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WEAK ALMOST PERIODIC AND OPTIMAL MILD SOLUTIONS OF FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this article, we prove the existence of optimal mild solutions for linear fractional evolution equations with an analytic semigroup in a Banach space. As in [16], we use the Gelfand-Shilov principle to prove existence, and then the Bochner almost periodicity condition to show that solutions are weakly almost periodic. As an application, we study a fractional partial differential equation of parabolic type.

1. Introduction

The object of this paper is to study the fractional evolution equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = f(t), \quad t > t_0 \tag{1.1}$$

in a Banach space X, where $0 < \alpha \leq 1$, u is an X-valued function on $\mathbb{R}^+ = [0, \infty)$, and f is a given abstract function on $\mathbb{R}^+$ with values in X. We assume that -A is a linear closed operator defined on a dense set S in X into X, $\{B(t) : t \in \mathbb{R}^+\}$ is a family of linear bounded operators defined on X into X. It is assumed that -A generates an analytic semigroup $Q(t)$ such that $\|Q(t)\| \leq M$ for all $t \in \mathbb{R}^+$, $Q(t)h \in S$, $\|AQ(t)h\| \leq \frac{M}{t}\|h\|$ for every $h \in X$ and all $t \in (0, \infty)$.

Let X be a uniformly convex Banach space equipped with a norm $\|\cdot\|$ and X^* its topological dual space. N'Guerekata [16] gave necessary conditions to ensure that the so-called optimal mild solutions of $u'(t) = Au(t) + f(t)$ are weakly almost periodic. Following Gelfand and Shilov [10], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s) ds,$$

also, the fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_a^t f(s)(t - s)^{-\alpha} ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [14, 18].

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Definition 1.1. By a solution of (1.1), we mean a function u with values in X such that:

(1) u is continuous function on $\mathbb{R}^+$ and $u(t) \in D(A)$,
(2) $\frac{d^\alpha u}{dt^\alpha}$ exists and continuous on $(0, \infty)$, $0 < \alpha < 1$, and u satisfies (1.1) on $(0, \infty)$.

It is suitable to rewrite equation (1.1) in the form

$$u(t) = u(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}[(B(s) - A)u(s) + f(s)]ds. \tag{1.2}$$

According to [5, 6, 7, 8, 9], a solution of equation (1.2) can be formally represented by

$$\begin{aligned} u(t) = &\int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) u(t_0) d\theta \\ &+ \alpha \int_{t_0}^{t} \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) F(s) d\theta ds, \end{aligned} \tag{1.3}$$

where $F(t) = B(t)u(t) + f(t)$ and ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \zeta_\alpha(\theta) d\theta = \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad 0 < \alpha \le 1, x > 0,$$

A continuous solution of the integral equation (1.3) is called a mild solution of (1.1).

The theory of almost periodic functions with values in a Banach space was developed by Bohr, Bochner, von Neumann, and others [1, 3]. See also [2, 4, 13, 16, 17, 19].

Definition 1.2. A function $f : \mathbb{R} \to X$ is called (Bochner) almost periodic if

(i) f is strongly continuous, and
(ii) for each $\epsilon > 0$ there exists $l(\epsilon) > 0$, such that every interval I of length $l(\epsilon)$ contains a number τ such that $\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| < \epsilon$.

2. Optimal mild solutions

As in N'Guerekata [16], let Ω_f denote the set of mild solutions $u(t)$ of (1.1) which are bounded over $\mathbb{R}$; that is

$$\mu(u) = \sup_{t\in\mathbb{R}} \|u(t)\| < \infty, \tag{2.1}$$

where $\mathbb{R} = (-\infty, \infty)$. We assume here that $\Omega_f \neq \emptyset$, and recall that a bounded mild solution $\tilde{u}(t)$ of (1.1) is called optimal mild solution of (1.1) if

$$\mu(\tilde{u}) \equiv \mu^* = \inf_{u\in\Omega_f} \mu(u). \tag{2.2}$$

Theorem 2.1. *Assume that $\Omega_f \neq \emptyset$ and $f : \mathbb{R} \to X$ is a nontrivial strongly continuous function, then* (1.1) *has a unique optimal mild solution.*

Compare with [22, Theorem 1.1, p.138] and [16, Theorem 1. p. 673]. Our proof is based on the following lemma.

