(t)-P(s)\|_{\mathcal{L}(X_0)} \leq \overline{b}|t-s|^\epsilon$;
(6) $\|P^{-1}(t)\|_{\mathcal{L}(X_0)} \leq \overline{a}$;
(7) $\|P^{-1}(t)\|_{\mathcal{L}(X_\mu)} \leq \overline{a}_\mu$;

for all $t \in J$.

Obviously, the existence of these constants is given by Proposition 2.6 and by the definition of W. It is convenient to observe, from the proof of that proposition, that it can be taken a constant c, not dependent on α and β, such that $a_{\alpha,\beta} \leq c$ and $b_{\alpha,\beta} \leq c$. Others constants, which depend on the constants defined above, can be defined in the next propositions. Concerning the way we proceed to obtain the estimates, it was necessary a little bit of analysis to allow that the interval of the estimates could be infinite.

Proposition 4.3. *Suppose* $\alpha \in (0,1]$. *Then*

$$\|AT_P(t,s)\|_{\mathcal{L}(X_\alpha,X_0)} \leq a_{\alpha,0}(1+(t-s)^{-1})^{1-\alpha}e^{-\Omega(t-s)}(1+m(t-s)^\epsilon)\,,$$

$t > s$, *in which* $\Omega = \omega - (1+\delta)(a_0\overline{a}\overline{b}\Gamma(\epsilon))^{\frac{1}{\epsilon}}$ *and* $m = a_0\overline{a}bc_1\Gamma(\epsilon)B(\epsilon,\alpha)$, *where* $c_1 = c_1(\epsilon,\alpha)$.

Proof. By the properties of the evolution operator, we have the relation:

$$T_P(t,s) = e^{-(t-s)P(t)A} + \int_s^t e^{-(t-\tau)P(t)A}(P(\tau)-P(t))AT_P(\tau,s)d\tau\,.$$

So, applying A and taking the norms,

$$\begin{aligned}
&\|AT_P(t,s)\|_{\mathcal{L}(X_\alpha,X_0)}\\
&\le \|Ae^{-(t-s)P(t)A}\|_{\mathcal{L}(X_\alpha,X_0)}\\
&\quad + \int_s^t \|Ae^{-(t-\tau)P(t)A}\|_{\mathcal{L}(X_0)}\|P(\tau)-P(t)\|_{\mathcal{L}(X_0)}\|AT_P(\tau,s)\|_{\mathcal{L}(X_\alpha,X_0))}d\tau\,,
\end{aligned}$$

in which the insertion of A in the integral is valid because A is closed. With the above constants and changing the variables to $r=t-s$, $\tau'=\tau-s$ and calling $u(r)=\|e^{\omega r}AT_P(r+s,s)\|_{\mathcal{L}(X_\alpha,X_0)}$, we obtain

$$u(r)\le a_{\alpha,0}(1+r^{-1})^{1-\alpha}+a_0\overline{a}\overline{b}\int_0^r (r-\tau')^{\epsilon-1}u(\tau')d\tau'\,.$$

So, dropping the $'$ and applying the singular Gronwall inequality

$$\begin{aligned}
&u(r)\le a_{\alpha,0}(1+r^{-1})^{1-\alpha}+a_{\alpha,0}a_0\overline{a}\overline{b}c_1\Gamma(\epsilon)\\
&\qquad\times\int_0^t (r-\tau)^{\epsilon-1}(1+\tau^{-1})^{1-\alpha}\exp[(1+\delta)(a_0\overline{a}\overline{b}\Gamma(\epsilon))^{1/\epsilon}(r-\tau)]d\tau\,.
\end{aligned}$$

