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EXISTENCE OF MILD SOLUTIONS OF A SEMILINEAR EVOLUTION DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper we prove the existence of a mild solution for a semilinear evolution differential inclusion with nonlocal condition and governed by a family of linear operators, not necessarily bounded or closed, in a Banach space. No compactness assumption is assumed on the evolution operator generated by the family operators. Also, we prove that the set of mild solutions is compact.

1. Introduction

The study of Cauchy problems with nonlocal conditions, which is a generalization for the classical Cauchy problems with initial condition, was motivated by physical problems (see [2, 17]). The pioneering work on nonlocal conditions problems is due to Byszewski [6, 7, 8, 9]. In the few past years, several papers have been devoted to study the existence of solutions for differential equations or differential inclusions with nonlocal conditions. Among others, we refer the reader to [1, 4, 5, 12, 16, 20].

Let E be a Banach space, $I = [0, b]$, $b > 0$, $C(I, E)$ be the Banach space of all continuous functions from I to E with the norm of uniform convergence. Let $\mathcal{L}(E)$ be the space of bounded linear operators on E and $\{A(t) : t \in I\}$ be a family of densely defined linear operator (not necessarily bounded or closed) on E and $T : \Delta = \{(t, s) : 0 \le s \le t \le b\} \to \mathcal{L}(E)$ be the evolution operator generated by the family $\{A(t) : t \in I\}$.

Let F be a Carathèodory type multifunction from $I \times E$ to the collection of all nonempty convex compact subsets of E, and let $g : C(I, E) \to E$ be a function.

Consider the following semilinear differential inclusion with nonlocal condition:

$$\begin{gathered} x'(t) \in A(t)x(t) + F(t, x(t)), \quad t \in I \\ x(0) = g(x) \end{gathered} \tag{1.1}$$

By a mild solution of problem (1.1) we mean a continuous function $x : I \to E$ such that

$$x(t) = T(t, 0)g(x) + \int_0^t T(t, s)f(s)ds, \quad t \in I$$

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where f is a Bochner integrable function such that $f(s) \in F(s, x(s))$, a.e. $t \in I$.

In the present paper we employ the methods of Kamenskii, Obukhowskii and Zecca [18] and Cardinali and Rubbioni [10] to prove the existence of mild solution for (p) without a compactness assumption on the evolution operator T which is generated by the family $\{A(t) : t \in I\}$. The price that we pay to achieve this aim is that we assume that F satisfies a compactness condition involving the Hausdorff measure of noncompactness.

We would like to mention that in a recent paper Fan, Dong and Li [16] proved the existence of mild solution for the semilinear differential equations with nonlocal conditions in Banach space

$$\begin{gathered} x'(t) = A\big(x(t)\big) + f(t, x(t)), \quad t \in (0, b] \\ x(0) = g(x), \end{gathered} \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{U(t) : t \in I\}$, $f : I \times E \to E$ is a Carathèodory type function and satisfies a compactness condition involving the Hausdorff measure of noncompactness and g is a continuous compact function.

Note that Corollary 3.3 generalizes the result of Fan, Dong and Li [16] to the case when the function f become a multifunction. Moreover, since the nonlocal conditions Cauchy problems is a generalization for the classical Cauchy problems with initial conditions, our work generalizes many results in the literature, see for example [10, 14, 18]. For differential inclusions with initial condition, we refer the reader to [13, 15].

Our basic tools are the methods and results for semilinear differential inclusions, the properties of non-compact measure and fixed point techniques.

2. Preliminaries and notation

Let $I = [0, b]$, $b > 0$, $(E, \|\cdot\|)$ be a real Banach space, $C(I, E)$ the space of E-valued continuous functions on I with the uniform norm $\|x\|_C = \sup\{\|x(t)\|,\ t \in I\}$, $L^1(I, E)$ the space of $E-$valued Bochner integrable functions on I with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\|dt$ and $P_k(E)$ $(P_{ck}(E))$ the collection of nonempty compact (convex compact) subsets of E

Definition 2.1. Let X and Y be two topological spaces and let $P(Y)$ the family of nonempty subsets of Y. A multifunction $G : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. The multifunction G is called closed if its graph $\Gamma_G = \{(x, y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. For details and equivalent definitions see [11].

Definition 2.2. Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : P(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega)$$

for every $\Omega \in P(E)$.

