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# RADIAL POSITIVE SOLUTIONS FOR A NONPOSITONE PROBLEM IN A BALL

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Abstract. In this paper, we study the existence of radial positive solutions for a nonpositone problem when the nonlinearity is superlinear and may have more than one zero.

## 1. Introduction

We study the existence of radial positive solutions for the boundary-value problem

$$\begin{gathered} -\Delta u(x) = \lambda f(u(x)) \quad x \in \Omega, \\ u(x) = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega$ is the unit ball of $\mathbb{R}^N$, $\lambda > 0$ and $N \geq 2$.

The existence of radial positive solutions of (1.1) is equivalent to the existence of positive solutions of the problem

$$\begin{gathered} -u''(r) - \frac{N-1}{r}u'(r) = \lambda f(u(r)) \quad r \in (0,1), \\ u'(0) = 0, \quad u(1) = 0. \end{gathered} \tag{1.2}$$

When $f$ is a monotone nondecreasing nonlinearity and has only one zero, problem (1.2) has been studied by Castro and Shivaji [2] in the ball, and by Arcoya and Zertiti [1] in the annulus.

Our main objective in this article is to prove that the result of the existence of radial positive solutions of problem (1.1) remains valid when $f$ has more than one zero and is not strictly increasing entirely on $[0,+\infty)$; see [2, Theorem 1.1]. More precisely, we assume that the map $f : [0,+\infty) \to \mathbb{R}$ satisfies the following hypotheses

- (F1) $f \in C^1([0,+\infty),\mathbb{R})$ such that $f' \geq 0$ on $[\beta,+\infty)$, where $\beta$ is the greatest zero of $f$.
- (F2) $f(0) < 0$,
- (F3) $\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty$,
- (F4) For some $k \in (0,1)$, $\lim_{d\to+\infty} \big(\frac{d}{f(d)}\big)^{N/2}\big(F(kd) - \frac{N-2}{2N}\, df(d)\big) = +\infty$ where, $F(x) = \int_0^x f(r)dr$.

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**Remark.** We note that in hypothesis (F1) there is no restriction on the function $f(u)$ for $0 < u < \beta$.

**Remark.** It is well known (see [3]) that all positive solutions u of problem (1.1) are radial with $\frac{\partial u}{\partial r} < 0$ in $\Omega$. This fact permits the authors in [1, 2] to apply the shooting method when $f$ satisfies some suitable hypotheses.

In this paper, we follow the work of Castro and Shivaji [2] and we used similar ideas (adapted to our case) in the work of Iaia and Pudipeddi [4]. So we apply the shooting method. For this, we consider the initial auxiliary boundary-value problem

$$\begin{gathered} -u''(r) - \frac{N-1}{r}u'(r) = \lambda f(u(r)) \quad r \in (0,1), \\ u'(0) = 0, \quad u(0) = d, \end{gathered}$$

where $d$ is the parameter of the shooting method.

It has been shown [2] that for each real number $d$, the above auxiliary problem has a unique solution $u(t,d,\lambda)$ satisfying the following property: if $(d_n, \lambda_n) \to (d,\lambda)$ as $n \to +\infty$, then $\{u(.,d_n,\lambda_n)\}$ converges to $u(.,d,\lambda)$ on $[0,1]$.

## 2. The main Result

In this section, we shall give the main result of this work. More precisely, we have the following theorem.

**Theorem 2.1.** *Assume that the hypotheses (F1)–(F4) are satisfied. Then there exists a positive real number* $\lambda_0$ *such that if* $\lambda \in ]0,\lambda_0[$*, problem* (1.1) *has at least one radial positive solution which is decreasing on* $[0,1]$.

The proof of this theorem is based on the three next technical Lemmas in this section. We note that the proofs of the first two Lemmas are analogous with those of [2, Lemmas 3.1 and 3.2]. On the opposite, the proof of the last Lemma, in our work, is different from that of [2, Lemma 3.3]. This is due to that in our case $f$ may have many zeros and isn't increasing entirely on $[0,+\infty)$.

