Electronic Journal of Differential Equations, Vol. 2009(2009), No. 36, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO CAUCHY-DIRICHLET PROBLEMS FOR SECOND-ORDER HYPERBOLIC EQUATIONS IN CYLINDER WITH NON-SMOOTH BASE

NGUYEN MANH HUNG, BUI TRONG KIM

Abstract. This paper concerns a Cauchy-Dirichlet problem for second-order hyperbolic equations in infinite cylinders with the base containing conical points. Some results on the asymptotical expansions of generalized solutions of this problem are given.

1. Introduction

Boundary-value problems for partial differential equations and systems in domains with smooth boundary have been nearly completely studied in the works [1, 2]. General boundary-value problems for elliptic equations and systems in domains with conical points were considered by Kondratiev [6], Nazarov and Plamenevsky [8]. The initial boundary-value problems for non-stationary equations and systems on non-smooth domains have been studied by many authors [3, 4, 5, 7, 9]. The Neumann problem for hyperbolic systems in domains with conical point was described in [7] and the same problem for the classical heat equation in a dihedral angle was investigated in [9]. The first initial boundary-value problems for strongly hyperbolic systems in an cylinder with conical point on the boundary of base have been investigated in [3], where the problem was only investigated in the finite cylinder.

In this paper we consider a Cauchy-Dirichlet problem for second-order hyperbolic equations in infinite cylinders with non-smooth base. First, we study the existence, uniqueness and smoothness with respect to time variable of a generalized solution in the Sobolev space by Galerkin's approximate method. After that, we take the term containing the derivative in time of the unknown function to the right-hand side of the equation such that the problem can be considered as an elliptic one. We can apply the results of elliptic boundary-value problems to deal with the asymptotic of the solutions.

The main goal of this paper is obtaining asymptotical expansions of solutions of the problem. In section 2 we introduce some notations and the formulation of the problem. We receive results on the unique existence and the smoothness with

---

2000 *Mathematics Subject Classification.* 35L05, 35L15, 35L20.
Key words and phrases. Generalized solution; asymptotic; conical point on the boundary; non-smooth domains.

Submitted August 18, 2008. Published March 3, 2009.

respect to time variable of solutions in section 3 and the asymptotical expansions of the solutions in section 4. Finally, in the last section we apply the results of section 4 to the problems of mathematical physics.

2. Formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$. Set $\Omega_t = \Omega \times (0,t)$ for each $t \in (0,\infty)$, $\Omega_\infty = \Omega \times (0,\infty)$, $S_t = \partial\Omega \times (0,t)$ and $S_\infty = \partial\Omega \times (0,\infty)$.

We use the following notation: For each multi-index $\alpha = (\alpha_1,\dots.\alpha_n) \in N^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}} = u_{x_1^{\alpha_1}\dots x_n^{\alpha_n}}$ is the generalized derivative up to order α with respect to $x = (x_1,\dots,x_n)$; $u_{t^k} = \frac{\partial^k u}{\partial t^k}$ is the generalized derivative up to order k with respect to t.

We begin by recalling some functional spaces which will be used frequently in this paper. $W^l(\Omega)$ is the space consisting of all functions $u(x)$, $x \in \Omega$, with the norm

$$\|u\|_{W^l(\Omega)} = \Big(\sum_{|\alpha|=0}^{l}\int_\Omega |D^\alpha u|^2 dx\Big)^{1/2}.$$

$\mathring{W}^l(\Omega)$ is the completion of $\mathring{C}C^\infty(\Omega)$ in the norm of the space $W^l(\Omega)$.
$W^l_\beta(\Omega)$ is the space consisting of all functions $u(x) = (u_1(x),\dots,u_s(x))$ which have generalized derivatives $D^\alpha u_i$, $|\alpha| \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{W^l_\beta(\Omega)} = \sum_{|\alpha|=0}^{l}\int_\Omega r^{2(\beta+|\alpha|-l)}|D^\alpha u|^2 dx < +\infty.$$

$W^{l,k}(e^{-\gamma t},\Omega_\infty)$ is the space consisting of functions $u(x,t)$, $(x,t) \in \Omega_\infty$, with the norm

$$\|u\|_{W^{l,k}(e^{-\gamma t},\Omega_\infty)} = \Big(\int_{\Omega_\infty}\big(\sum_{|\alpha|=0}^{l}|D^\alpha u|^2 + \sum_{j=1}^{k}|u_{t^j}|^2\big)e^{-2\gamma t}\,dx\,dt\Big)^{1/2}, \quad k \ge 1.$$

$W^{l,0}(e^{-\gamma t},\Omega_\infty)$ is the space consisting of functions $u(x,t)$, $(x,t) \in \Omega_\infty$, with the norm

$$\|u\|_{W^{l,0}(e^{-\gamma t},\Omega_\infty)} = \Big(\int_{\Omega_\infty}\big(\sum_{|\alpha|=0}^{l}|D^\alpha u|^2\,dx\,dt\Big)^{1/2}.$$

$\mathring{W}^{l,k}(e^{-\gamma t},\Omega_\infty)$ is the closure in $W^{l,k}(e^{-\gamma t},\Omega_\infty)$ of the set consisting of infinite differentiable in Ω_∞ functions which belong to $W^{l,k}(e^{-\gamma t},\Omega_\infty)$ and vanish near S_∞.
$W^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)$ is the space consisting of functions $u(x,t)$ satisfying

$$\|u\|^2_{W^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)} = \int_{\Omega_\infty}\big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)}|D^\alpha u|^2 + \sum_{j=1}^{k}|u_{t^j}|^2\big)e^{-2\gamma t}\,dx\,dt < \infty.$$

Denote by $L^\infty(0,\infty;X)$ the space consisting of measurable functions $u:(0,\infty) \to X, t \mapsto u(x,t)$ satisfying

$$\|u\|_{L^\infty(0,\infty;X)} = \operatorname*{ess\,sup}_{t>0}\big\|u(x,t)\big\|_X < +\infty.$$

Let $L(x,t,D)$ be a differential operator

$$L(x,t,D) \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big) + a, \tag{2.1}$$

where $a_{ij} \equiv a_{ij}(x,t), i,j = 1,\dots,n$ are infinitely differentiable bounded complex-valued functions on $\overline{\Omega}_\infty$, $a_{ij} = \overline{a}_{ji}$, $a \equiv a(x,t)$ are infinitely differentiable bounded real-valued functions on $\overline{\Omega}_\infty$. Suppose that $a_{ij}, i,j = 1,\dots,n$, are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0|\xi|^2 \tag{2.2}$$

for all $\xi \in \mathbb{R}^n\backslash\{0\}$ and $(x,t) \in \overline{\Omega}_\infty$, where μ_0 is a positive constant.