Definition 2.3. A measure of noncompactness β is called:

(i) monotone if $\Omega_0, \Omega_1 \in P(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in P(E)$;

(iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As an example of the measure of noncompactness possessing all these properties is the Hausdorff of MNC which is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net}\}.$$

For more information about the measure of noncompactness we refer the reader to [3].

Definition 2.4. A multifunction $G : E \to P_k(E)$ is said to be χ-condensing if for every bounded subset $\Omega \subseteq E$ the relation

$$\chi(G(\Omega)) \geq \chi(\Omega)$$

implies the relative compactness of Ω.

Definition 2.5. A countable set $\{f_n : n \geq 1\} \subseteq L^1(I, E)$ is said to be semicompact if

- (i) it is integrably bounded: $\|f_n(t)\| \leq \omega(t)$ for a.e. $t \in I$ and every $n \geq 1$ where $\omega \in L^1(I, \mathbb{R}^+)$
- (ii) the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E for a.e. $t \in I$.

Now, let for every $t \in I = I$, $A(t) : E \to E$ be a linear operator such that

- (i) For all $t \in I$, $D(A(t)) = D(A) \subseteq E$ is independent of t.
- (ii) For each $s \in I$ and each $x \in E$ there is a unique solution $v : [s, b] \to E$ for the evolution equation

$$\begin{gathered} v'(t) = A(t)v(t), \quad t \in [s, b] \\ v(s) = x. \end{gathered} \tag{2.1}$$

In this case an operator T can be defined as

$$T : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E), \quad T(t, s)(x) = v(t),$$

where v is the unique solution of (2.1) and $\mathcal{L}(E)$ is the family of linear bounded operators on E.

Definition 2.6. The operator T is called the evolution operator generated by the family $\{A(t) : t \in I\}$. It is known that (see [19])

1. (1) $T(s, s) = I_E$
2. (2) $T(t, r)T(r, s) = T(t, s)$, for all $0 \leq s \leq r \leq t \leq b$.
3. (3) each operator $T(t, s)$ is strongly differentiable and

$$\frac{\partial T(t,s)}{\partial t} = A(t)T(t, s) \quad \frac{\partial T(t,s)}{\partial s} = -T(t, s)A(s)$$

Definition 2.7. The operator $G : L^1(I, X) \to C(I, X)$ defined by

$$Gf(t) = \int_0^t T(t, s)f(s)ds \tag{2.2}$$

is called the generalized Cauchy operator, where $T(.,.)$ is the evolution operator generated by the family of operators $\{A(t) : t \in I\}$.

In the sequel we will need the following known results.

Lemma 2.8 (cite[Proposition 4.2.1]k1)**.** *Every semicompact set is weakly compact in the space $L^1(I, E)$.*

Lemma 2.9 ([10, Theorem 2])**.** *The generalized Cauchy operator G satisfies the properties*

(G1) *there exists $\zeta \geq 0$ such that*
$$\|Gf(t) - Gh(t)\| \leq \zeta \int_0^t \|f(s) - h(s)\| ds$$
for every $f, h \in L^1(I, E)$, $t \in I$.

(G2) *for any compact $K \subseteq E$ and sequence $(f_n)_{n\geq 1}$, $f_n \in L^1(I, E)$ such that for all $n \geq 1$, $f_n(t) \in K$, a. e. $t \in I$, the weak convergence $f_n \rightharpoonup f_0$ in $L^1(I, E)$ implies the convergence $Gf_n \to Gf_0$ in $C(I, E)$.*

Lemma 2.10 ([18, Theorem 5.1]). *Let $S : L^1(I, E) \to C(I, E)$ be an operator satisfying condition (G_2) and the following Lipschits condition (weaker than* (G1)*)*

(G1')
$$\|Sf - Sg\|_{C(I,E)} \leq \zeta \|f - g\|_{L^1(I,E)}.$$

Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(I, E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(I, E)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to f_0 in $L^1(I, E)$ then $Sf_n \to Sf_0$ in $C(I, E)$.

Lemma 2.11 ([18, Theorem 4.2.2]). *Let $S : L^1(I, E) \to C(I, E)$ be an operator satisfying conditions* (G1), (G2) *and let the set $\{f_n\}_{n=1}^{\infty}$ be integrably bounded with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in I$, where $\eta(.) \in L^1_+(I, \mathbb{R}^+)$ and χ is the Hausdorff MNC. Then*
$$\chi(\{Sf_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s) ds$$
for all $t \in I$, where $\zeta \geq 0$ is the constant in condition (G1).