Lemma 2.2 ([12, Corollary 8.2.1])**.** *If K is a non-empty convex and closed subset of a uniformly convex Banach space X and $v \notin K$, then there exists a unique $k_0 \in K$ such that $|v - k_0| = \inf_{k\in K} |v - k|$.*

Proof of Theorem 2.1. It suffices to prove that Ω_f is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use lemma 2.2 to deduce the uniqueness of the optimal mild solution, see [16]. For the convexity of Ω_f, we consider two distinct bounded mild solutions $u_1(t)$ and $u_2(t)$, and a real number $0 \le \lambda \le 1$ and let $u(t) = \lambda u_1(t) + (1-\lambda)u_2(t), t \in \mathbb{R}$. For every $t_0 \in \mathbb{R}$, $u(t)$ is continuous and (see [16]) has the integral representation

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds, \quad t \ge t_0, \tag{2.3}$$

where

$$T(t) = \int_0^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta, \quad S(t) = \alpha\int_0^\infty \theta t^{\alpha-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta.$$

We have $u(t_0) = \lambda u_1(t_0)+(1-\lambda)u_2(t_0)$, then $u(t)$ is a mild solution of (1.1). We note that $u(t)$ is bounded over $\mathbb{R}$ since $\mu(u) = \sup_{t\in\mathbb{R}} \|u(t)\| \le \lambda\mu(u_1)+(1-\lambda)\mu(u_2) < \infty$, we conclude that $u(t) \in \Omega_f$. Now we show that Ω_f is closed. Let $u_n \in \Omega_f$ a sequence such that $\lim_{n\to\infty} u_n(t) = u(t), t \in \mathbb{R}$. For all $t_0 \in \mathbb{R}$ and $t \ge t_0$ we have

$$u_n(t) = T(t-t_0)u_n(t_0) + \int_{t_0}^{t} S(t-s)[B(s)u_n(s) + f(s)]ds, \tag{2.4}$$

It is clearly that $T(t-t_0)$ and $S(t-s)$ are continuous operators, then for every fixed t and t_0 with $t \ge t_0$, we have

$$\begin{aligned}
\lim_{n\to\infty} T(t-t_0)u_n(t_0) &= \lim_{n\to\infty} \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)u_n(t_0)d\theta \\
&= \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)d\theta \lim_{n\to\infty} u_n(t_0) \\
&= T(t-t_0)\lim_{n\to\infty} u_n(t_0) \\
&= T(t-t_0)u(t_0).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{t_0}^{t} S(t-s)[B(s)u_n(s) + f(s)]ds &= \int_{t_0}^{t} S(t-s)[\lim_{n\to\infty} B(s)u_n(s) + f(s)]ds \\
&= \int_{t_0}^{t} S(t-s)F(s)ds.
\end{aligned}$$

Then we deduce that

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds,$$

for all $t_0 \in \mathbb{R}, t \ge t_0$, which means that $u(t)$ is a mild solution of (1.1). Finally we show that $u(t)$ is bounded over $\mathbb{R}$. We can write (2.3) as

$$\begin{aligned}
u(t) &= T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds - u_n(t) + u_n(t) \\
&= T(t-t_0)[u(t_0) - u_n(t_0)] + \int_{t_0}^{t} S(t-s)(B(u-u_n))(s)ds + u_n(t),
\end{aligned}$$

for $n = 1, 2, \dots$, and every $t_0 \in \mathbb{R}$ such that $t \geq t_0$. Since $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, it follows that $\|T(t)\| \leq M$, again, since $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1$ (see [9, p. 54]), it follows that $\|S(t)\| \leq \alpha M t^{\alpha-1}$. Let $\|B\| \leq C$. These estimates lead to

$$\|u(t)\| \leq M\|u(t_0) - u_n(t_0)\| + \alpha MC \int_{t_0}^t (t-s)^{\alpha-1}\|u(s) - u_n(s)\|ds + \|u_n(t)\|.$$