Following [2], we introduce the notation and the following preliminaries. Denote by $\theta$ the greatest zero of $F$. From (F4), we can choose $\gamma \geq \sup\{\frac{\beta}{k}, \frac{\theta}{k}\}$ such that

$$2NF(kd) - (N-2)df(d) \geq 1, \quad \forall d \geq \gamma. \tag{2.1}$$

Moreover, for $d \geq \gamma$ there exists $t_0 \in (0,1]$ such that

$$u(t_0,d,\lambda) = kd \quad \text{and} \quad kd \leq u(t,d,\lambda) \leq d, \quad \forall t \in [0,t_0]. \tag{2.2}$$

Taking into account that $f$ is nondecreasing on $[kd,d] \subset (\beta,+\infty)$ and that

$$u'(t,d,\lambda) = -\lambda t^{-(N-1)} \int_0^t r^{N-1} f(u(r))dr,$$

we obtain from this and (2.2)

$$\frac{\lambda f(kd)}{N}t \leq -u' \leq \frac{\lambda f(d)}{N}t$$

and integrating on $[0,t_0]$

$$c_1\Big(\frac{d}{\lambda f(kd)}\Big)^{1/2} \geq t_0 \geq c_1\Big(\frac{d}{\lambda f(d)}\Big)^{1/2}, \tag{2.3}$$

where $c_1 = (2N(1-k))^{1/2}$. Next given $d \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define

$$E(t,d,\lambda) = \frac{u'(t,d,\lambda)^2}{2} + \lambda F(u(t,d,\lambda)).$$
$$H(t,d,\lambda) = tE(t,d,\lambda) + \frac{N-2}{2} u(t,d,\lambda)u'(t,d,\lambda).$$

Then, we show the following identity of Pohozaev type:

$$\begin{aligned} t^{N-1}H(t,d,\lambda) &= \widehat{t}^{N-1}H(\widehat{t},d,\lambda) \\ &\quad + \lambda \int_{\widehat{t}}^{t} r^{N-1}\{NF(u(r,d,\lambda)) - \frac{N-2}{2} f(u(r,d,\lambda))u(r,d,\lambda)\}dr. \end{aligned}$$

Taking $\widehat{t} = 0$ and $t = t_0$ in the previous identity and using (2.2) and (2.3), we show as in [2]:

$$t_0^{N-1}H(t_0,d,\lambda) \geq c_2\lambda^{1-\frac{N}{2}}\{F(kd) - \frac{N-2}{2N} f(d)d\}\{\frac{d}{f(d)}\}^{N/2}, \tag{2.4}$$

where $c_2 = c_1^N$.

**Lemma 2.2.** *There exists $\lambda_1 > 0$ such that $u(t,\gamma,\lambda) \geq \beta$ for all $\lambda \in (0,\lambda_1)$ and for all $t \in [0,1]$.*

*Proof.* Let $t_1 = \sup\{t \leq 1 : u(r,\gamma,\lambda) \geq \beta, \forall r \in (0,t)\}$. Since $f \geq 0$ on $[\beta,+\infty)$ and $u'(t,\gamma,\lambda) = -\lambda t^{-(N-1)} \int_0^t r^{N-1} f(u(r,\gamma,\lambda))dr$, $u$ is decreasing on $[0,t_1]$. Again, for all $t \in [0,t_1]$, we have

$$|u'(t,\gamma,\lambda)| \leq \frac{\lambda\, f\,(\gamma)}{N} < \gamma - \beta,$$

by assuming that $\lambda < \lambda_1 = \frac{N(\gamma-\beta)}{f(\gamma)}$. After that, by using the mean value theorem, we obtain for all $\lambda \in (0,\lambda_1)$:

$$u(t_1,\gamma,\lambda) - u(0,\gamma,\lambda) = u(t_1,\gamma,\lambda) - \gamma \geq -(\gamma-\beta)t_1.$$

If $t_1 < 1$, then $u(t_1,\gamma,\lambda) > \beta$, which contradicts the definition of $t_1$. Thus $t_1 = 1$ and the lemma is proved. $\square$

**Lemma 2.3.** *There exists $\lambda_2 > 0$ such that if $\lambda \in (0,\lambda_2)$, then*

$$u(t,d,\lambda)^2 + u'(t,d,\lambda)^2 > 0, \quad \forall t \in [0,1],\ \forall d \geq \gamma.$$

*Proof.* For $t \geq t_0$, the identity of Pohozaev type gives

$$t^{N-1}H(t) = t_0^{N-1}H(t_0) + \lambda \int_{t_0}^{t} r^{N-1}\{NF(u) - \frac{N-2}{2} f(u)u\}dr.$$