Theorem 2.12 ([18, Corollary 3.3.1]). *If U is a closed convex subset of a Banach space and R is a closed β-condensing multifunction from U to the family of nonempty convex compact subsets of U, where β is a nonsingular MNC defined on the subsets of U, then R has a fixed point.*

Theorem 2.13 ([18, Proposition 3.5.1]). *Let W be a closed subset of a Banach space E and $R : W \to P_{ck}(E)$ be a closed multifunction which is β-condensing on every bounded subset of W, where $P_{ck}(E)$ is the family of nonempty convex compact subsets of E and β is a monotone measure of noncompactness. If the set of fixed points for R is a bounded subset of E then it is compact.*

3. Main Results

In the following theorem we prove the existence of mild solution for (1.1).

Theorem 3.1. *Let $I = [0, b]$, $\{A(t) : t \in I\}$ be a family of linear (not necessarily bounded) operators, $A(t) : D(A) \subset E \to E$, $D(A)$ not depending on t and dense subset of E and $T : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E)$ be the evolution operator generated by the family $\{A(t) : t \in I\}$.*

Let F be a multifunction defined from $I \times E$ to the family of nonempty closed convex subsets of E such that

(H1) *for every x the multifunction $t \to F(t, x)$ admits a strongly measurable selection and for a.e. $t \in I$ the multifunction $x \to F(t, x)$ is upper semicontinuous;*

(H2) *there exists a function* $m \in L^1(I, \mathbb{R}^+)$ *such that for every* $x \in E$
$$\|F(t,x)\| \leq m(t)(1 + \|x\|) \quad \text{for a.e. } t \in I;$$
(H3) *there exists a function* $h \in L^1(I, \mathbb{R}^+)$ *such that for every bounded* $D \subset E$*:*
$$\chi(F(t,D)) \leq h(t)\chi(D) \quad \text{for a.e. } t \in I,$$
where χ *is the Hausdorff measure of noncompactness;*

(H4) *let* $g : C(I,E) \to E$ *be a continuous function that maps every bounded set into relatively compact subset of* E *and that*
$$\|g(x)\| \leq c\|x\|_C + d, \quad \forall x \in C(I,E),$$
for some positive constants c *and* d*.*

Then the nonlocal condition Cauchy problem (1.1) *has a mild solution provided that*
$$M(c + \|m\|_{L^1}) \neq 1, \tag{3.1}$$
where $M = \sup_{(t,s)\in\Delta} \|T(t,s)\|_{\mathcal{L}(E)}$

Proof. From the strong continuity of the evolution operator T on the compact set Δ, a positive real number M can be found such that
$$\|T(t,s)\| \leq M, \quad \forall (t,s) \in \Delta$$
Thus, the number M mentioned in the statement of the theorem is well defined. From assumptions (H1) and (H2), see [10, Lemma 4], it follows that the super position multioperator
$$\operatorname{sel}_F : C(I,E) \to 2^{L^1(I,E)}$$
defied by
$$\operatorname{sel}_F(x) = \mathcal{S}^1_{F(.,x(.))} = \{f \in L^1(I,E) : f(t) \in F(t,x(t)), \quad \text{a.e. } t \in I\}$$
is well defined and weakly closed in the following sense: if $(x_n)_{n\geq 1}, (f_n)_{n\geq 1}$ are two sequences in $C(I,E)$ and $L^1(I,E)$ respectively such that $f_n \in \operatorname{sel}_F(x_n)$, for all $n \geq 1$ and if $x_n \to x_0$ in $C(I,E)$ and f_n converges weakly to f_0 in $L^1(I,E)$ then $f_0 \in \operatorname{sel}_F(x_0)$. Therefore, the multifunction R defined on $C(I,E)$ by
$$R(x) = \{y \in C(I,E) : y(t) = T(t,0)g(x) + \int_0^t T(t,s)f(s)ds, \ f \in \operatorname{sel}_F(x)\}, \tag{3.2}$$
has nonempty values. It is easy to see that any fixed point for R is a mild solution for (1.1). So, our goal is to prove that R satisfies the conditions of Theorem 2.12 in the preliminaries. The proof will be given in steps.

