

**EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR
FOURTH-ORDER BOUNDARY-VALUE PROBLEMS
IN BANACH SPACES**

YUJUN CUI, YUMEI ZOU

ABSTRACT. This article concerns the fourth-order boundary-value problem

$$\begin{aligned}x^{(4)}(t) &= f(t, x(t), x''(t)), \quad t \in (0, 1), \\x(0) &= x(1) = x''(0) = x''(1) = \theta,\end{aligned}$$

in a Banach space. We present some spectral conditions, on the nonlinearity $f(t, u, v)$, to guarantee the existence and uniqueness of solutions. Our method is different from the one used in the references, even the above problem in a scalar space.

1. INTRODUCTION

In this paper, we consider the existence and uniqueness of solutions for the fourth-order boundary-value problem in a Banach space E :

$$\begin{aligned}x^{(4)}(t) &= f(t, x(t), x''(t)), \quad t \in (0, 1), \\x(0) &= x(1) = x''(0) = x''(1) = \theta,\end{aligned}\tag{1.1}$$

where $f : J \times E \times E \rightarrow E$ is continuous, $J = [0, 1]$, θ is the zero element of E . This problem models deformations of an elastic beam in equilibrium state, whose two ends are simply supported. Due to its importance in physics, this problem has been studied by many authors using Schauder's fixed point theorem and Leray-Schauder degree theory (see [1, 2, 3, 4] and references therein). In [1], Aftabizadeh showed the existence of a solution to (1.1) under the restriction that f is a bounded function. In [2, Theorem 1], Yang extended Aftabizadeh's result and showed existence under the growth condition

$$|f(t, u, v)| \leq a|u| + b|v| + c,\tag{1.2}$$

where a, b and c are positive constants such that $\frac{a}{\pi^4} + \frac{b}{\pi^2} < 1$.

However, the above-mentioned references consider (1.1) only in scalar space. On the other hand, the theory of ordinary differential equations in abstract spaces is becoming an important branch of mathematics in last thirty years because of its application in partial differential equations and ODEs in appropriately infinite

2000 *Mathematics Subject Classification.* 47H10.

Key words and phrases. Spectral conditions; Mönch fixed-point theorem.

©2009 Texas State University - San Marcos.

Submitted March 14, 2008. Published February 19, 2009.

Supported by grant 10671167 from the National Science Foundation of China.

dimensional spaces (see, for example [5, 6, 7]). As a result the goal of this paper is to fill up the gap in this area; that is, to investigate the existence and uniqueness of solutions of (1.1) in a Banach space E .

The goal of this paper is to introduce a new method to investigate the existence of solutions for (1.1) when f satisfies the growth condition (1.2). To do this, we transform (1.1) into a second-order integro-differential equation. Then we construct a particular nonempty closed convex set and apply the Mönch fixed-point theorem to obtain existence results for (1.1).

The technique used in this paper includes the well-known Gelfand's formula, a bounded closed convex set, and Mönch fixed-point theorem.

This paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to the existence results of (1.1) and their proofs. Section 4 is devoted to the uniqueness results (1.1) and their proofs.

2. PRELIMINARIES

Let E be a real Banach space with norm $\|\cdot\|$. Evidently, $C[J, E]$ is a Banach space with norm $\|x\|_C = \max_{t \in J} \|x(t)\|$.

A function x is said to be a solution of (1.1) if $x \in C^2[J, E] \cap C^4[(0, 1), E]$ and satisfies (1.1).

For a bounded set V in a Banach space, we denote $\alpha(V)$ the Kuratowski measure of noncompactness (see [5, 6, 7], for further understanding). In the paper, we will denote $\alpha(\cdot)$ and $\alpha_C(\cdot)$ the Kuratowski measure of noncompactness of a bounded subset in E and in $C[I, E]$, respectively.

Next, we list four lemmas which will be used in Section 3.

Lemma 2.1. *Let $V \subset C[J, E]$ be bounded and equicontinuous on J . Then $\alpha(V(t))$ is continuous on I and*

$$\alpha_C(V) = \max_{t \in J} \alpha(V(t)), \quad \alpha\left(\left\{\int_J x(t)dt : x \in V\right\}\right) \leq \int_J \alpha(V(t))dt.$$

where $V(t) = \{x(t) : x \in V\}$.