Either

$$u(r)\le a_{\alpha,0}(1+r^{-1})^{1-\alpha}(1+a_0\overline{a}\overline{b}c_1\Gamma(\epsilon)B(\epsilon,\alpha)r^\epsilon\exp[(1+\delta)(a_0\overline{a}\overline{b}\Gamma(\epsilon))^{\frac{1}{\epsilon}}r]),$$

or

$$u(r)\le a_{\alpha,0}(1+r^{-1})^{1-\alpha}\exp[(1+\delta)(a_0\overline{a}\overline{b}\Gamma(\epsilon))^{\frac{1}{\epsilon}}r](1+a_0\overline{a}\overline{b}c_1\Gamma(\epsilon)B(\epsilon,\alpha)r^\epsilon)\,.$$

Then, coming back the variables, the proof is complete. □

Proposition 4.4. *Suppose* $0<\alpha\le\beta\le 1$. *Then*

$$\|T_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\beta)}\le (1+(t-s)^{-1})^{\beta-\alpha}e^{-\Omega(t-s)}p_1(t-s),$$

$t>s$, *where* $p_1(t-s)=(m_1+(m_2(t-s)^\epsilon+m_3(t-s)^{2\epsilon})(1+t-s))$, $m_1=b_{\alpha,\beta}$, $m_2=B(1-\beta+\epsilon,\alpha)b_{\alpha,\beta}\overline{b}a_{\alpha,0}$, $m_3=mB(1-\beta+\epsilon,\alpha+\epsilon)b_{0,\beta}\overline{b}a_{\alpha,0}$.

Proof. We have

$$\begin{aligned}
&\|T_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\beta)}\\
&\le \|e^{-(t-s)P(t)A}\|_{\mathcal{L}(X_\alpha,X_\beta)}\\
&\quad+\int_s^t \|e^{-(t-\tau)P(t)A}\|_{\mathcal{L}(X_0,X_\beta)}\|P(\tau)-P(t)\|_{\mathcal{L}(X_0)}\|AT_P(\tau,s)\|_{\mathcal{L}(X_\alpha,X_0)}d\tau\,.
\end{aligned}$$

So, with the notation of the above proposition,

$$\begin{aligned}
&\|T_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\beta)}\\
&\le b_{\alpha,\beta}(1+(t-s)^{-1})^{\beta-\alpha}e^{-\omega(t-s)}+b_{0,\beta}\overline{b}a_{\alpha,0}e^{-\Omega(t-s)}\\
&\quad\times\int_s^t (1+(t-\tau)^{-1})^\beta(t-\tau)^\epsilon(1+(\tau-s)^{-1})^{1-\alpha}(1+m(\tau-s)^\epsilon)d\tau\,.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_s^t (1+(t-\tau)^{-1})^\beta(t-\tau)^\epsilon(1+(\tau-s)^{-1})^{1-\alpha}(1+m(\tau-s)^\epsilon)d\tau\\
&\le (1+(t-s)^{-1})^{\beta-\alpha}(1+(t-s))(B(1-\beta+\epsilon,\alpha)(t-s)^\epsilon\\
&\quad+mB(1-\beta+\epsilon,\alpha+\epsilon)(t-s)^{2\epsilon})\,,
\end{aligned}$$

the proof is complete. □

Proposition 4.5. *Suppose $0 < \alpha \le 1$ and $0 \le \mu < \epsilon$. Then*
$$\|AT_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\mu)} \le (1+(t-s)^{-1})^{1+\mu-\alpha} e^{-\Omega(t-s)} p_2(t-s)\,,$$
where $p_2(t-s) = (n_1 + (n_2 + (t-s)^\epsilon + n_3(t-s)^{2\epsilon})(1+t-s))$, $n_1 = a_{\alpha,\mu}$, $n_2 = a_{0,\mu}a_{\alpha,0}\bar{b}B(\epsilon,\alpha)$, $n_3 = a_{0,\mu}a_{\alpha,0}\bar{b}B(\epsilon-\mu,\alpha+\epsilon)m$.