Step 1. The values of R are convex subsets in $C(I,E)$. Let $x \in C(I,E)$, $y_1, y_2 \in R(x)$ and $0 < \alpha < 1$. From the definition of M we get
$$\begin{aligned}
(1-\alpha)y_1 + \alpha y_2 &= (1-\alpha)T(t,0)g(x) + \alpha T(t,0)g(x) \\
&\quad + \int_0^t ((1-\alpha)T(t,s)f(s) + \alpha T(t,s)h(s))ds \\
&= T(t,0)g(x) + \int_0^t T(t,s)((1-\alpha)f(s) + \alpha h(s))ds \in R(x),
\end{aligned}$$
where $f, h \in \operatorname{sel}_F(x)$. Which means that $R(x)$ is convex for each $x \in C(I,E)$.

Step 2. The values of R are compact subsets in $C(I,E)$. Let $x \in C(I,E)$ and $(z_n)_{n\geq 1}$ be a sequence in $R(x)$. To prove that $R(x)$ is compact we have to show that $(z_n)_{n\geq 1}$ has a subsequence converging to a point z in $R(x)$. By (3.2) for each $n \geq 1$,

$$z_n(t) = T(t,0)g(x) + \int_0^t T(t,s)f_n(s)ds, \quad \text{where } f_n(s) \in F(s, x(s)), \text{ a.e. } t \in I.$$

By (H2)

$$\|f_n(t)\| \leq \|F(t, x(t)\| \leq m(t)(1 + \|x\|_C), \quad \text{a.e. } t \in I.$$

Thus the set $\{f_n : n \geq 1\}$ is integrably bounded in $L^1(I, E)$. Further, the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E for a.e. $t \in I$ because by (H3) we have

$$\chi(\{f_n(t) : n \geq 1\}) \leq \chi(F(t, x(t))) \leq h(t)\chi(\{x(t)\}) = 0, \quad \text{a.e. } t \in I.$$

Hence the set $\{f_n : n \geq 1\}$ is semicompact in $L^1(I, E)$. From Lemma 2.8, it follows that $\{f_n : n \geq 1\}$ is weakly compact in $L^1(I, E)$. So, without loss of generality we can assume that the sequence $(f_n)_{n\geq 1}$ converges weakly to a function f in $L^1(I, E)$ with $f(t) \in F(t, x(t))$ a.e. $t \in I$. By applying Lemma 2.10, $Gf_n \to Gf$, where G is the generalized Cauchy operator defined by (2.2). Thus

$$\lim_{n\to\infty} z_n(t) = T(t,0)g(x) + Gf(t) = T(t,0)g(x) + \int_0^t T(t,s)f(s)ds.$$

So that $z \in R(x)$. Hence $R(x)$ is compact.

Step 3. R is closed, i.e. its graph is closed. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two sequences in $C(I, E)$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ in $C(I, E)$ and $y_n \in R(x_n)$ for all $n \geq 1$. By means of the definition of the generalized Cauchy operator, we can write

$$y_n = T(t,0)g(x_n) + Gf_n(t), \quad \forall n \geq 1,\ \forall t \in I,$$

where $f_n \in \operatorname{sel}_F(x_n)$. Since g is continuous on $C(I, E)$, $\lim_{n\to\infty} g(x_n) = g(x)$ in E. So,

$$\lim_{n\to\infty} T(t,0)g(x_n) = T(t,0)g(x).$$

To apply Lemma 2.10 we have to show that the set $\{f_n : n \geq 1\}$ is semicompact. Since the sequence $(x_n)_{n\geq 1}$ converges uniformly to x, we can find a positive real number k such that $\|x_n\|_C \leq k$, for all $n \geq 1$. By (H2), for every $n \geq 1$,

$$\begin{aligned}\|f_n(t)\| &\leq \sup\{\|z\| : z \in F(t, , x_n(t))\} \\ &\leq (1 + \|x_n\|_C)m(t) \leq (1 + k)m(t), \quad \text{a.e. } t \in I.\end{aligned}$$