We consider the following problem in the infinite cylinder Ω_∞:

$$L(x,t,D)u - u_{tt} = f(x,t), \tag{2.3}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{2.4}$$

$$u|_{S_\infty} = 0. \tag{2.5}$$

A function $u(x,t)$ is called a generalized solution of the problem (2.3)–(2.5) in $W^{1,1}(e^{-\gamma t},\Omega_\infty)$ if $u(x,t) \in \mathring{W}^{1,1}(e^{-\gamma t},\Omega_\infty)$, $u(x,0) = 0$ and for each $T > 0$ the following equality holds:

$$\int_{\Omega_\infty} u_t\overline{\eta}_t\,dx\,dt - \int_{\Omega_\infty}\Big(\sum_{i,j=1}^{n} a_{ij}u_{x_j}\overline{\eta}_{x_i} - au\overline{\eta}\Big)\,dx\,dt = \int_{\Omega_\infty} f\ \overline{\eta}\,dx\,dt \tag{2.6}$$

for all test functions $\eta = \eta(x,t) \in \mathring{W}^{1,1}(e^{-\gamma t},\Omega_\infty)$ such that $\eta(x,t) = 0$ with $t \in [T,\infty)$. Set

$$B[u,v](t) = \sum_{i,j=1}^{n}\int_\Omega a_{ij}u_{x_j}\overline{v}_{x_i}dx.$$

The following lemma can be proved similarly to Garding's inequality.

Lemma 2.1. *Assume that coefficients $a_{ij} = a_{ij}(x,t)$, $i,j = 1\dots n$, $a = a(x,t)$ of the operator $L(x,t,D)$ satisfy condition (2.2) and $a_{ij}(x,t)$ are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty)$. Then there exist two constants $\mu_0 > 0$, $\lambda_0 \geq 0$ such that*

$$B[u,u](t) \geq \mu_0\|u(x,t)\|^2_{W^1(\Omega)} - \lambda_0\|u(x,t)\|^2_{L_2(\Omega)}$$

for all functions $u = u(x,t) \in \mathring{W}^{1,0}(e^{-\gamma t},\Omega_\infty)$.

Remark. It follows from the above lemma that if the function $a(x,t)$ in (2.1) satisfies

$$a(x,t) \leq -\lambda_0, \text{for all } (x,t) \in \Omega_\infty,$$

then

$$B_1[u,u](x,t) \equiv B[u,u](t) - \int_\Omega a(x,t)|u(x,t)|^2dx \geq \mu_0\|u(x,t)\|^2_{W^1(\Omega)} \tag{2.7}$$

for all functions $u = u(x,t) \in \mathring{W}^{1,0}(e^{-\gamma t},\Omega_\infty)$.

3. Solvability of the problem

In this section we investigate the smoothness of generalized solutions with respect to time. We begin by studying uniqueness of the problem.

Theorem 3.1. *Assume that for a positive constant μ,*

$$\sup\Big\{\big|\frac{\partial a_{ij}}{\partial t}\big|,\big|\frac{\partial a}{\partial t}\big| : (x,t)\in\Omega_\infty,\ i,j=1\dots,n\Big\}\le\mu.$$

In addition, suppose $a(x,t)\le-\lambda_0$, for all $(x,t)\in\Omega_\infty$. Then (2.3)-(2.5) *has at most one generalized solution in $W^{1,1}(e^{-\gamma t},\Omega_\infty)$ for $\gamma>0$ arbitrary.*

Proof. Suppose that there are two solutions u_1,u_2 in $\mathring{W}^{1,1}(e^{-\gamma t},\Omega_\infty)$. Putting $u=u_1-u_2$, so for each $T>0$ the following equality holds:

$$\int_{\Omega_\infty} u_t\overline{\eta}_t\,dx\,dt-\int_{\Omega_\infty}\Big(\sum_{i,j=1}^n a_{ij}u_{x_j}\overline{\eta}_{x_i}-au\overline{\eta}\Big)\,dx\,dt=0$$

for all test functions $\eta=\eta(x,t)\in\mathring{W}^{1,1}(e^{-\gamma t},\Omega_\infty)$ such that $\eta(x,t)=0$ with $t\in[T,\infty)$. For b with $0<b<T$, we set

$$\eta(x,t)=\begin{cases}\int_b^t u(x,s)ds, & 0\le t\le b,\\ 0, & t>b.\end{cases}$$

It is easy to check that $\eta(x,t)\in\mathring{W}^{1,1}(e^{-\gamma t},\Omega_\infty),\eta(x,t)=0$ with $t\in[T,\infty)$ and $\eta_t(x,t)=u(x,t)$. We have

$$\int_{\Omega_\infty}\eta_{tt}\overline{\eta}_t\,dx\,dt-\int_{\Omega_\infty}\Big(\sum_{i,j=1}^n a_{ij}\eta_{tx_j}\overline{\eta}_{x_i}-a\eta_t\overline{\eta}\Big)\,dx\,dt=0.$$

Adding this equality and its complex conjugate, using $a_{ij}=\overline{a}_{ji},i,j=1,\dots n$ and integrating by parts with respect to t, we obtain

$$\int_\Omega|\eta_t(x,b)|^2dx+B_1[\eta,\eta](x,0)+\int_{\Omega_\infty}\Big(\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial t}\eta_{x_j}\overline{\eta_{x_i}}-\frac{\partial a}{\partial t}\eta\overline{\eta}\Big)\,dx\,dt=0. \tag{3.1}$$

Putting $v_i(x,t)=\int_t^0 u_{x_i}(x,s)ds, i=1,\dots,n,\ v_0(x,t)=\int_t^0 u(x,s)ds$, we can write

$$\eta_{x_i}(x,t)=\int_b^t u_{x_i}(x,s)ds=v_i(x,b)-v_i(x,t),\quad \eta_{x_i}(x,0)=v_i(x,b).$$

$$\eta(x,t)=\int_b^t u(x,s)ds=v_0(x,b)-v_0(x,t),\eta(x,0)=v_0(x,b).$$

Substituting those into (3.1), then using the Cauchy's inequality and (2.7), we obtain

$$\begin{aligned}&\int_\Omega|\eta_t(x,b)|^2dx+\mu_0\sum_{i=0}^n\int_\Omega|v_i(x,b)|^2dx\\ &\le C_1b\sum_{i=0}^n\int_\Omega|v_i(x,b)|^2dx+C_2\int_0^b\Big(\sum_{i=0}^n\int_\Omega|v_i(x,t)|^2dx\Big)dt\\ &\le C_1b\sum_{i=0}^n\int_\Omega|v_i(x,b)|^2dx+C_2\int_0^b\Big(\int_\Omega|\eta_t(x,t)|^2dx+\sum_{i=0}^n\int_\Omega|v_i(x,t)|^2dx\Big)dt\end{aligned} \tag{3.2}$$

where C_1, C_2 are positive constants. Put

$$J(t) = \int_\Omega |\eta_t(x,b)|^2 dx + \sum_{i=0}^n \int_\Omega |v_i(x,b)|^2 dx.$$

From (3.2) we get

$$J(b) \le C \int_0^b J(t) dt$$

for all $b \in [0, \mu_0/2C_1]$, where C is a positive constant. This implies that $J(t) \equiv 0$ on $[0, \mu_0/2C_1]$ by Gronwall-Bellman's inequality. It follows $u_1 \equiv u_2$ on $[0, \mu_0/2C_1]$, where C_1 does not depend on b. By similar arguments for two functions u_1, u_2 on $[\mu_0/2C_1, \tau]$, we can show that after finite steps $u_1 \equiv u_2$ on $[\mu_0/2C_1, \tau]$. Since $\tau > 0$ is arbitrary, so $u_1 = u_2$ in $W^{1,1}(e^{-\gamma t}, \Omega_\infty)$. The proof is complete. $\square$

Now, we establish the existence of generalized solution of the mentioned problem by Galerkin's approximate method. We use the notation:

$$\gamma_0 = \frac{n\mu}{2\mu_0},$$

where n is dimensional number of the space $\mathbb{R}^n$, μ is the constant in theorem 3.1 and μ_0 is the constant in lemma 2.1. We have following theorem.