Proof. We have
$$\begin{aligned}
&\|AT_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\mu)}\\
&\le \|Ae^{-(t-s)P(t)A}\|_{\mathcal{L}(X_\alpha,X_\mu)}\\
&\quad + \int_s^t \|Ae^{-(t-s)P(t)A}\|_{\mathcal{L}(X_0,X_\mu)} \|P(\tau)-P(t)\|_{\mathcal{L}(X_0)} \|AT_P(\tau,s)\|_{\mathcal{L}(X_\alpha,X_0)} d\tau
\end{aligned}$$
which implies
$$\begin{aligned}
&\|AT_P(t,s)\|_{\mathcal{L}(X_\alpha,X_\mu)}\\
&\le a_{\alpha,\mu}(1+(t-s)^{-1})^{1+\mu-\alpha}e^{-\omega(t-s)} + a_{0,\mu}a_{\alpha,0}e^{-\Omega(t-s)}\\
&\quad\times \int_s^t (1+(t-\tau)^{-1})^{1+\mu}(1+(\tau-s)^{-1})^{1-\alpha}(t-\tau)^\epsilon(1+m(\tau-s)^\epsilon)d\tau\,.
\end{aligned}$$
Proceeding as before, the proof is complete. □

Proposition 4.6. *Suppose $\epsilon \in (0,1]$, $0 < \mu < \epsilon$ and $\mu < \alpha \le 1$. Then*
$$\begin{aligned}
&\|A(T_P(t,s) - T_Q(t,s))\|_{\mathcal{L}(X_\alpha,X_0)}\\
&\le e^{-\Omega(t-s)} \max_{\tau\in[s,t]} \|P(\tau)-Q(\tau)\|_{\mathcal{L}(X_\mu)}(1+(t-s)^{-1})^{1-\alpha}p(t-s),
\end{aligned}$$
where $p(t-s) = \sum_{\text{index}}(1+t-s)(t-s)^{(\alpha_1+\alpha_2)\epsilon+\delta}B(\alpha_1\epsilon+\mu,\alpha_2\epsilon+\alpha+\delta-\mu)c_{\alpha_1,\alpha_2,\delta}$ and such that the index set is $0 \le \alpha_1 \le 1$, $0 \le \alpha_2 \le 2$, $0 \le \delta \le 1$, $\alpha_1,\alpha_2,\delta \in \mathbb{Z}$, and the coefficients $c_{\alpha_1,\alpha_2,\delta}$ can be determined in the last inequality of the proof below.

Proof. The properties of the evolution operator give
$$T_P(t,s) - T_Q(t,s) = -\int_s^t T_Q(t,\tau)(P(\tau)-Q(\tau))AT_P(\tau,s)d\tau\,.$$
Then
$$\begin{aligned}
&\|A(T_P(t,s)-T_Q(t,s))\|_{\mathcal{L}(X_\alpha,X_0)}\\
&\le -\int_s^t \|AT_Q(t,\tau)\|_{\mathcal{L}(X_\mu,X_0)}\|(P(\tau)-Q(\tau))\|_{\mathcal{L}(X_\mu)}\|AT_P(\tau,s)\|_{\mathcal{L}(X_\alpha,X_\mu)}d\tau\,.
\end{aligned}$$
yielding
$$\|A(T_P(t,s)-T_Q(t,s))\|_{\mathcal{L}(X_\alpha,X_0)} \le e^{-\Omega(t-s)} \max_{\tau\in[s,t]}\|P(\tau)-Q(\tau)\|_{\mathcal{L}(X_\mu)}a_{\mu,0}I_1$$
in which
$$\begin{aligned}
I_1 = \int_s^t &(1+(t-\tau)^{-1})^{1-\mu}(1+m(t-\tau)^\epsilon(1+(\tau-s)^{-1})^{1+\mu-\alpha}\\
&\times (n_1 + (n_2(\tau-s)^\epsilon + n_3(\tau-s)^{2\epsilon})(1+\tau-s))d\tau\,.
\end{aligned}$$

Observing that

$$\begin{aligned}&\int_s^t (1+(t-\tau)^{-1})^{1-\mu}(t-\tau)^{\alpha_1\epsilon}(1+(\tau-s)^{-1})^{1+\mu-\alpha}(\tau-s)^{\alpha_2\epsilon+\delta}d\tau\\&\leq (1+(t-s)^{-1})^{1-\alpha}(1+t-s)(t-s)^{(\alpha_1+\alpha_2)\epsilon}B(\alpha_1\epsilon+\mu,\alpha_2\epsilon+\alpha+\delta-\mu)\,.\end{aligned}$$

The result is concluded. □

5. Analyticity

The construction of a convenient topology gives the necessary tool to ask about the regularity of the evolution operator in relation to the coefficient operator, which is done now.