Choose n large enough, for every $\epsilon_1, \epsilon_2 > 0$ we get

$$\mu(u) \leq \epsilon_1 + \epsilon_2 + \mu(u_n) < \infty.$$

Thus $u \in \Omega_f$. This completes the proof. □

3. Weak almost periodic solutions

To formulate a property of almost periodic functions, which is equivalent to Definition 1.2, we discuss the concept of normality of almost periodic functions. Namely, let $f(t)$ be almost periodic in $t \in \mathbb{R}$, then for every sequence of real numbers (s_n') there exists a subsequence (s_n) such that $f(t + s_n)$ is uniformly convergent in $t \in \mathbb{R}$. see Hamaya [11, p. 188]. It is well known [15, 16, 21, 22] that: $f : \mathbb{R} \to X$ is weakly almost periodic if for every sequence of real numbers (s_n') there exists a subsequence (s_n) such that every $(f(t + s_n))$ is convergent in the weak sense, uniformly in $t \in \mathbb{R}$. In other words, for every $u^* \in X^*$, the sequence $(\langle u^*, f(t + s_n)\rangle)$ is uniformly convergent in $t \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes duality $\langle X^*, X\rangle$. For each $Q(t), t \in \mathbb{R}^+$, $Q^*(t)$ denotes the adjoint operator of $Q(t)$.

Theorem 3.1. *Let $f : \mathbb{R} \to X$ be almost periodic and a nontrivial strongly continuous function, also assume that $f \in L^1(R)$ and $Q^*(t) \in L(X^*)$ for every $t \in \mathbb{R}^+$, then the optimal mild solution of* (1.1) *is weakly almost periodic.*

Proof. As in N'Guerekata [16], let $u(t)$ be the unique optimal mild solution of (1.1), by Theorem 2.1

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^t S(t - s)F(s)ds,$$

for all $t_0 \in \mathbb{R}$, $t \geq t_0$. Let (s_n') be an arbitrary sequence of real numbers. Since f is almost periodic, we can extract a subsequence $(s_n) \subset (s_n')$ such that $\lim_{n\to\infty} f(t + s_n) = g(t)$ uniformly in $t \in \mathbb{R}$. We note that $g(t)$ is also strongly continuous. For fixed $t_0 \in \mathbb{R}$, we can obtain a subsequence of (s_n), which again we will denote (s_n), such that

$$\operatorname*{weak-lim}_{n\to\infty} u(t_0 + s_n) = v_0 \in X.$$

Since X is a reflexive Banach space, then the function

$$y(t) = T(t - t_0)v_0 + \int_{t_0}^t S(t - s)(Bu + g)(s)ds,$$

is strongly continuous. It is a mild solution of

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = g(t), \quad t \in \mathbb{R}.$$

□

We need the following lemmas.

Lemma 3.2. *For each $t \in \mathbb{R}$, we have*
$$\operatorname*{weak-lim}_{n\to\infty} u(t+s_n) = y(t).$$

Proof. We can write
$$u(t+s_n) = T(t-t_0)u(t_0+s_n) + \int_{t_0}^{t} S(t-s)[(Bu)(s) + f(s+s_n)]ds,$$
$n = 1, 2, \dots$ (see for instance [20, p. 721]). Let $u^* \in X^*$, then we have
$$\langle u^*, T(t-t_0)u(t_0+s_n)\rangle - \langle u^*, T(t-t_0)v_0\rangle = \langle T^*(t-t_0)u^*, u(t_0+s_n) - v_0\rangle,$$
for every $n = 1, 2, \dots$, we deduce that the sequence $(T(t-t_0)u(t_0+s_n))$ converges to $T(t-t_0)v_0$ in the weak sense. Also we have
$$\begin{aligned}
&\int_{t_0}^{t} S(t-s)[(Bu)(s) + f(s+s_n)]ds - \int_{t_0}^{t} S(t-s)[(Bu)(s) + g(s)]ds \\
&\le \|\int_{t_0}^{t} S(t-s)[f(s+s_n) - g(s)]ds\| \\
&\le \alpha M \int_{t_0}^{t} (t-s)^{\alpha-1}\|f(s+s_n) - g(s)\|ds.
\end{aligned}$$
This leads to
$$\lim_{n\to\infty} \int_{t_0}^{t} S(t-s)[(Bu)(s) + f(s+s_n)]ds = \int_{t_0}^{t} S(t-s)[(Bu)(s) + g(s)]ds,$$
in the strong sense, then consequently in the weak sense in X. $\square$

Lemma 3.3. $\mu(y) = \mu(u) = \mu^*$.