Theorem 3.2. *Assume that $a(x,t) \le -\lambda_0$, for all $(x,t) \in \Omega_\infty$ and the following conditions are fulfilled:*

(ii) $\sup\big\{\big|\frac{\partial^k a_{ij}}{\partial t^k}\big|, \big|\frac{\partial^k a}{\partial t^k}\big|\big\} : (x,t) \in \Omega_\infty,\ i,j = 1,\dots,n;\ k \le h-1\big\} \le \mu, h \ge 1,$
(iii) $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega)), k \le h,$
(iv) $f_{t^k}(x,0) = 0, k \le h-1.$

Then (2.3)–(2.5) has a unique generalized solution $u(x,t)$ in $W^{1,1}(e^{-\gamma t}, \Omega_\infty)$ for every $\gamma > \gamma_0$. Moreover, $u(x,t)$ has derivatives with respect to t up to order h belonging to $\mathring{W}^{1,1}(e^{-(2h+1)\gamma t}, \Omega_\infty)$ and the following inequality holds:

$$\|u_{t^h}\|^2_{W^{1,1}(e^{-(2h+1)\gamma t}, \Omega_\infty)} \le C \sum_{k=0}^h \|f_{t^k}\|^2_{L^\infty(0,\infty; L^2(\Omega))},$$

where C is a positive constant independent of u and f.

Proof. The uniqueness follows from theorem 3.1. The existence is obtained using Galerkin's method. Let $\{\varphi_k\}_{k=1}^\infty \subset \mathring{C}^\infty(\Omega)$ be an orthogonal system in $L_2(\Omega)$ such that its linear closure in $W^1(\Omega)$ is the space $\mathring{W}^1(\Omega)$. For each integer N we consider the function $u^N(x,t) = \sum_{k=1}^N C_k^N(t)\varphi_k(x)$, where $(C_1^N(t), \dots, C_N^N(t))$ is the solution of the ordinary differential system

$$\int_\Omega u_{tt}^N \overline{\varphi}_l dx + \int_\Omega (\sum_{i,j=1}^n a_{ij} u_{x_j}^N \overline{\varphi}_{lx_i} - a u^N \overline{\varphi}_l) dx = -\int_\Omega f \overline{\varphi}_l dx; \quad l = 1,\dots,N, \tag{3.3}$$

$$C_k^N(0) = \frac{d}{dt} C_k^N(0) = 0, \quad k = 1, \dots N. \tag{3.4}$$

Let us multiply (3.3) by $dC_k^N(t)/dt$ and take the sum with respect to l from 1 to N. Then we integrate the equality obtained with respect to t from 0 to t and add this equality to its complex conjugate. Finally, integrating by part and applying condition (3.4), we obtain

$$\begin{aligned}&\int_{\Omega}|u_t^N(x,t)|^2dx+\int_{\Omega}\big(\sum_{i,j=1}^n a_{ij}u^N_{x_j}\overline{u^N_{x_i}}-au^N\overline{u^N}\big)|_{t=t}dx\\&=\int_{\Omega_t}\Big(\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial t}u^N_{x_j}\overline{u^N_{x_i}}-\frac{\partial a}{\partial t}u^N\overline{u^N}\Big)\,dx\,dt-2\operatorname{Re}\int_{\Omega_t}f\overline{u_t^N}\,dx\,dt.\end{aligned}$$

Using (2.7) and Cauchy's inequality, one has

$$\begin{aligned}&\|u_t^N(x,t)\|^2_{L_2(\Omega)}+\mu_0\|u^N(x,t)\|^2_{W^1(\Omega)}\\&\le\int_{\Omega_t}\big(n\mu\sum_{i=1}^n|u_{x_i}|^2+\mu|u^N|^2+\delta|u_t^N|^2\big)\,dx\,dt+\frac{t}{\delta}\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))}\\&\le\delta\int_0^t\big(\|u_t^N\|^2_{L_2(\Omega)}+\frac{n\mu}{\delta}\|u^N(x,t)\|^2_{W^1(\Omega)}\big)dt+\frac{t}{\delta}\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))},\end{aligned}\tag{3.5}$$

where δ is a positive constant. Choosing $\delta=\frac{n\mu}{\mu_0}$ and putting

$$J^N(t)=\|u_t^N(x,t)\|^2_{L_2(\Omega)}+\mu_0\|u^N(x,t)\|^2_{W^1(\Omega)},$$

from inequality (3.5) we obtain

$$J^N(t)\le\frac{n\mu}{\mu_0}\int_0^t J^N(\tau)d\tau+t\Big(\frac{\mu_0}{n\mu}\|f\|^2_{L_\infty(0,\infty;L_2(\Omega))}\Big).$$

From this inequality and Gronwall-Bellman's inequality it follows that

$$J^N(t)\le C_1e^{\frac{n\mu}{\mu_0}t}\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where C_1 is a positive constant independent of N and f. Since $e^{-\gamma t}\le 1$ with $\gamma>0$ and $t\ge0$, putting $C_0=\min\{\mu_0,1\}$ we have

$$\begin{aligned}C_0\|u^N\|^2_{W^{1,1}(e^{-\gamma t},\Omega_\infty)}&\le\|u_t^N(x,t)\|^2_{L_2(\Omega)}+\mu_0\|u^N(x,t)\|^2_{W^1(\Omega)}\\&\le C_1e^{\frac{n\mu}{\mu_0}t}\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))}.\end{aligned}\tag{3.6}$$

Let γ be a positive constant such that $\gamma>\gamma_0=\frac{n\mu}{2\mu_0}$. Multiplying both sides of (3.6) by $e^{-2\gamma t}$ and integrating with respect to t from 0 to ∞, we get

$$\|u^N\|^2_{W^{1,1}(e^{-\gamma t},\Omega_\infty)}\le C_2\|f\|^2_{L^\infty(0,\infty;L_2(\Omega))},\tag{3.7}$$

where C_2 is a positive constant independent of N and f.

From (3.7) it follows that there exists a subsequence of the sequence $\{u^N\}$ with converges weakly to a function $u(x,t)$ in the space $W^{1,1}(e^{-\gamma t},\Omega_\infty)$. We can check that $u(x,t)$ is a generalized solution of the problem.

Now we prove the smoothness with respect to time variable of the generalized solution. We use the induction to show that the following inequalities are true for all integer $g\ge0$:

$$\|u^N_{t^g}(x,t)\|^2_{W^1(\Omega)}\le C_3e^{\frac{(2g+1)n\mu+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^g\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},\tag{3.8}$$

$$\|u^N_{t^g}(x,t)\|^2_{W^{1,1}(e^{-(2g+1)\gamma t},\Omega_\infty)}\le C_4\sum_{k=0}^g\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},\tag{3.9}$$

where $0 < \epsilon < \mu_0$, C_i is a positive constant independent of N and f; $i = 3, 4$.