Lemma 2.2. *Let H be a set of countable strongly measurable functions $x : J \rightarrow E$. Assume, in addition, there exists $M \in L[J, R^+]$, for any $x \in H$, such that $\|x(t)\| \leq M(t)$, a.e. $t \in J$, holds. Then $\alpha(H(t)) \in L[J, R^+]$, moreover,*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq 2 \int_J \alpha(H(t))dt.$$

Lemma 2.3. *Mönch fixed-point theorem. Let Q be a closed convex set of E and $u \in Q$. Assume that the continuous operator $F : Q \rightarrow Q$ has the following property: $V \subset Q$ countable, $V \subset \overline{\text{co}}(\{u\} \cup F(V)) \implies V$ is relatively compact. Then F has a fixed point in Q .*

Lemma 2.4 ([8]). *Let P be a cone of a real Banach space E and $B : P \rightarrow P$ a completely continuous operator. Assume that B is order-preserving and positively homogeneous of degree 1 and that there exist $v \in P \setminus \{\theta\}$, $\lambda > 0$ such that $Bv \geq \lambda v$. Then $r(B) \geq \lambda$, where $r(B)$ denotes the spectral radius of B .*

3. EXISTENCE RESULTS

For convenience, let us list some conditions to be used later.

(H1) $f \in C[J \times E \times E, E]$. Assume that there exist $a, b \in (0, +\infty)$, $L(t) \in C[J, [0, +\infty)]$ such that

$$\|f(t, x, y)\| \leq a\|x\| + b\|y\| + L(t), \quad \forall (t, x, y) \in J \times E \times E, \quad \frac{a}{\pi^4} + \frac{b}{\pi^2} < 1.$$

(H2) For any bounded sets $A, B \subset E$, f is uniformly continuous on $I \times A \times B$. Assume that there exist $c, d \in [0, +\infty)$ such that

$$\alpha(f(t, A, B)) \leq c\alpha(A) + d\alpha(B), \quad \forall t \in J, \quad \frac{c}{\pi^4} + \frac{d}{\pi^2} < 1.$$

(H2') There exist $c, d \in [0, +\infty)$ such that

$$\alpha(f(t, A, B)) \leq c\alpha(A) + d\alpha(B), \quad \forall t \in J, \quad \frac{4c}{\pi^4} + \frac{2d}{\pi^2} < 1.$$

Now let $G(t, s)$ be the Green's function of the linear problem $\varphi'' = 0$, $t \in (0, 1)$ together with $\varphi(0) = \varphi(1) = 0$, which is explicitly given by

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Set

$$(Tx)(t) = \int_0^1 G(t, s)x(s)ds, \quad t \in J, \quad x \in C[I, E].$$

Obviously, $T : C[J, E] \rightarrow C[J, E]$ is continuous. Let $u = x''$. Since $x(0) = x(1) = \theta$, we have

$$x(t) = (Tu)(t) = \int_0^1 G(t, s)u(s)ds. \quad (3.1)$$

Using the above transformation and (3.1), Problem (1.1) becomes

$$u''(t) = f(t, (Tu)(t), u(t)), \quad (3.2)$$

$$u(0) = u(1) = \theta. \quad (3.3)$$

From these two equalities, we have

$$u(t) = \int_0^1 G(t, s)f(s, (Tu)(s), u(s))ds.$$

Now, define an operator A on $C[J, E]$ by

$$(Au)(t) = \int_0^1 G(t, s)f(s, \int_0^1 G(s, \tau)u(\tau)d\tau, u(s))ds.$$

The following Lemma can be easily obtained.

Lemma 3.1. *Assume that (H1) holds. Then $A : C[J, E] \rightarrow C^2[J, E]$ and*

- (i) $A : C[J, E] \rightarrow C^2[J, E]$ is continuous and bounded;
- (ii) Problem (1.1) has a solution in $C^2[J, E]$ if and only if A has a fixed point in $C[J, E]$.

Let

$$(K\varphi)(t) = \int_0^1 G(t,s)\varphi(s)ds, \quad \forall \varphi \in C[J, R],$$

$$B\varphi = aK^2\varphi + bK\varphi, \quad a, b > 0, \quad \forall \varphi \in C[J, R].$$

It is easy to see that $K, B : C[J, R] \rightarrow C[J, R]$ are two continuous linear positive operators and $r(K) = \frac{1}{\pi^2}$.