Theorem 5.1. *Suppose $\epsilon \in (0,1)$, $\mu \in (0,\epsilon)$ and J is a finite interval. Then the map*

$$\begin{gathered}P \to \{T_P(t,s) : t > s, t, s \in J\} :\\ W \subset C^\epsilon(J,\mathcal{L}(X_0)) \cap C(J,\mathcal{L}(X_\mu)) \to C(\Delta,\mathcal{L}(X_\alpha))\end{gathered}$$

is analytic if $\alpha \in (0,1)$ and if $X_\alpha \subset X_0$, continuously and densely, then

$$\begin{gathered}(P,\xi) \to \{T_P(t,s)\xi : t \geq s, t, s \in J\} :\\ W \times X_\alpha \subset C^\epsilon(J,\mathcal{L}(X_0)) \cap C(J,\mathcal{L}(X_\mu)) \times X_\alpha \to C(\overline{\Delta},\mathcal{L}(X_\alpha))\end{gathered}$$

is also analytic.

Furthermore, let $J = [0,T]$, $T < \infty$, and $G(P,f)(t) = \int_0^t T_P(t,s)f(s)ds$. The map

$$(P,f) \to G(P,f) : W \times C(J,X_\beta) \to C(J,X_\alpha)$$

is analytic if $\alpha \in [0,1)$ and $\mu < \beta \leq 1$ and $\beta \geq \alpha$.

Proof. The well definition of the the first map follows from the properties of the evolution operators which say that $[(t,s) \to T_P(t,s)] \in C(\Delta,\mathcal{L}(X_0))$ and $[(t,s) \to T_P(t,s)] \in C(\Delta,\mathcal{L}(X_1))$. So, by interpolation arguments, $[(t,s) \to T_P(t,s)] \in C(\Delta,\mathcal{L}(X_\alpha))$. The others follow from similar arguments.

It is well known that if X, Y are complex Banach spaces, $U \subset X$ is an open set such that the map $f : U \subset X \to Y$ is locally bounded and complex Gâteaux differentiable, then f is analytic. Thus, consider any $P, Q \in W$ and take an open ball $B(P,r)$ with center P and radius r such that $\{R(t)A | t \in, R \in B(P,r)\}$ is a family of sectorial operators in X_0 with domain X_1. As a result, there exists $a_0 > 0$ such that $\|T_R(t,s)\|_{\mathcal{L}(X_\alpha)} \leq a_0$ for all $R \in B(P,r)$. Therefore the function $R \to T_R$ is locally bounded. Recall the last section and substitute the family of sectorial operators $\mathcal{S}$, defined in the initial part of that section, by the ball $B(P,r)$. So use here, the constants defined in those propositions. Consider also the complex neighborhood $O = \{\lambda | P + \lambda Q \in B(P,r)\}$. For all $\lambda \in O$, we have

$$\frac{\partial}{\partial t}T_{P+\lambda Q}(t,s) + (P(t)+\lambda Q(t))AT_{P+\lambda Q} = 0\,, \quad t > s\,.$$

Then

$$\begin{aligned}&\frac{\partial}{\partial t}(T_{P+\lambda Q}(t,s) - T_P(t,s)) + P(t)A(T_{P+\lambda Q}(t,s) - T_P(t,s))\\&= \lambda Q(t)A(T_{P+\lambda Q}(t,s) - T_P(t,s)) - \lambda Q(t)AT_P(t,s), \quad t > s\,.\end{aligned}$$