For $g = 0$ inequalities (3.8) and (3.9) are true according to relations (3.6) and (3.7). Assume that $s \geq 1$ and inequalities (3.8) and (3.9) hold for all $g \leq s-1$. From identity (3.3) we have

$$\int_\Omega u^N_{t^{s+2}}\overline{\varphi}_l dx + \int_\Omega \Big(\sum_{i,j=1}^n \frac{\partial^s}{\partial t^s}\big(a_{ij}u^N_{x_j}\overline{\varphi}_{lx_i} - au\overline{\varphi}_l\big)\Big)dx = -\int_\Omega f_{t^s}\overline{\varphi}_l dx.$$

Multiplying both sides of this identity by $d^{s+1}C^N_l/dt^{s+1}$ and taking the sum with respect to l from 1 to N and integrating the result over $(0,t)$. Then adding this identity with its complex conjugate, we get

$$\begin{aligned}
&\int_{\Omega_t} \frac{\partial}{\partial t}\big(u^N_{t^{s+1}}\bar{u}^N_{t^{s+1}}\big)\,dx\,dt \\
&+ 2\operatorname{Re}\int_{\Omega_t}\Big(\sum_{i,j=1}^n \big(a_{ij}u^N_{x_j}\big)_{t^s}\big(\bar{u}^N_{x_i}\big)_{t^{s+1}} - \big(au^N\big)_{t^s}\bar{u}^N_{t^{s+1}}\Big)\,dx\,dt \\
&= -2\operatorname{Re}\int_{\Omega_t} f_{t^s}\bar{u}^N_{t^{s+1}}\,dx\,dt.
\end{aligned} \tag{3.10}$$

Denote

$$\binom{k}{s} = \frac{s!}{k!(s-k)!}.$$

Since $a_{ij} = \bar{a}_{ji}$ with $i, j = 1, 2, \dots, n$, we have

$$\begin{aligned}
&2\operatorname{Re}\sum_{i,j=1}^n (a_{ij}u^N_{x_j})_{t^s}(\bar{u}^N_{x_i})_{t^{s+1}} - (au^N)_{t^s}\bar{u}^N_{t^{s+1}} \\
&= \frac{\partial}{\partial t}\Big(\sum_{i,j=1}^n a_{ij}(u^N_{x_j})_{t^s}(\bar{u}^N_{x_i})_{t^s} - au^N_{t^s}\bar{u}^N_{t^s}\Big) - \operatorname{Re}\sum_{i,j=1}^m \frac{\partial a_{ij}}{\partial t}(u^N_{x_j})_{t^s}(\bar{u}^N_{x_i})_{t^s} + \frac{\partial a}{\partial t}u^N_{t^s}\bar{u}^N_{t^s} \\
&+ 2\operatorname{Re}\Big(\sum_{i,j=1}^n\sum_{k=1}^s \binom{k}{s}\frac{\partial}{\partial t}\big(\frac{\partial^k a_{ij}}{\partial t^k}(u^N_{x_j})_{t^{s-k}}(\bar{u}^N_{x_i})_{t^s}\big) - \sum_{k=1}^s\binom{k}{s}\frac{\partial}{\partial t}\big(\frac{\partial^k a}{\partial t^k}u^N_{t^{s-k}}\bar{u}^N_{t^s}\big)\Big) \\
&- 2\operatorname{Re}\Big(\sum_{i,j=1}^n\sum_{k=1}^s\binom{k}{s}\big(\frac{\partial^{k+1}a_{ij}}{\partial t^{k+1}}(u^N_{x_j})_{t^{s-k}}(\bar{u}^N_{x_i})_{t^s}\big) - \sum_{k=1}^s\binom{k}{s}\frac{\partial^{k+1}a}{\partial t^{k+1}}u^N_{t^{s-k}}\bar{u}^N_{t^s}\Big) \\
&- 2\operatorname{Re}\Big(\sum_{i,j=1}^n\sum_{k=1}^s\binom{k}{s}\frac{\partial^k a_{ij}}{\partial t^k}(u^N_{x_j})_{t^{s-k+1}}(\bar{u}^N_{x_i})_{t^s} - \sum_{k=1}^s\binom{k}{s}\frac{\partial^k a}{\partial t^k}u^N_{t^{s-k+1}}\bar{u}^N_{t^s}\Big).
\end{aligned} \tag{3.11}$$

From the condition $u^N(x,0) = 0$ we get

$$\begin{aligned}
&\int_{\Omega_t}\frac{\partial}{\partial t}\Big(\sum_{i,j=1}^n a_{ij}(u^N_{x_j})_{t^s}(\bar{u}^N_{x_i})_{t^s} - au^N_{t^s}\bar{u}^N_{t^s}\Big)\,dx\,dt \\
&= \int_\Omega\Big(\sum_{i,j=1}^n a_{ij}(u^N_{x_j})_{t^s}(\bar{u}^N_{x_i})_{t^s} - au^N_{t^s}\bar{u}^N_{t^s}\Big)\Big|_{t=0}^{t=t}dx = B_1(u^N_{t^s}, u^N_{t^s})(x,t).
\end{aligned}$$

Therefore, from identities (3.10) and (3.11) it follows that

$$
\begin{aligned}
&\int_\Omega |u^N_{t^{s+1}}(x,t)|^2 dx + B_1(u^N_{t^s}, u^N_{t^s})(x,t)\\
&= -2\operatorname{Re}\sum_{k=1}^{s}\binom{k}{s}\int_\Omega\Big(\sum_{i,j=1}^{n}\frac{\partial^k a_{ij}}{\partial t^k}(u^N_{x_j})_{t^{s-k}}(\bar{u^N_{x_i}})_{t^s} - \frac{\partial^k a}{\partial t^k}u^N_{t^{s-k}}\bar{u^N_{t^s}}\Big)|_{t=t}dx\\
&\quad + 2\operatorname{Re}\sum_{k=1}^{s}\binom{k}{s}\int_{\Omega_t}\Big(\sum_{i,j=1}^{n}\frac{\partial^{k+1} a_{ij}}{\partial t^{k+1}}(u^N_{x_j})_{t^{s-k}}(\bar{u^N_{x_i}})_{t^s} - \frac{\partial^{k+1} a}{\partial t^{k+1}}u^N_{t^{s-k}}\bar{u^N_{t^s}}\Big)\,dx\,dt\\
&\quad + 2\operatorname{Re}\sum_{k=1}^{s}\binom{k}{s}\int_{\Omega_t}\Big(\sum_{i,j=1}^{n}\frac{\partial^k a_{ij}}{\partial t^k}(u^N_{x_j})_{t^{s-k+1}}(\bar{u^N_{x_i}})_{t^s} - \frac{\partial^k a}{\partial t^k}u^N_{t^{s-k+1}}\bar{u^N_{t^s}}\Big)\,dx\,dt\\
&\quad + \operatorname{Re}\int_{\Omega_t}\Big(\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial t}(u^N_{x_j})_{t^s}(\bar{u^N_{x_i}})_{t^s} + \frac{\partial a}{\partial t}u^N_{t^s}\bar{u^N_{t^s}}\Big)\,dx\,dt - 2\operatorname{Re}\int_{\Omega_t} f_{t^s}\bar{u^N_{t^{s+1}}}\,dx\,dt.
\end{aligned}
$$