Extending $f$ by $f(x) = f(0) < 0$, for all $x \in (-\infty,0]$, there exists $B < 0$ such that

$$NF(s) - \frac{N-2}{2} f(s)s \geq B, \quad \forall s \in \mathbb{R}.$$

By (F4), we can take $\gamma$ sufficiently large such that

$$\{F(kd) - \frac{N-2}{2N} df(d)\}\{\frac{d}{f(d)}\}^{N/2} \geq 1 \quad \forall d \geq \gamma,$$

and using inequality (2.4), we obtain

$$t^{N-1}H(t) \geq c_2\lambda^{1-\frac{N}{2}}\{F(kd) - \frac{N-2}{2N} df(d)\}\{\frac{d}{f(d)}\}^{N/2} + \lambda B \frac{t^N - t_0^N}{N}, \tag{2.5}$$

then
$$t^{N-1}H(t) \geq \lambda(c_2\lambda^{-\frac{N}{2}} + \frac{B}{N}), \quad \forall t \in [t_0, 1].$$
Hence, there exists $\lambda_2$ such that for all $\lambda \in (0, \lambda_2)$, $H(t) > 0$ for all $t \in [0, 1]$ and for all $d \geq \gamma$. This implies that $u^2(t) + u'(t)^2 > 0$, for all $t \in [0, 1]$, for all $d \geq \gamma$. □

**Lemma 2.4.** *Given any $\lambda > 0$, there exists $d \geq \gamma$ such that $u(t, d, \lambda) < 0$ for some $t \in [0, 1]$.*

*Proof.* Let $d \geq \gamma$. We put $\bar{t} = \sup\{t \in (0, 1) : u(., d, \lambda)$ is decreasing on $(0, t)\}$. Let $\omega$ be such that
$$\omega'' + \frac{N-1}{r}\omega' + \varrho\omega = 0,$$
$$\omega(0) = 1, \quad \omega'(0) = 0,$$
where $\varrho$ is chosen such that the first zero of $\omega$ is $\frac{\bar{t}}{4}$.

We argue by contradiction. Suppose that $u(t, d, \lambda) \geq 0$ for all $t \in [0, 1]$ and all $d \geq \gamma$. By (F3), there exists $d_0 \geq \gamma$ such that
$$\frac{f(x)}{x} \geq \frac{\varrho}{\lambda}, \quad \forall x \geq d_0. \tag{2.6}$$
On the other hand, since $(d\omega)'' + \frac{N-1}{r}(d\omega)' + \varrho(d\omega) = 0$ and $u'' + \frac{N-1}{r}u' + \lambda f(u) = 0$, we obtain
$$t^{N-1}[u(t)v(t)' - v(t)u(t)'] = \int_0^t s^{N-1}[\lambda\frac{f(u(s))}{u(s} - \varrho]u(s)v(s)ds,$$
where $v = d\omega$. Therefore, if $u(t, d, \lambda) \geq d_0$, for all $t \in [0, \frac{\bar{t}}{4}]$, we obtain from (2.6),
$$\int_0^t s^{N-1}[\lambda\frac{f(u(s))}{u(s)} - \varrho]u(s)v(s)ds \geq 0,$$
so that
$$u(t)v'(t) - v(t)u'(t) > 0, \quad \forall t \in (0, \frac{\bar{t}}{4}]. \tag{2.7}$$
Thus, taking into account that $v(\frac{\bar{t}}{4}) = 0$, $v'(\frac{\bar{t}}{4}) < 0$, we have
$$u(\frac{\bar{t}}{4})v'(\frac{\bar{t}}{4}) - v(\frac{\bar{t}}{4})u'(\frac{\bar{t}}{4}) < 0.$$
This is a contradiction with (2.7). Hence, there exists $t^*$ in $(0, \bar{t}/4)$ such that $u(t^*, d, \lambda) = d_0$ and since $d_0 \geq \gamma > \beta$ there exists $\widehat{t} \in (t^*, \bar{t})$ such that
$$\beta \leq u(t, d, \lambda) \leq d_0, \quad \forall t \in (t^*, \widehat{t}). \tag{2.8}$$
Now, we consider the point $t_0$ defined in (2.2). It is clear that $t_0 < \bar{t}$.

On $[0, t_0]$, since $F$ is nondecreasing on $[\beta, +\infty[$ and $u(t, d, \lambda) \geq kd \geq \beta$, for all $t \in (0, t_0]$, we have
$$E(t, d, \lambda) = \frac{u'(t, d, \lambda)^2}{2} + \lambda F(u(t, d, \lambda)) \geq \lambda F(kd). \tag{2.9}$$
On the other hand, since $u(t, d, \lambda)u'(t, d, \lambda) \leq 0$ for all $t \in (t_0, \bar{t}]$, we have
$$\begin{aligned} t^N E(t, d, \lambda) &= t^{N-1}H(t, d, \lambda) - \frac{N-2}{2}t^{N-1}u(t, d, \lambda)u'(t, d, \lambda) \\ &\geq t^{N-1}H(t, d, \lambda), \end{aligned}$$

hence, by (2.5), we obtain

$$t^N E(t,d,\lambda) \geq c_2 \lambda^{1-\frac{N}{2}} \{F(kd) - \frac{N-2}{2N} df(d)\}\{\frac{d}{f(d)}\}^{N/2} + \lambda B \frac{t^N - t_0^N}{N}. \quad (2.10)$$