We write this equation in the integral form,

$$T_{P+\lambda Q}(t,s) = T_P(t,s) - \lambda \int_s^t T_P(t,\tau)Q(\tau)AT_P(\tau)d\tau + \Psi(\lambda),$$

where

$$\Psi(\lambda) = -\lambda \int_s^t T_P(t,\tau)Q(\tau)A(T_{P+\lambda Q}(\tau,s) - T_P(\tau,s))d\tau\,.$$

By the estimates concerning the evolution operators, we obtain

$$\|\Psi(\lambda)\|_{\mathcal{L}(X_\alpha)} \le |\lambda|^2 e^{-\Omega(t-s)} \max_{\tau\in[s,t]}\{\|Q(\tau)\|_{\mathcal{L}(X_\mu)}\} \max_{\tau\in[s,t]}\{\|Q(\tau)\|_{\mathcal{L}(X_0)}\} I^*\,,$$

where

$$I^* = \int_s^t (1+(t-\tau)^{-1})^{\alpha}(1+(\tau-s)^{-1})^{1-\alpha} p_1(t-\tau)p(\tau-s)d\tau\,.$$

So $I^* \le B(\alpha,1-\alpha)p_1(t-s)p(t-s)$. Then the limit of

$$(T_{P+\lambda Q}(t,s) - T_P(t,s))/\lambda$$

exists uniformly for $t,s\in J$, $t>s$, in any finite interval J, if $\Omega<0$, and in an arbitrary interval (finite or infinite), if $\Omega>0$. Anyway, the function $P\to T_P$ is complex Gâuteaux differentiable from W to $C(\Delta,\mathcal{L}(X_0))$ in any finite interval J. The other case follows straightforward from the above and from the linearity of $T_P(t,s)\xi$ relative to ξ. As a consequence of this proof, we obtain the derivative

$$\partial_P T_P(t,s)H = -\int_s^t T_P(t,\tau)H(\tau)AT_P(\tau,s)d\tau\,,$$

where $H\in C^\epsilon(J,\mathcal{L}(X_0))\cap C(J,\mathcal{L}(X_\mu))$.

Now we prove the last assertion. For $\lambda\in O$, we have

$$\frac{G(P+\lambda Q, f+\lambda g) - G(P,f)}{\lambda} = \frac{G(P+\lambda Q, f) - G(P,f)}{\lambda} + G(P+\lambda Q, g)$$

The evaluation of the limit for $\lambda\to 0$ of the first part is done likewise for $(\xi,P)\to \{\int_0^t T_P(t,s)\xi dx,\, t\in J\}$ and the second follows straightforward from the observation that $(P,g)\to G(P,g)$ is continuous. □

Corollary 5.2. *For $P\in W$ such that $\|T_P(t,x)\|_{(X_\alpha,X_\beta)} = O(e^{-\Omega(t-s)})$, $\Omega>0$, the interval J in Theorem 5.1 can be taken infinite.*

6. Application

In this section we present an application of Theorem 5.1. It applies naturally in obtaining results about the dependence of the solution of reaction-diffusion equations in respect to the parameters of the equation.

Let n be an integer, $3\ge n\ge 1$, $\Omega\subset R^n$, a C^∞ domain (see Triebel [13] for definition), and $L_2(\Omega,\mathbb{C})$, $W^{2,2}(\Omega,\mathbb{C})$ the usual spaces of Lebesgue and Sobolev.

It is well known that the Laplacian operator Δ, which is defined over the regular functions that satisfies the Dirichlet conditions $u|_{\partial\Omega}=0$ is closed in $L_2(\Omega,\mathbb{C})$. Its domain $D(-\Delta)$ is the space $W_0^{2,2}(\Omega,\mathbb{C}) = \{f\in W^{2,2}(\Omega,\mathbb{C})\ |f|_{\partial\Omega}=0\}$ and the norm of this space is equivalent to the norm of the graph $-\Delta$.