From this equality, (2.7) and Cauchy's inequality, we obtain

$$
\begin{aligned}
&\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + \mu_0\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}\\
&\le \epsilon\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} + C_1(\epsilon)\sum_{k=0}^{s-1}\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}\\
&\quad + C_2(\epsilon)\sum_{k=0}^{s-1}\int_0^t\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}dt\\
&\quad + \int_{\Omega_t}\Big(\big(n\mu(2s+1)+\epsilon\big)\sum_{i=1}^{n}|(u_{x_i})_{t^s}|^2 + \big(\mu(2s+1)+\epsilon\big)|u_{t^s}|^2\Big)\,dx\,dt\\
&\quad + \int_{\Omega_t}\delta|u_{t^{s+1}}|^2\,dx\,dt + \int_{\Omega_t}\frac{1}{\delta}|f_{t^s}|^2\,dx\,dt\\
&\le \epsilon\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} + C_1(\epsilon)\sum_{k=0}^{s-1}\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}\\
&\quad + C_2(\epsilon)\sum_{k=0}^{s-1}\int_0^t\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}\,dx\,dt + \int_0^t\big(n\mu(2s+1)+\epsilon\big)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}dt\\
&\quad + \int_0^t\delta\|u_{t^{s+1}}\|^2_{L_2(\Omega)}dt + \int_{\Omega_t}\frac{1}{\delta}|f_{t^s}|^2\,dx\,dt,
\end{aligned}
$$

where ϵ is a positive constant and $C_i(\epsilon)$ is positive constant that depends ϵ, $i=1,2$. From this inequality we obtain

$$
\begin{aligned}
&\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}\\
&\le C_1(\epsilon)\sum_{k=0}^{s-1}\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}\\
&\quad + \delta\int_0^t\frac{(n\mu(2s+1)+\epsilon)}{\delta}\|u_{t^s}(x,t)\|^2_{W^1(\Omega)} + \|u_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)}\Big)dt\\
&\quad + \int_{\Omega_t}\frac{1}{\delta}|f_{t^s}|^2\,dx\,dt + C_2(\epsilon)\sum_{k=0}^{s-1}\int_0^t\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}dt,
\end{aligned}
\tag{3.12}
$$

Substituting $\delta = \frac{(n\mu(2s+1)+\epsilon)}{\mu_0-\epsilon}$ into (3.12), one can see that

$$\begin{aligned}
&\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} \\
&\leq \frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon}\int_0^t \Big(\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}\Big)dt \\
&\quad + C\Big(\sum_{k=0}^{s-1}\|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)} + \sum_{k=0}^{s-1}\int_0^t \|u^N_{t^k}(x,t)\|^2_{W^1(\Omega)}dt + \int_{\Omega_t}|f_{t^s}|^2\,dx\,dt\Big),
\end{aligned}$$

where C is a positive constant independent of N and f. From this inequality and by the inductive hypothesis for (3.8), we get

$$\begin{aligned}
&\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} \\
&\leq \frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon}\int_0^t \Big(\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}\Big)dt \\
&\quad + C_1(1+t)e^{\frac{n\mu(2s-1)+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^{s-1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))} + C_2 t\|f_{t^s}\|^2_{L^\infty(0,\infty;L_2(\Omega))},
\end{aligned} \tag{3.13}$$

where C_i are positive constants independent of N and f; $i=1,2$. Put

$$\begin{gathered}
J^N_s(t) = \|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)}, \\
\phi(t) = C_1(1+t)e^{\frac{n\mu(2s-1)+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^{s-1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))} + C_2 t\|f_{t^s}\|^2_{L^\infty(0,\infty;L_2(\Omega))}.
\end{gathered}$$

From (3.13) we have

$$J^N_s(t) \leq \frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon}\int_0^t J^N_s(\tau)d\tau + \phi(t).$$

From this inequality and Gronwall-Bellman inequality we obtain

$$J^N_s(t) \leq Ce^{\frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^{s}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where C is a positive constant is independent of N and f. Therefore,

$$\begin{aligned}
&\|u^N_{t^{s+1}}(x,t)\|^2_{L_2(\Omega)} + (\mu_0-\epsilon)\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} \\
&\leq Ce^{\frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^{s}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},
\end{aligned} \tag{3.14}$$

where C is positive constant independent of N and f. Therefore,

$$\|u^N_{t^s}(x,t)\|^2_{W^1(\Omega)} \leq C_1 e^{\frac{(2s+1)n\mu+\epsilon}{\mu_0-\epsilon}t}\sum_{k=0}^{s}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))}, \tag{3.15}$$

where C_1 is a positive constant independent of N and f. Hence (3.8) holds for s.

Since $\gamma > \gamma_0 = \frac{n\mu}{2\mu_0}$, there exists a constant ϵ such that

$$2(2s+1)\gamma_0 = \frac{n\mu(2s+1)}{\mu_0} < \frac{n\mu(2s+1)+\epsilon}{\mu_0-\epsilon} < 2(2s+1)\gamma. \tag{3.16}$$

Multiplying both sides of (3.15) by $e^{-2(2s+1)\gamma t}$ and integrating it with respect to t from 0 to ∞. Then applying (3.16), we obtain

$$\|u^N_{t^s}(x,t)\|^2_{W^{1,1}(e^{-(2s+1)\gamma t},\Omega_\infty)} \le C_2 \sum_{k=0}^{s} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where C_2 is a positive constant independent of N and f. Hence (3.9) holds for s and (3.9) is proved.

Since C_2 from inequality (3.9) are independent of N, relation (3.9) yields the inequality. The proof is complete. □

4. Asymptotical expansions of solutions

Let Ω be a bounded domain in $\mathbb{R}^n (n \ge 2)$ with the boundary $\partial\Omega$. We suppose that $\partial\Omega\setminus\{0\}$ is a smooth manifold and Ω in a neighborhood of the origin 0 coincides with the cone $K = \{x : x/|x| \in G\}$, where G is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. Set $Q_\infty = \Omega \times (0,\infty)$ and $S_\infty = \partial\Omega \times (0,+\infty)$. We will use notations: $D^\alpha = \partial^{|\alpha|}/\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$ for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $u_{t^k} = \partial^k u/\partial t^k$, $r = |x| = \big(\sum_{k=1}^n x_k^2\big)^{\frac12}$.

Suppose that $w = (w_1, \dots, w_{n-1})$ is a local coordinate system on the unit sphere S^{n-1}. Let $L_0(0,t,D)$ be the principal part of the operator $L(x,t,D)$ at the coordinate origin. We can write $L_0(0,t,D)$ in the form

$$L_0(0,t,D) = r^{-2} Q(w,t,D_w, rD_r),$$

where $Q(w,t,D_w,rD_r)$ is the linear operator with smooth coefficients, $D_r = i\partial/\partial_r$ $D_w = \partial/\partial w_1 \dots \partial w_{n-1}$. Consider the spectral problem

$$Q(\omega,t,\lambda,D_w)v(w) = 0, w \in G, \tag{4.1}$$

$$v|_{\partial G} = 0. \tag{4.2}$$

It is well known that for every $t \in [0,\infty)$ its spectrum is discrete [1]. In the cone K we consider Dirichlet problem for the equation

$$L_0(0,t,D)u = r^{-i\lambda(t)-2} \sum_{s=0}^{M} \ln^s r f_s(\omega,t), \tag{4.3}$$

The following lemma can be found in [8].