This shows that the family $\{f_n : n \geq 1\}$ is integrably bounded. Furthermore, by (H3)

$$\begin{aligned}\chi(\{f_n(t) : n \geq 1\}) &\leq \chi(F(t, D_t)), \quad D_t = \{x_n(t) : n \geq 1\} \\ &\leq h(t)\chi(D_t), \quad \text{for a.e. } t \in I.\end{aligned}$$

Since $(x_n)_{n\geq 1}$ converges to x in $C(I, E)$, the set D_t is relatively compact for each $t \in I$ and consequently $\chi(D_t) = 0$. Thus for a.e. $t \in I$, the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E. Then the set $\{f_n : n \geq 1\}$ is semicompact in $L^1(I, E)$. Invoking Lemma 2.8 this set is weakly compact in $L^1(I, E)$. So, without loss of generality we can assume that f_n converges weakly to a function $f \in L^1(I, E)$. From [10, Lemma 4], we get $f(t) \in F(t, x(t))$, a.e. $t \in I$. Furthermore, by Lemma 2.10,

$$\lim_{n\to\infty} Gf_n = Gf.$$

Thus

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) = T(t,0)g(x) + Gf \\ &= T(t,0)g(x) + \int_0^t T(t,s)f(s)ds, \end{aligned}$$

which yields to, $y \in R(x)$ and so R has a closed graph.

Step 4. R is condensing with respect to a nonsingular measure of noncompactness defined on $C(I,E)$

Let us define a measure of noncompactness ν on $C(I,E)$ as foloows: for each bounded subset Ω of $C(I,E)$ we put

$$\nu(\Omega) = \max_{D\in\Delta(\Omega)} (\gamma(D), \quad \operatorname{mod}_C(D)) \in \mathbb{R}^2, \tag{3.3}$$

where $\Delta(\Omega)$ is the collection of all the denumerable subsets of Ω;

$$\gamma(D) = \sup_{t\in I} e^{-Lt}\chi(\{x(t) : x \in D\}); \tag{3.4}$$

where $\operatorname{mod}_C(D)$ is the modulus of equicontinuity of the set of functions D given by the formula

$$\operatorname{mod}_C(D) = \lim_{\delta\to 0} \sup_{x\in D} \max_{|t_1-t_2|\le\delta} \|x(t_1) - x(t_2)\|; \tag{3.5}$$

and $L > 0$ is a positive real number chosen so that

$$q = 2M \sup_{t\in I} \int_0^t e^{-L(t-s)} h(s)ds < 1 \tag{3.6}$$

where M is defined above by $M = \sup_{(t,s)\in\Delta} \|T(t,s)\|$.

It is known that (see [18, Example 2.1.4]) ν is monotone, nonsingular and regular measure of noncompactness on the space $C(I,E)$.

To show that R is ν-condensing let Ω be a bounded subset of $C(I,E)$ such that

$$\nu(R(\Omega)) \ge \nu(\Omega).$$

We have to prove that Ω is relatively compact. Since ν is regular it suffices to prove that $\nu(\Omega) = (0,0)$. For this purpose it is enough to show that $\nu(R(\Omega)) = (0,0)$.

Let $D = \{y_n : n \ge 1\}$ be a countable subset of $R(\Omega)$ which achieves that maximum. Then there exists a countable set $\{x_n : n \ge 1\}$ of Ω such that $y_n \in R(x_n)$, $\forall n \ge 1$. Hence for every $n \ge 1$ and every $t \in I$

$$y_n(t) = T(t,0)g(x_n) + \int_0^t T(t,s)f_n(s)ds = T(t,0)g(x_n) + Gf_n(t), \tag{3.7}$$

where $f_n \in \operatorname{sel}_F(x_n)$. From the assumption that $\nu(R(\Omega)) \ge \nu(\Omega)$ and by (3.3) we obtain

$$\begin{aligned} (\gamma(\{y_n : n \ge 1\}), \operatorname{mod}_C(\{y_n : n \ge 1\})) &= \nu(R(\Omega)) \ge \nu(\Omega) \\ &\ge (\gamma(\{x_n : n \ge 1\}), \operatorname{mod}_C(\{x_n : n \ge 1\})). \end{aligned}$$

Thus

$$\gamma(\{y_n : n \ge 1\}) \ge \gamma(\{x_n : n \ge 1\}) \tag{3.8}$$

and

$$\operatorname{mod}_C(\{y_n : n \ge 1\}) \ge \operatorname{mod}_C(\{x_n : n \ge 1\}) \tag{3.9}$$