Lemma 3.2. *Assume that (H1) holds. Then there exists a norm $\|\cdot\|_{C[J]}^*$ which is equivalent with norm $\|\cdot\|_{C[J]}$ ($\|y\|_{C[J]} = \max_{t \in J} |y(t)|$, $y \in C[J, R]$) and has the following properties:*

- (i) $\|By\|_{C[J]}^* \leq \frac{r(B)+1}{2} \|y\|_{C[J]}^*$, for all $y \in C[J, R]$.
- (ii) $\|\varphi\|_{C[J]}^* \leq \|\psi\|_{C[J]}^*$ for any $\varphi, \psi \in C[J, R]$ with $0 \leq \varphi(t) \leq \psi(t)$, for all $t \in J$.

Proof. Since $r(K) = 1/\pi^2$, we have $r(B) = \frac{a}{\pi^4} + \frac{b}{\pi^2} < 1$. Set

$$\varepsilon = \frac{1}{2}(1 - r(B)).$$

Since $r(B) < 1$, by Gelfand's formula, we have that there exists a natural number N such that for $n \geq N$,

$$\|B^n\| \leq [r(B) + \varepsilon]^n = \left[\frac{r(B) + 1}{2}\right]^n.$$

For every $y \in C[J, R]$, define

$$\|y\|_{C[J]}^* = \sum_{i=1}^N \left[\frac{r(B) + 1}{2}\right]^{N-i} \|B^{i-1}y\|_{C[J]}, \quad (3.4)$$

where $T_1^0 = I$ is the identity operator. By (3.4), It is easy to see that

$$\left[\frac{r(B) + 1}{2}\right]^N \|y\|_{C[J]} \leq \|y\|_{C[J]}^* \leq \sum_{i=1}^N \left[\frac{r(B) + 1}{2}\right]^{N-i} \|B^{i-1}\| \cdot \|y\|_{C[J]},$$

which implies $\|\cdot\|_{C[J]}^*$ is a norm in $C[J, R]$ and equivalent to the norm $\|\cdot\|_{C[J]}$. For any $y \in C[J, R]$, we have

$$\begin{aligned} \|By\|_{C[J]}^* &= \sum_{i=1}^N \left[\frac{r(B) + 1}{2}\right]^{N-i} \|B^i y\|_{C[J]} \\ &= \frac{r(B) + 1}{2} \sum_{i=1}^{N-1} \left[\frac{r(B) + 1}{2}\right]^{N-i-1} \|B^i y\|_{C[J]} + \|B^N y\|_{C[J]} \\ &\leq \frac{r(B) + 1}{2} \sum_{i=1}^{N-1} \left[\frac{r(B) + 1}{2}\right]^{N-i-1} \|B^i y\|_{C[J]} + \left[\frac{r(B) + 1}{2}\right]^N \|y\|_{C[J]} \\ &= \frac{r(B) + 1}{2} \sum_{i=1}^N \left[\frac{r(B) + 1}{2}\right]^{N-i} \|B^{i-1} y\|_{C[J]} = \frac{r(B) + 1}{2} \|y\|_{C[J]}^*. \end{aligned}$$

Therefore, (i) holds. The result (ii) is easily obtained by the positivity of the operator B and the monotonicity of the norm $\|\cdot\|_{C[J]}$. \square

The following theorem is our main result.

Theorem 3.3. *Assume that (H1), (H2) hold. Then (1.1) has at least one solution.*

Proof. Set

$$M = \int_0^1 s(1-s)L(s)ds.$$

It is clear that $M < +\infty$. Select

$$R_1 > \max\left\{1, \frac{2\pi^4 M^*}{\pi^4 - a - b\pi^2}\right\},$$

where $M^* = \|M\|_{C[J]}^*$. Let

$$W = \{x \in C[J, E] : \|y(t)\|_{C[J]}^* \leq R_1, y(t) = \|x(t)\|\}.$$

It is clear that W is a bounded closed convex set in $C[J, E]$.