Lemma 4.1. *Assume that $f_s(\omega,t)$, $s = 0,\dots,M$ are infinitely differentiable functions with respect to ω. Then there exists the solution of the Dirichlet problem for* (4.3) *in the form*

$$u(x,t) = r^{-i\lambda(t)} \sum_{s=0}^{M+\mu} \ln^s r g_s(\omega,t), \tag{4.4}$$

where g_s, $s = 0,\dots M+\mu$, are infinitely differentiable functions with respect to ω, $\mu = 1$ if λ_0 is simple eigenvalue of problem (4.1)-(4.2)*, and $\mu = 0$ if λ_0 is not a eigenvalue of this problem.*

Now we will study the asymptotical expansions of solutions of problem (2.3)–(2.5). Denote by K_∞ a infinite cylinder with base $K:\ K_\infty = K \times (0,\infty)$. Rewrite the equation (2.3) in the form

$$L_0(0,t,D)u = F(x,t) \tag{4.5}$$

where $F(x,t) = (u_{tt} + f) + [L_0(0,t,D) - L(x,t,D)]u$.

Lemma 4.2. *Assume that $u(x,t)$ is a generalized solution of problem* (2.3)–(2.5) *in the space $W^{1,1}(e^{-\gamma t}, K_\infty)$ such that $u \equiv 0$ whenever $|x| > R$, R a positive constant, and $u_{t^k} \in W^{2+l,0}_{\beta}(e^{-(2k+1)\gamma t}, K_\infty)$, $F_{t^k} \in W^{l,0}_{\beta'}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h$, $\beta' < \beta \le l+1$. In addition, suppose that the straight lines*

$$\operatorname{Im}\lambda = -\beta + l + 2 - \frac{n}{2}, \quad \operatorname{Im}\lambda = -\beta' + l + 2 - \frac{n}{2}$$

do not contain points of spectrum of problem (4.1)-(4.2) *for every $t \in [0,\infty)$, and in the strip*

$$-\beta + l + 2 - \frac{n}{2} < \operatorname{Im}\lambda < -\beta' + l + 2 - \frac{n}{2}$$

there exists only simple eigenvalue $\lambda(t)$ of problem (4.1)-(4.2). *Then the following representation holds*

$$u(x,t) = c(t) r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t),$$

where $\phi(x,t)$ is an infinitely differentiable function of (ω,t), $c_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, and $(u_1)_{t^k} \in W^{l+2,0}_{\beta'}(e^{-(k+1)\gamma t}, K_\infty)$ for $k \le h$.

Proof. From the result of [8] it follows that for almost every $t \in (0,\infty)$ we have

$$u(x,t) = c(t) r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t), \tag{4.6}$$

where $\phi(\omega,t)$ is the energy function of the problem (4.1)-(4.2) which corresponds to the eigenvalue $\lambda(t)$, $u_1 \in W^{2+l,0}_{\beta'}(e^{-(k+1)\gamma t}, K_\infty)$ and

$$c(t) = i\int_K F(x,t) r^{-i\overline{\lambda(t)}+2-n}\psi(x,t)dx,$$

where $\psi(x,t)$ is the energy function of the problem conjugating to the problem (4.1)-(4.2) which corresponds to the eigenvalue $\overline{\lambda(t)}$. Since $\operatorname{Im}\overline{\lambda(t)} > \beta' - l - 2 + \frac{n}{2}$, from $F(x,t) \in W^{l,0}_{\beta'}(e^{-(2k+1)\gamma t}, K_\infty)$ it follows that $c(t) \in L_{2,\mathrm{loc}}(0,\infty)$. Hence the assertion is proved for $h = 0$.

Assume that the assertion is true for $0, 1, \dots, h-1$. Denoting u_{t^h} by v. From (2.3) and (4.5) we obtain

$$L_0(0,t,D)v = F_{t^h} - \sum_{k=1}^{h}\binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}}, \tag{4.7}$$

where

$$L_{0t^k} = \sum_{i,j=1}^{n} \frac{\partial^k a_{pq}(0,t)}{\partial t^k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}.$$

Putting $S_0(\omega,t) = r^{-i\lambda(t)}\phi(\omega,t)$. From (4.7) it follows that

$$\begin{aligned}
&\sum_{k=1}^{h}\binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}} \\
&= \sum_{k=1}^{h}\binom{h}{k} L_{0t^k}(0,t,D)\big[(cS_0)_{t^{h-k}}\big] + \sum_{k=1}^{h}\binom{h}{k} L_{0t^k}(0,t,D)(u_1)_{t^{h-k}}.
\end{aligned}$$

Using the inductive hypothesis and by arguments used in the proof of case $h = 0$ we find that

$$u_{t^h} = v = \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}} (S_0)_{t^k} + d(t) S_0 + u_2,$$

where $d(t) \in L_{2,\mathrm{loc}}(0,\infty)$, $u_2 \in W^{2+l,0}_{\beta'}(e^{-(2h+1)\gamma t}, K_\infty)$. Putting $S_1 = S_0^{-1}(u_1)_{t^{h-1}}$, $S_2 = S_0^{-1}u_2 - S_0^{-2}(S_0)_t(u_1)_{t^{h-1}}$. Since $(u_1)_{t^{h-1}}, u_2 \in W^{l,0}_{\beta'}(e^{-(2h+1)\gamma t}, K_\infty)$ so $S_1, S_2 \in W^{0,0}_{-\frac{n}{2}}(e^{-(2h+1)\gamma t}, K_\infty)$. Therefore, $I(t) = c_{t^{h-1}}(t) - c_{t^{h-1}}(0) - \int_0^t d(\tau)d\tau = \int_0^t S_2(x,\tau)d\tau - S_1(x,t) + S_1(x,0) \in W^0_{-\frac{n}{2}}(K)$. Hence $I(t) \equiv 0$ and $c_{t^h} = d \in L_{2,\mathrm{loc}}[0,\infty)$, $(u_1)_{t^h} = u_2 \in W^{2m+l,0}_{\beta'}(e^{-(h+1)\gamma t}, K_\infty)$. This completes the proof. □

Theorem 4.3. *Let $u(x,t)$ be a generalized solution of (2.3)-(2.5) in the space $W^{1,1}(e^{-\gamma t}, K_\infty)$ such that $u \equiv 0$ whenever $|x| > R$, and $f_{t^k} \in L^\infty(0,\infty; W^l_0(K))$ for $k \le 2l + h + 1$, $f_{t^k}(x,0) = 0$ for $k \le 2l + h$. Assume that the straight lines*

$$\operatorname{Im}\lambda = 1 - \frac{n}{2}, \quad \operatorname{Im}\lambda = 2 + l - \frac{n}{2}$$

do not contain points of spectrum of (4.1)-(4.2) for every $t \in [0,\infty)$, and in the strip

$$1 - \frac{n}{2} < \operatorname{Im}\lambda < 2 + l - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of (4.1)-(4.2). Then the following representation holds

$$u(x,t) = \sum_{s=0}^{l} c_s(t) r^{-i\lambda(t)+s} P_{3l,s}(\ln r) + u_1(x,t), \tag{4.8}$$

where $P_{3l,s}$ is a polynomial of order less than $3l+1$ and coefficients infinitely differentiable functions of (ω,t), $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, $(u_1)_{t^k} \in W^{2+l,0}_0(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h + l$.