Using (H3) and the relation (3.4) we have, a.e. on I,

$$\begin{aligned}
\chi(\{f_n(s) : n \geq 1\}) &\leq \chi(F(s, \{x_n(s) : n \geq 1\})) \\
&\leq \chi(\{z : z \in F(s, x_n(s)) : n \geq 1\}) \\
&\leq h(s)\chi(\{x_n(s) : n \geq 1\}) \\
&\leq h(s)e^{Ls} \sup_{t\in I} e^{-Lt}\chi(\{x_n(t) : n \geq 1\}) \\
&\leq h(s)e^{Ls}\gamma(\{x_n : n \geq 1\}).
\end{aligned}$$

By Lemma 2.11,

$$\begin{aligned}
\chi(\{Gf_n(s) : n \geq 1\}) &\leq 2M \int_0^s h(t)e^{Lt}(\gamma(\{x_n : n \geq 1\}))dt \\
&= 2M\gamma(\{x_n : n \geq 1\}) \int_0^s h(t)e^{Lt}.
\end{aligned}$$

Thus, by (3.7) and (3.8) we get

$$\begin{aligned}
\gamma(\{x_n : n \geq 1\}) &\leq \gamma(\{y_n : n \geq 1\}) \\
&= \sup_{t\in I} e^{-Lt}\chi(\{y_n(t) : n \geq 1\}) \\
&\leq \sup_{t\in I} e^{-Lt}\chi(\{T(t,0)g(x_n) : n \geq 1\}) \\
&\quad + \sup_{t\in I} e^{-Lt}(\chi(\{Gf_n(t) : n \geq 1\})).
\end{aligned} \tag{3.10}$$

Since $\{x_n : n \geq 1\}$ is a bounded set, the condition (H4) assumes that the set $\{g(x_n) : n \geq 1\}$ is relatively compact in E. So, for each $t \in I$ the set $\{T(t,0)g(x_n) : n \geq 1\}$ is relatively compact in E. Thus

$$\chi(\{T(t,0)g(x_n) : n \geq 1\}) = 0, \quad \forall t \in I.$$

So, (3.10) gives us

$$\begin{aligned}
\gamma(\{x_n : n \geq 1\}) &\leq \gamma(\{y_n : n \geq 1\}) \\
&\leq \sup_{t\in I} e^{-Lt}(\chi(\{Gf_n(t) : n \geq 1\})) \\
&\leq \sup_{t\in I} e^{-Lt} 2M\gamma(\{x_n : n \geq 1\}) \int_0^t e^{Ls}h(s)ds \\
&= 2M\gamma(\{x_n : n \geq 1\}) \sup_{t\in I} e^{-Lt} \int_0^t e^{Ls}h(s)ds \\
&= 2M\gamma(\{x_n : n \geq 1\}) \sup_{t\in I} \int_0^t e^{-L(t-s)}h(s)ds \\
&= \gamma(\{x_n : n \geq 1\})q.
\end{aligned}$$

Because $q < 1$ we get $\gamma(\{x_n : n \geq 1\}) = 0$ and consequently $\gamma(\{y_n : n \geq 1\}) = 0$. Thus the set $\{y_n : n \geq 1\}$ is relatively compact in $C(I, E)$. Hence

$$\lim_{\delta\to 0} \sup_{n\geq 1} \max_{|t_1-t_2|<\delta} \|y_n(t_1) - y_n(t_2)\| = 0.$$

Thus, by (3.5), $\operatorname{mod}_C(\{y_n : n \geq 1\}) = 0$. So, $\nu(R(\Omega)) = (0,0)$ and then $\nu(\Omega) = (0,0)$.