Now we show that $AW \subset W$. For any $x \in W$, set $y(t) = \|x(t)\|$, then it follows from (H1) that

$$\begin{aligned} & \|Ax(t)\| \\ &= \left\| \int_0^1 G(t, s)f(s, \int_0^1 G(s, \tau)u(\tau)d\tau, u(s))ds \right\| \\ &\leq \int_0^1 G(t, s) \left[a \left\| \int_0^1 G(s, \tau)u(\tau)d\tau \right\| + b\|u(s)\| + L(s) \right] ds \\ &\leq a \int_0^1 G(t, s) \int_0^1 G(s, \tau)\|u(\tau)\|d\tau ds + b \int_0^1 G(t, s)\|u(s)\|ds + \int_0^1 s(1-s)L(s)ds \\ &\leq (By)(t) + M. \end{aligned}$$

Consequently, by Lemma 3.2, we have

$$\begin{aligned} \|\|Ax\|\|_{C[J]}^* &\leq \|By + M\|_{C[J]}^* \\ &\leq \|By\|_{C[J]}^* + M^* \\ &\leq \frac{r(B) + 1}{2} \|y\|_{C[J]}^* + M^* \\ &\leq \frac{1 + r(B)}{2} R_1 + \frac{1 - r(B)}{2} R_1 = R_1. \end{aligned}$$

Hence we obtain $Ax \in W$ for any $x \in W$.

We have by Lemma 3.1 that A is a continuous operator from W into W . It is easily seen that seeking a solution of (1.1) in $C^2[J, E] \cap C^4[(0, 1), E]$ is equivalent to finding a fixed point of operator A in $C[J, E]$. Thus, our results are established if we can prove the existence of a fixed point of A in $C[J, E]$.

Let $V \subset W$ be a countable set with

$$V \subset \overline{\text{co}}\{u\} \cup AV, \quad u \in W, \quad (3.5)$$

in which $AV = \{Ax : x \in V\}$. Now we show that V is relatively compact in $C[J, E]$. In fact, we have by (3.5) that $\alpha_C(V) \leq \alpha_C(AV)$. Also, it is easy to see that AV is a class of equicontinuous functions on J , hence V is also a class of equicontinuous functions on J and $\overline{V}(t) \subset \overline{\text{co}}\{u(t)\} \cup (AV)(t)$ [9]. This together with Lemma 2.1 yields

$$\alpha_C(V) = \max_{t \in J} \alpha(V(t)), \quad \alpha_C(AV) = \max_{t \in J} \alpha((AV)(t)), \quad \alpha(V(t)) \leq \alpha((AV)(t)), \quad (3.6)$$

where $(AV)(t) = \{x(t) : x \in AV\}$. We now estimate $\alpha(V(t))$. We have by the definition of A and Lemma 2.1,

$$\begin{aligned} \alpha(V(t)) &\leq \alpha((AV)(t)) \\ &= \alpha(\{ \int_0^1 G(t,s)f(s, \int_0^1 G(s,\tau)x(\tau)d\tau, x(s))ds : x \in V \}) \\ &\leq \int_0^1 G(t,s)\alpha(\{f(s, \int_0^1 G(s,\tau)x(\tau)d\tau, x(s)) : x \in V\})ds \\ &\leq c \int_0^1 G(t,s)\alpha(\{ \int_0^1 G(s,\tau)x(\tau)d\tau : x \in V \})ds + d \int_0^1 G(t,s)\alpha(V(s))ds \\ &\leq c \int_0^1 G(t,s) \int_0^1 G(s,\tau)\alpha(V(\tau))d\tau ds + d \int_0^1 G(t,s)\alpha(V(s))ds \\ &= (cK^2 + dK)\alpha(V(t)). \end{aligned}$$

If $\alpha(V(t)) \neq 0$ for some $t \in J$, then by Lemma 2.4, we have $r(cK^2 + dK) \geq 1$, which is a contradiction with $r(cK^2 + dK) = \frac{c}{\pi^4} + \frac{d}{\pi^2} < 1$. So $\alpha(V(t)) \equiv 0$ for $\forall t \in J$, which implies that V is relatively compact in $C[J, E]$. Consequently, Lemma 2.3 guarantees that A has at least one fixed point in W . The proof of the theorem is completed. \square

Theorem 3.4. *Assume that (H1), (H2') hold, then (1.1) has at least one solution.*