Proof. We will use induction on l. If $l = 0$ the statement follows from Lemma 4.2 with $\beta = 1$, $\beta' = 0$ and theorem 3.2. Let the statement be true for $j \le (l-1)$. We distinguish the following cases:

Case 1: $1 - \frac{n}{2} < \operatorname{Im}\lambda(t) < 2 + j - \frac{n}{2}$. From the induction hypothesis,

$$u(x,t) = \sum_{s=0}^{j} c_s(t) r^{-i\lambda(t)+s} P_{3j,s}(\ln r) + u_1(x,t), \tag{4.9}$$

where $P_{3j,s}$ is a polynomial of less than $3j+1$ and coefficients infinitely differentiable functions of (ω,t), $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, $(u_1)_{t^k} \in W^{2+j,0}_0(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h + j$. Therefore

$$L_0(0,t,D)u_1 = F_1 - LS + S_{tt},$$

where $F_1 = (u_1)_{tt} + f + [L_0(0,t,D) - L(x,t,D)]u_1$,

$$S = \sum_{s=0}^{j} c_s(t) r^{-i\lambda(t)+s} P_{3j,s}(\ln r).$$

Since $f_{t^k} \in L^\infty(0,\infty; W^{j+1}_0(K))$ for $k \le 2(j+1) + h + 1$ and $f_{t^k}(x,0) = 0$ for $k \le 2j + h + 1$, so $f_{t^k} \in L^\infty(0,\infty; W^j_0(K))$, $k \le 2j + (h+2) + 1$, and $f_{t^k}(x,0) = 0$,

$k \le 2j+h+1$. Therefore, $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$ and $(u_1)_{t^k} \in W_0^{j+2,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+j+2$. Hence it follows that $(F_1)_{t^k} \in W_0^{j+1,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le j+h+1$. On the other hand

$$-LS + S_{tt} = F_2 + \sum_{s=0}^{j+1} \widetilde{c}_s(t) r^{-i\lambda(t)-2+s} \widetilde{P}_{3j+2,s}(\ln r),$$

where $\widetilde{P}_{3j+2,s}$ is a polynomial having order less than $3j+3$ and its coefficients are infinitely differentiable functions of (ω, t), $(F_2)_{t^k} \in W_0^{j+1,0}(e^{-(2k+1)\gamma t}, K_\infty)$, and $(\widetilde{c}_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$ for $k \le h+j+1$. Therefore we obtain

$$L_0(0,t,D)u_1 = F_3 + \sum_{s=0}^{j+1} \widetilde{c}_s(t) r^{-i\lambda(t)-2+s} \widetilde{P}_{3j+2,s}(\ln r),$$

where $F_3 = F_1 + F_2 \in W_0^{j+1,0}(e^{-(2k+1)\gamma t}, K_\infty) \subseteq H_{-1}^{j,0}(e^{-(2k+1)\gamma t}, K_\infty)$. By Lemma 3.1 we find

$$u_1(x,t) = \sum_{s=0}^{j+1} \widetilde{c}_s(t) r^{-i\lambda(t)+s} \widetilde{P}_{3j+3,s}(\ln r) + u_2(x,t),$$

where $\widetilde{P}_{3j+3,s}$ is a polynomial having order less than $3j+4$ and its coefficients are infinitely differentiable functions of (ω, t), $(u_2)_{t^k} \in W_{-1}^{2+j,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+j+1$. Therefore $(u_2)_{t^k} \in W_0^{j+3,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+j+1$. Hence and from (4.9) it follows that

$$u(x,t) = \sum_{s=0}^{j+1} c_s(t) r^{-i\lambda(t)+s} P_{3j+3,s}(\ln r) + u_2(x,t),$$

where $P_{3j+3,s}$ is a polynomial having order less than $3j+4$ and its coefficients are infinitely differentiable functions of (ω, t), $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, and $(u_2)_{t^k} \in W_0^{j+3,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+j+1$.

Case 2: $2+j-\frac{n}{2} < \operatorname{Im}\lambda(t) < 3+j-\frac{n}{2}$. From theorem 3.2 we have $u_{t^k} \in W^{1,1}(e^{-(2k+1)\gamma t}, K_\infty)$. Hence (see [3]) $u_{t^k} \in W_1^{2,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+2l$. On the other hand, the strip $1-\frac{n}{2} \le \operatorname{Im}\lambda \le 2-\frac{n}{2}$ does not contain points of spectrum of the problem (4.1)-(4.2) for every $t \in (0,\infty)$. Hence and from theorems on the smoothness of solutions of elliptic problems in domains with conical points (see [8]) it follows that $u_{t^k} \in W_0^{2,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+2l$.

We will prove that if $f_{t^k} \in L^\infty(0,\infty; W_0^j(K))$ for $k \le 2j+h+1$ and $f_{t^k}(x,0) = 0$ for $k \le 2j+h$, then $u_{t^k} \in W_0^{2+j,0}(e^{-(2k+1)\gamma t}, K_\infty)$, $k \le h+2l-j$. This assertion was proved for $j=0$. Assume that it is true for $j-1$. Since $f_{t^k} \in L^\infty(0,\infty; W_0^{j-1}(K))$ for $k \le 2(j-1)+(h+2)+1$ and $f_{t^k}(x,0) = 0$ for $k \le 2(j-1)+h+2$, then from inductive hypothesis it follows that $u_{t^k} \in W_0^{j+1,0}(e^{-(2k+1)\gamma t}, K_\infty)$, $k \le h+2l-j+3$. Therefore, $u_{t^{k+2}} \in W_{-1}^{j-1,-1}(e^{-(2k+3)\gamma t}, K_\infty)$ for $k \le h+2l-j$. Hence and from the fact that the strip

$$j+1-\frac{n}{2} < \operatorname{Im}\lambda < j+2-\frac{n}{2}$$

does not contain points of spectrum of (4.1)-(4.2) for every $t \in [0,\infty)$, we obtain $u_{t^k} \in W_{-1}^{j+1,0}(e^{-(2k+1)\gamma t}, K_\infty)$, $k \le h+2l-j$. Hence $u_{t^k} \in W_0^{j+2,0}(e^{-(2k+1)\gamma t}, K_\infty)$ for $k \le h+2l-j$.

By Lemma 4.2 and from above arguments we obtain

$$u(x,t) = c(t)r^{-i\lambda(t)}\varphi(\omega,t) + u_1(x,t),$$

where φ is an infinitely differentiable function of (ω,t) which does not depend on the solution, $c_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, and $(u_1)_{t^k} \in W_0^{2+l,0}(e^{-(k+1)\gamma t}, K_\infty)$ for $k \le h+l$.
case 3: There exists t_0 such that $\operatorname{Im}\lambda(t_0) = 2+j-\frac{n}{2}$. We can assume that $2+j-\epsilon-\frac{n}{2} < \operatorname{Im}\lambda(t) < 3+j-\epsilon-\frac{n}{2}, 0<\epsilon<1$. By arguments used in case 1 and 2 we obtain (4.8). The proof is complete. □

Theorem 4.4. *Let $u(x,t)$ be a generalized solution of* (2.3)-(2.5) *in the space $W^{1,1}(e^{-\gamma t},\Omega_\infty)$, and $f_{t^k} \in L^\infty(0,\infty; W_0^l(\Omega))$ for $k \le 2l+h+1$, $f_{t^k}(x,0) = 0$ for $k \le 2l+h$. Assume that the straight lines*

$$\operatorname{Im}\lambda = 1-\frac{n}{2}, \quad \operatorname{Im}\lambda = 2+l-\frac{n}{2}$$

do not contain points of spectrum of (4.1)-(4.2) *for every $t \in [0,\infty)$, and in the strip*

$$1-\frac{n}{2} < \operatorname{Im}\lambda < 2+l-\frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of (4.1)-(4.2). *Then the following representation holds*

$$u(x,t) = \sum_{s=0}^{l} c_s(t) r^{-i\lambda(t)+s} P_{3l,s}(\ln r) + u_1(x,t), \tag{4.10}$$

where $P_{3l,s}$ is a polynomial of order less than $3l+1$ and coefficients infinitely differentiable functions of (ω,t), $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, $(u_1)_{t^k} \in W_0^{2+l,0}(e^{-(2k+1)\gamma t},\Omega_\infty)$ for $k \le h+l$.