Proof. Since $u(t)$ is an optimal mild solution of (1.1), we have $\mu^* = \mu(u) = \sup_{t\in\mathbb{R}} \|u(t)\|$. Let $u^* \in X^*$, then by lemma 3.2 we obtain
$$\lim_{n\to\infty} \langle u^*, u(t+s_n)\rangle = \langle u^*, y(t)\rangle,$$
for every $t \in \mathbb{R}$. For each $n = 1, 2, \dots$, we have
$$\|\langle u^*, u(t+s_n)\rangle\| \le \|u^*\|\|u(t+s_n)\| \le \|u^*\|\mu^*.$$
Therefore, $\|\langle u^*, y(t)\rangle\| \le \|u^*\|\mu^*$ for every $t \in \mathbb{R}$, and consequently $\|y(t)\| \le \mu^*$ for every $t \in \mathbb{R}$, so that $\mu(y) < \mu^*$. We suppose that $\mu(y) < \mu^*$. Note that $\lim_{n\to\infty} g(t-s_n) = f(t)$ uniformly in $t \in \mathbb{R}$ because $f(t)$ is almost periodic. Since X is a reflexive Banach space, we can extract from the sequence (s_n), a subsequence which we still denote (s_n) such that $(y(t_0-s_n))$ is weakly convergent to $z \in X$. We have
$$\lim_{n\to\infty} y(t-s_n) = T(t-t_0)z + \int_{t_0}^{t} S(t-s)F(s)ds$$
in the weak sense for every $t \in \mathbb{R}$. Now we consider the function
$$Z(t) = T(t-t_0)z + \int_{t_0}^{t} S(t-s)F(s)ds.$$
It is a bounded mild solution of equation (1.1). Similarly as above, we have $\mu(Z) \le \mu(y)$; therefore, $\mu(Z) < \mu^*$, which is absurd by definition of μ^*. $\square$

Lemma 3.4. $\mu(y) = \inf_{v\in\Omega_g} \mu(v)$; *i.e., $y(t)$ is an optimal mild solution of the equation*

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = g(t), \quad t \in \mathbb{R}. \tag{3.1}$$

Proof. By lemma 3.3, $y(t)$ is bounded over $\mathbb{R}$. Also $y(t)$ is a mild solution of (3.1) which implies $y(t) \in \Omega_g$. It remains to prove that $y(t)$ is optimal. Suppose it is not. Since $\Omega_g \neq \emptyset$, by Theorem 2.1, there exists a unique optimal solution $v(t)$ of (3.1). We have $\mu(v) < \mu(y)$ and

$$v(t) = T(t - t_0)v(t_0) + \int_{t_0}^{t} S(t - s)(Bu + g)(s)ds,$$

for all $t_0 \in \mathbb{R}, t \geq t_0$. We can find a subsequence $(s_{n_k}) \subset (s_n)$ such that

$$\operatorname*{weak-lim}_{k\to\infty} v(t - s_{n_k}) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)F(s)ds \equiv V(t).$$

Noting that $V(t) \in \Omega_f$ and $\mu(V) \leq \mu(v) < \mu(y)$, which is absurd. Therefore, $y(t)$ is an optimal mild solution of (3.1), and in fact the only one by Theorem 2.1. □

Proof of Theorem 3.1. To prove that $u(t)$ is weakly almost periodic, it suffices to show that

$$\operatorname*{weak-lim}_{n\to\infty} u(t + s_n) = y(t)$$

uniformly in $t \in \mathbb{R}$. Suppose that this does not hold; then there exists $u^* \in X^*$ such that

$$\lim_{n\to\infty} \langle u^*, u(t + s_n)\rangle = \langle u^*, y(t)\rangle$$

is not uniform in $t \in \mathbb{R}$. Consequently, we can find a number $\gamma > 0$, and a sequence (t_k) with two subsequences (s'_k) and (s''_k) of (s_n) such that

$$|\langle u^*, u(t + s'_k) - u(t + s''_k)\rangle| > \gamma \tag{3.2}$$

for all $k = 1, 2, \dots$. Again, let us extract two subsequences of (s'_k) and (s''_k) respectively, with the same notation, such that

$$\lim_{k\to\infty} f(t + t_k + s'_k) = g_1(t), \quad \mathit{text} \quad \lim_{k\to\infty} f(t + t_k + s''_k) = g_2(t)$$

both uniformly in $t \in \mathbb{R}$, because f is almost periodic. As we did previously, we may obtain

$$\operatorname*{weak-lim}_{k\to\infty} f(t + t_k + s'_k) = T(t - t_0)z_1 + \int_{t_0}^{t} S(t - s)[(Bu)(s) + g_1(s)]ds \equiv y_1(t),$$

and

$$\operatorname*{weak-lim}_{k\to\infty} f(t + t_k + s''_k) = T(t - t_0)z_2 + \int_{t_0}^{t} S(t - s)[(Bu)(s) + g_2(s)]ds \equiv y_2(t)$$