Proof. The proof is similar to that of Theorem 3.3. The difference is that we use Lemma 2.2, instead of Lemma 2.1, to prove $\alpha(V(t)) \equiv 0$, for all $t \in I$; for example,

$$\begin{aligned} \alpha(V(t)) &\leq \alpha((AV)(t)) = \alpha(\{ \int_0^1 G(t,s)f(s, \int_0^1 G(s,\tau)x(\tau)d\tau, x(s))ds : x \in V \}) \\ &\leq 2 \int_0^1 G(t,s)\alpha(\{f(s, \int_0^1 G(s,\tau)x(\tau)d\tau, x(s)) : x \in V\})ds \\ &\leq 2c \int_0^1 G(t,s)\alpha(\{ \int_0^1 G(s,\tau)x(\tau)d\tau : x \in V \})ds + 2d \int_0^1 G(t,s)\alpha(V(s))ds \\ &\leq 4c \int_0^1 G(t,s) \int_0^1 G(s,\tau)\alpha(V(\tau))d\tau ds + 2d \int_0^1 G(t,s)\alpha(V(s))ds \\ &= (4cK^2 + 2dK)\alpha(V(t)). \end{aligned}$$

\square

4. UNIQUENESS RESULTS

Theorem 4.1. *$f \in C[J \times E \times E, E]$. Assume that there exist $\alpha, \beta \in (0, +\infty)$ such that*

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \alpha \|u_1 - u_2\| + \beta \|v_1 - v_2\|, \quad \forall t \in J, u_1, v_1, u_2, v_2 \in E,$$

and

$$\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1.$$

Then (1.1) has a unique solution.

Proof. From Lemma 3.1, to prove (1.1) has a unique solution, we have only to show that A has a unique fixed point in $C[I, E]$. For $x(t), y(t) \in C[I, E]$, we have

$$\begin{aligned} & \| (A^n x)(t) - (A^n y)(t) \| \\ &= \left\| \int_0^1 G(t, s) \left[f(s, \int_0^1 G(s, \tau) (A^{n-1} x)(\tau) d\tau, (A^{n-1} x)(s)) \right. \right. \\ &\quad \left. \left. - f(s, \int_0^1 G(s, \tau) (A^{n-1} y)(\tau) d\tau, (A^{n-1} y)(s)) \right] ds \right\| \\ &\leq \int_0^1 G(t, s) \left[\alpha \int_0^1 G(s, \tau) \| (A^{n-1} x)(\tau) \right. \\ &\quad \left. - (A^{n-1} y)(\tau) \| d\tau + \beta \| (A^{n-1} x)(\tau) - (A^{n-1} y)(\tau) \| \right] ds \\ &= (\alpha K^2 + \beta K) (\| (A^{n-1} x)(s) - (A^{n-1} y)(s) \|) \\ &:= B_1 (\| (A^{n-1} x)(s) - (A^{n-1} y)(s) \|). \end{aligned}$$

By induction, we have

$$\| (A^n x)(t) - (A^n y)(t) \| \leq B_1^n (\| x(s) - y(s) \|) \leq \| B_1^n \| \| x - y \|_C.$$

Thus $\| A^n x - A^n y \|_C \leq \| B_1^n \| \| x - y \|_C$.

Since $r(K) = 1/\pi^2$, We have $r(B_1) = \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$. Set $\varepsilon = \frac{1}{2}(1 - r(B_1))$. By Gelfand's formula, we have that there exists a natural number N_1 such that

$$\| B_1^{N_1} \| \leq [r(B_1) + \varepsilon]^{N_1} = \left[\frac{r(B_1) + 1}{2} \right]^{N_1} < 1.$$

Thus the Banach contraction mapping principle implies A^{N_1} has a unique fixed point x^* in $C[I, E]$, and so A has a unique fixed point x^* in $C[I, E]$. Therefore, $x(t) = \int_0^1 G(t, s)x^*(s)ds$ is the unique solution of (1.1). \square

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YUJUN CUI

INSTITUTE OF MATHEMATICS, SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, QINGDAO 266510, CHINA

E-mail address: cyj720201@163.com, sdcyj720163.com

YUMEI ZOU
INSTITUTE OF MATHEMATICS, SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, QINGDAO
266510, CHINA
E-mail address: `sdzouym@126.com`