Proof. Surrounding the point 0 by a neighbourhood U_0 with small diameter that the intersection of Ω and U_0 coincides with K. Consider a function $u_0 = \varphi_0 u$, where $\varphi_0 \in \mathring{C}^\infty(U_0)$ and $\varphi_0 \equiv 1$ in some neighbourhood of 0. The function u_0 satisfies the system

$$L(x,t,D)u_0 - (u_0)_{tt} = \varphi_0 f + L'(x,t,D)u,$$

where $L'(x,t,D)$ is a linear differential operator having order less than 2. Coefficients of this operator depend on the choice of the function φ_0 and equal to 0 outside U_0. Hence and from arguments analogous to the proof of Theorem 4.1, we obtain

$$\varphi_0 u(x,t) = \sum_{s=0}^{l} c_s(t) r^{-i\lambda(t)+s} P_{3l,s}(\ln r) + u_2(x,t), \tag{4.11}$$

where $P_{3l,s}$ is a polynomial of order less than $3l+1$ and coefficients infinitely differentiable functions of (ω,t), $(c_s)_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, $(u_2)_{t^k} \in W_0^{2+l,0}(e^{-(k+1)\gamma t},\Omega_\infty)$ for $k \le h+l$.

The function $\varphi_1 u = (1-\varphi_0)u$ equals to 0 in some neighbourhood of the conical point. We can apply the known theorem on the smoothness of solutions of elliptic problems in a smooth domain to this function and obtain $\varphi_1 u \in W_0^{2+l}(\Omega)$ for almost every $t \in (0,\infty)$. Hence we have $(\varphi_1 u)_{t^k} \in W_0^{2+l,0}(e^{-(k+1)\gamma t},\Omega_\infty)$ for $k \le h+l$. Since $u = \varphi_0 u + \varphi_1 u$ so from (4.11) we obtain (4.10). The proof is complete. □

5. An example

In this section we apply the previous results to the Cauchy-Dirichlet problem for the wave equation. Let Ω be a bounded domain in $\mathbb{R}^2$. It is shown that the asymptotic of the generalized solution of the problem depends on the structure of the boundary of the domain, and the right-hand side. We consider the Cauchy-Dirichlet problem for wave equation in Ω_∞:

$$\Delta u - u_{tt} = f(x,t) \tag{5.1}$$

with initial conditions

$$u|_{t=0} = u_t|_{t=0} = 0 \tag{5.2}$$

and boundary condition

$$u|_{S_\infty} = 0, \tag{5.3}$$

where Δ is the Laplace operator.

Assume that in a neighborhood of the origin of coordinates, the boundary $\partial\Omega$ coincides with a rectilinear angle having measure w_0. Then spectral problem (4.1)-(4.2) is Sturm-Liouville problem:

$$v_{ww} - \lambda^2 v = 0, 0 < w < w_0, \tag{5.4}$$

$$v(0) = v(w_0) = 0. \tag{5.5}$$

Eigenvalues of (5.4)-(5.5) are $\lambda_k = \pm i(\pi k/w_0)$, k is a positive integer. They are simple eigenvalues. Then it follows that $mathop\mathrm{Im}\lambda_k = \pm(\pi k/w_0)$.

If $w_0 > \pi$, then $0 < \pi/w_0 < 1$. On the other hand $0 < \omega_0 < 2\pi$ so $(k\pi/w_0) > 1$ for all $k \geq 2$. Therefore, in the trip $0 \leq \mathrm{Im}\lambda \leq 1$ there exists only one simple eigenvalue $\lambda(t) = i\pi/w_0$ of the problem (5.4)-(5.5). From Theorem 4.2 we obtain the following result.

Theorem 5.1. *Let $u(x,t)$ be a generalized solution of* (5.1)-(5.3) *in the space $W^{1,1}(e^{-\gamma t}, \Omega_\infty)$. In addition, suppose that $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega))$ for $k \leq h+1$, $f_{t^k}(x,0) = 0$ for $k \leq h$. Then the following representation holds*

$$u(x,t) = c(t) r^{\pi/w_0} P(\ln r) + u_1(x,t),$$

where P is a polynomial having order less than 1 *and its coefficients are infinitely differentiable functions of (ω,t), $c_{t^k} \in L_{2,\mathrm{loc}}(0,\infty)$, $(u_1)_{t^k} \in W_0^{2,0}(e^{-(2k+1)\gamma t}, \Omega_\infty)$ for $k \leq h$.*

References

[1] M. S. Agranovich and M. I. Vishik. *Elliptic problems with a parameter and parabolic problems of general type.* Usp. Mat. Nauk,19, No3, 53-161 (1964).

[2] R. Dautray and J. L. Lions (1990). *Mathematical analysis and numerical methods for science and technology.* Springer-Verlag, vol. 3.

[3] N. M. Hung (1999). *Asymptotic behaviour of solutions of the first buondary-value problem for strongly hyperbolic systems near a conical point at the boundary of the domain.* Math. Sbornik, 19, 103-126.

[4] N. M. Hung and N. T. Anh (2008) *Regularity of solutions of initial-boundary-value problems for parabolc equations in domains with conical points.* J. Differential Equations, Volume 245, Issue 7, 1 October 2008, 1801-1818.

[5] N. M. Hung and Yao J. C. (2008) *Cauchy-Dirichlet problem for second-order hyperbolic equations in cylinder with non-smooth base.* Nonlinear Analysis, Volume 70, Issue 2, 15 January 2009, 741-756.

[6] V. G. Kondratiev. *The boundary problems for elliptic equations in domains with conical or angled points.* Trudy Moskov. Mat. Obshch, T. 16, 209-292, (1967).

[7] A. Kokatov and B. A. Plamenevssky. *On the asymptotic on solutions to the Neumann problem for hyperbolic systems in domain with conical point.* English transl., St. Peterburg Math. J, 16, No 3, 477-506, (2005).
[8] S. A. Nazrov and B. A. Plamenevsky. *Elliptic problems in domains with piecewise-smooth boundary.* Nauka, Moscow, (1990), (in Russian).
[9] V. A. Solonnikov. *On the solvability of classical initial boundary-value problem for the heat equation in a dihedral angle.* Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. 127, 7-48, (1983), (in Russian).

Nguyen Manh Hung
Department of Mathematics, Hanoi National University of Education, Vietnam
E-mail address: `hungnmmath@hnue.edu.vn`

Bui Trong Kim
Department of Mathematics, Hanam College of Education, Vietnam
E-mail address: `buitrongkim@gmail.com`