Step 5. To apply the fixed point theorem (Theorem 2.12.), we consider the closed ball in $C(I,E)$:

$$\overline{B(0,r)} = \{x \in C(I,E) : \|x\|_C \le r\},$$

where r is a constant chosen so that

$$r \ge \frac{Md + M\|m\|_{L^1}}{1 - Mc - M\|m\|_{L^1}} \tag{3.11}$$

Note that from (3.1) the number r is well defined. Let us show that the multifunction R maps the closed ball $\overline{B(0,r)}$ into itself. Let $x \in \overline{B(0,r)}$ and $y \in R(x)$. Then for every $t \in I$

$$\|y(t)\| \le \|T(t,0)g(x)\| + \int_0^t \|T(t,s)\|\|f(s)\|ds,$$

where $f(s) \in F(s,x(s))$, for a.e. $s \in I$. Using (H2), (H4) and the fact that $M = \sup_{(t,s)\in\Delta} \|T(t,s)\|$, for any $t \in I$, we have

$$\begin{aligned}
\|y(t)\| &\le M(c\|x\|_C + d) + M\int_0^t m(s)(\|x(s)\| + 1)ds \\
&\le Mcr + Md + M(r+1)\|m\|_{L^1} \\
&= r(Mc + M\|m\|_{L^1}) + Md + M\|m\|_{L^1} \\
&\le r. \quad \text{(from (3.11))}
\end{aligned}$$

It results that $\|y\|_C \le r$. Then $y \in \overline{B(0,r)}$. Applying Theorem 2.12, there is a $x \in C(I,E)$ such that $x \in R(x)$. Clearly the function x is a solution for (1.1) This proves the theorem. □

Theorem 3.2. *In addition of the assumptions in Theorem 3.1, suppose that*

$$M(c + \|m\|_{L^1}) < 1 \tag{3.12}$$

where the numbers M, c and the function m are as in Theorem 3.1. Then the set of mild solutions of (1.1) *is a compact subset in $C(I,E)$.*

Proof. We proved through Theorem 3.1 that the values of the multifunction R defined on $C(I,E)$ by

$$R(x) = \{y \in C(I,E) : y(t) = T(t,0)g(x) + \int_0^t T(t,s)f(s)ds, \quad f \in \mathrm{sel}_F(x)\}$$

are nonempty convex compact subsets in $C(I,E)$.

Since the set of mild solutions of (p) is the set of fixed points of R, then Theorem 2.13 implies that the set of mild solutions of (p) will be compact if we show that the set of fixed points of R is a bounded subset in $C(I,E)$. So, let x be a fixed point for R. From conditions (H2) and (H4), for all $t \in I$,

$$\begin{aligned}
\|x(t)\| &\le \|T(t,0)g(x)\| + \Big\|\int_0^t T(t,s)f(s)ds\Big\|, \quad f \in \mathrm{sel}_F(x) \\
&\le M(c\|x\|_C + d) + M(\|m\|_{L^1}(1 + \|x\|_C)).
\end{aligned}$$

Thus,

$$\|x\|_C(1 - Mc - M\|m\|_{L^1}) \le Md + M\|m\|_{L^1}.$$

So,

$$\|x\|_C \le \frac{Md + M\|m\|_{L^1}}{1 - M(c + \|m\|_{L^1})}.$$

This means that the set of fixed points of the multifunction R is bounded. Thanks Theorem 2.13, the set of mild solutions for (1.1) is compact. □

In the following corollary we generalize the result of Fan, Dong and Li [16]. This generalization deals to prove that the set of mild solutions of (1.2) is nonempty in the case when the function f become a multifunction.

Corollary 3.3. *Given a C_0-semigroup $Z = \{U(t) : t \in I\}$, A be the infinitesimal generator of Z and F and g be as in Theorem 3.1. Then there is a mild solution for the semilinear differential inclusion:*

$$x^{\cdot}(t) \in Ax(t) + F(t, x(t)), \quad x(0) = g(x).$$

Proof. Consider the family $\{A(t); t \in I\}$, where $A(t) = A$ for every $t \in I$ and let $T : D \to \mathcal{L}(E)$ be an operator defined by $T(t, s) = U(t - s)$, $(t, s) \in \Delta$. Then, see [6], the operator T is an evolution operator generated by the family $\{A(t); t \in I\}$. By Theorem 3.1 there is a continuous function $x : I \to E$ such that

$$\begin{aligned} x(t) &= T(t,0)g(x) + \int_0^t T(t,s)f(s)ds; \quad t \in I \\ &= U(t)g(x) + \int_0^t U(t-s)f(s)ds; \quad t \in I, \end{aligned}$$

where $f(s) \in F(s, x(s))$, a.e. This means that x is a mild solution for the semilinear differential inclusion

$$\begin{gathered} x^{\cdot}(t) \in Ax(t) + F(t, x(t)); \quad t \in I \\ x(0) = g(x). \end{gathered}$$

□

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