Let $N\ge 1$, N integer, and I_N the identity matrix of order N over $\mathbb{C}^N\times\mathbb{C}^N$. Also, define by $-I_N\Delta$ the operator which diagonal is the Laplacian. Clearly, $-I_N\Delta$

is closed in $L_2(\Omega, \mathbb{C}^N)$ and its domains is equal to $W_0^{2,2}(\Omega, \mathbb{C}^N)$, whose the N components satisfy the Dirichlet conditions.

We consider now the complex interpolation functor $[\,,\,]_\theta$, $\theta \in (1/4, 1)$. The interpolation space theory states that the space $[L_2(\Omega, \mathbb{C}), W_0^{2,2}(\Omega, \mathbb{C})]_\theta$ is an interpolation space likewise it was defined in the Section 2 (see [13, p. 321, theorem (a)]). Thus, $X_\theta = [L_2(\Omega, \mathbb{C}^N), W_0(\Omega, \mathbb{C}^N)]_\theta$ also satisfies the same definition of Section 2, because $L_2(\Omega, \mathbb{C}^N)$ is isomorphic to $L_2(\Omega, \mathbb{C}) \times \cdots \times L_2(\Omega, \mathbb{C})$, N times. Similarly, $W_0^{2,2}(\Omega, \mathbb{C}^N)$ is isomorphic to $W_0^{2,2}(\Omega, \mathbb{C}) \times \cdots \times W_0^{2,2}(\Omega, \mathbb{C})$. Moreover, the complex interpolation of the Cartesian product is the Cartesian product of the complex interpolation.

In what follows, let $\epsilon \in (0, 1)$ and J be the interval $[0, T]$, $T > 0$. Let $C^\epsilon(J, M_N)$ be the set of continuous Hölder functions over the space of the square complex matrices $M_N \in \mathbb{C}^N \times \mathbb{C}^N$, and $C_+^\epsilon(J, M_N)$ the open set in $C^\epsilon(J, M_N)$, such that the operator $P(t)$ has non-zero positive eigenvalues for all $t \in J$.

Finally, let $f : J \to L_2(\Omega, \mathbb{C}^N)$, Hölder continuous and such that $f : J \to X_\theta$ is continuous. Using these conditions, we shall apply Theorem 5.1 to the system

$$
\begin{gathered}
u_t + P(t)(I_N(-\Delta))u = f(t) \\
u|_{\partial\Omega} = 0, \\
u(0) = \xi
\end{gathered}
$$

which has a solution, and it can be written as

$$
u(t) = T_P(t, 0)\xi + \int_0^t T_P(t, s) f(t) ds
$$

We remark, firstly, that the conditions (ii)) and (iii) in the definition of W, see Proposition 3.3, follow trivially from the fact that X_θ is a linear space. The condition (i) follows from the main theorem in Oliveira [8]. Hence, according to Theorem 5.1, the mapping $(P, f) \to u(.; P, f, \xi)$ is analytic, from $C_+^\epsilon(J, M_N) \times C(J, X_\mu) \times X_\theta$ to $C(J, X_\theta)$, $\theta \in (\mu, 1]$.

Obviously, this application includes the case

$$
\begin{gathered}
u_t + P(\lambda)(I_N(-\Delta))u = f(t, \pi) \\
u|_{\partial\Omega} = 0 \\
u(0) = \xi
\end{gathered}
$$

in which $\lambda \in \Lambda$ and $\pi \in \Pi$, where Λ and Π are Banach spaces. Also, supposing the mappings $\lambda \to P(\lambda)$ and $\pi \to f(., \pi)$ are analytic, it allows to conclude the analyticity of u in respect the parameters λ and π. Observe that a theorem, obtained by Henry [3, Lemma 3.4.2], for the dependency of the parameters with the operator, covers the case when $P(\lambda)$ is diagonal matrix and, therefore, the N components of the equation system can be decoupled.

By repeating Henry's argument [3, chapter 3, Theorem 3.4.4], the present application can be extended to the semilinear case in which f also depends of the solution with the restriction that the image of $f(u)$ must have greater regularity in X_μ, $\mu > 0$. In addition, we note that the Semilinear Geometric Theory of Henry can be constructed with interpolation spaces as referred here.

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