Now from (F4), (2.9) and (2.10) we obtain $\lim_{d\to+\infty} E(t,d,\lambda) = +\infty$ uniformly with respect to $t \in [0,\bar{t}]$. Hence there exists $d_1 \geq d_0$ such that for all $d \geq d_1$,

$$E(t,d,\lambda) \geq \lambda F(d_0) + \frac{2}{(\widehat{t} - t^*)^2} d_0^2, \quad \forall t \in [0,\bar{t}].$$

Taking into account (2.8), this implies

$$\begin{aligned} u'(t,d,\lambda)^2 &\geq \frac{4}{(\widehat{t} - t^*)^2} d_0^2 + 2\lambda(F(d_0) - F(u(t,d,\lambda))) \\ &\geq \frac{4}{(\widehat{t} - t^*)^2} d_0^2, \quad \forall t \in (t^*, \widehat{t}), \end{aligned}$$

which implies $u'(t,d,\lambda) \leq -\frac{2}{\widehat{t}-t^*} d_0$, for all $t \in (t^*, \widehat{t})$.

The mean value theorem gives us $c \in (t^*, \frac{t^*+\widehat{t}}{2})$ such that

$$u(\frac{\widehat{t}+t^*}{2}) - u(t^*) = u'(c)\frac{\widehat{t}-t^*}{2} \leq -\frac{2d_0}{\widehat{t}-t^*}.\frac{\widehat{t}-t^*}{2} = -d_0.$$

Hence $u(\frac{t^*+\widehat{t}}{2}) \leq 0$ and since $u'(\frac{t^*+\widehat{t}}{2}) \leq -\frac{2}{\widehat{t}-t^*} d_0 < 0$, there exists $T \in (0,1)$ such that $u(T,d,\lambda) < 0$ which is a contradiction. So the proof is finished. □

**Remark 2.5.** In [2], to prove the last lemma, the authors use the fact that if $u(t,d,\lambda) \leq \beta$, where $\beta$ is the only zero of $f$, then $F(u(t,d,\lambda)) \leq 0$, which simplifies the proof.

In our case, even if $u(t,d,\lambda) \leq \beta$ we do not have $F(u(t,d,\lambda)) \leq 0$, and we have overcome this problem.

*Proof of theorem 2.1.* As in [2], we take $\lambda \in (0,\lambda_0)$ where $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. Let $\widehat{d} = \sup\{d \geq \gamma : u(t,d,\lambda) \geq 0\ ,\ \forall t \in (0,1]\}$.

Lemma 2.2 leads to the fact that the set $\{d \geq \gamma : u(t,d,\lambda) \geq 0, \forall t \in [0,1]\}$ is nonempty. By Lemma 2.4, we have $\widehat{d} < +\infty$, and we claim that $u(.,\widehat{d},\lambda)$ is the desired solution, which satisfies the following properties

(i) $u(t,\widehat{d},\lambda) > 0$, for all $t \in [0,1)$,
(ii) $u(1,\widehat{d},\lambda) = 0$,
(iii) $u'(1,\widehat{d},\lambda) < 0$,
(iv) $u$ is decreasing in $[0,1]$.

To prove (i), we assume that there exists $T_1 < 1$ such that $u(T_1,\widehat{d},\lambda) = 0$. By Lemma 2.3, $u'(T_1,\widehat{d},\lambda) \neq 0$. We can assume (see [2]) that $u'(T_1,\widehat{d},\lambda) < 0$, then there exists $T_2 \in (T_1,1)$ such that $u(T_2,\widehat{d},\lambda) < 0$, which is a contradiction with the definition of $\widehat{d}$.

For(ii), we assume $u(1,\widehat{d},\lambda) > 0$, then there exists $\eta > 0$ such that $u(t,\widehat{d},\lambda) \geq \eta$ for all $t \in (0,1]$. Again there exists $\delta > 0$ such that $u(t,\widehat{d}+\delta,\lambda) \geq \frac{\eta}{2}$ for all $t \in (0,1]$, which is a contradiction with the definition of $\widehat{d}$.

Statement (iii) is a consequence of Lemma 2.3; and (iv) is a result of [3]. □

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