for each $t \in \mathbb{R}$, where $y_1(t)$ and $y_2(t)$ are optimal mild solutions in Ω_{g_1} and Ω_{g_2}, respectively. Since $\lim_{k\to\infty} f(t + t_k + s_k)$ exists uniformly in $t \in \mathbb{R}$, and $(s'_k), (s''_k)$ are two subsequences of (s_k), we will get

$$\sup_{s\in\mathbb{R}} \|f(s + s'_k) - f(s + s''_k)\| < \epsilon$$

if $k \geq k_0(\epsilon)$ and consequently

$$\sup_{s\in\mathbb{R}} \|f(t + t_k + s'_k) - f(t + t_k + s''_k)\| < \epsilon$$

for $k \geq k_0(\epsilon)$, which shows that $g_1(s) = g_2(s)$ for all $s \in \mathbb{R}$. By the uniqueness of the optimal mild solution we get $y_1(t) = y_2(t)$, $t \in \mathbb{R}$. But $y_1(0) = \text{weak-lim}_{k\to\infty} u(t_k + s'_k)$ and $y_2(0) = \text{weak-lim}_{k\to\infty} u(t_k + s''_k)$. Clearly $y_1(0) = y_2(0)$ contradicts the inequality (3.2) above. This completes the proof. □

4. Application

Consider the partial differential equation of fractional order

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q|\leq 2m} a_q(x) D_x^q u(x,t) = \int_{\mathbb{R}^n} K(x,\eta,t) u(\eta,t) d\eta + f(x,t), \tag{4.1}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $D_x^q = D_{x_1}^{q_1} \dots D_{x_n}^{q_n}$, $D_{x_i} = \frac{\partial}{\partial x_i}$, $q = (q_1, \dots, q_n)$ is an n-dimensional multi-index, $|q| = q_1 + \dots + q_n$. Let $L_2(\mathbb{R}^n)$ be the set of all square integrable functions on $\mathbb{R}^n$. We denote by $C^m(\mathbb{R}^n)$ the set of all continuous real-valued functions defined on $\mathbb{R}^n$ which have continuous partial derivatives of order less than or equal to m. By $C_0^m(\mathbb{R}^n)$ we denote the set of all functions $f \in C^m(\mathbb{R}^n)$ with compact supports. Let $H_0^m(\mathbb{R}^n)$ be the completion of $C_0^m(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_m^2 = \sum_{|q|\leq m} \int_{\mathbb{R}^n} |D_x^q f(x)|^2 dx.$$

It is supposed that
(i) The operator $A = -\sum_{|q|=2m} a_q(x) D_x^q$ is uniformly parabolic on $\mathbb{R}^n$. In other words, all the coefficients $a_q, |q| = 2m$, are continuous and bounded on $\mathbb{R}^n$ and

$$(-1)^m \sum_{|q|=2m} a_q(x)\xi^q \geq c|\xi|^{2m}, \quad c > 0,$$

for all $x \in \mathbb{R}^n$ and all $\xi \neq 0, \xi \in \mathbb{R}^n$, where $\xi^q = \xi_1^{q_1} \dots \xi_n^{q_n}$ and $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$.
(ii) All the coefficients $a_q, |q| = 2m$, satisfy a uniform Hölder condition on $\mathbb{R}^n$, $\int_{\mathbb{R}^n} K^2(x,\eta,t)d\eta < \infty$. It's proved, see [5, p. 438], that the operator A defined by (i) with domain of definition $S = H^{2m}(\mathbb{R}^n)$ generates an analytic semigroup $Q(t)$ defined on $L_2(\mathbb{R}^n)$, and that $H^{2m}(\mathbb{R}^n)$ is dense in $X = L_2(\mathbb{R}^n)$. Which achieves the proof of the existence of (bounded) mild solutions of the equation (4.1).
(iii) f is a nontrivial strongly continuous function defined on $\mathbb{R}^n \times \mathbb{R}^+$ satisfying: For every $\epsilon > 0$ there exists $\beta > 0$ such that every interval $[a, a+\beta]$ contains at least a point τ such that

$$\int_{\mathbb{R}^n} |f(x,t+\tau) - f(x,t)|^2 dx < \epsilon,$$

for all $t \in \mathbb{R}^+$ and all $x \in \mathbb{R}^n$. Applying Theorems 2.1, 3.1, stated above, we deduce that (4.1) has a unique optimal mild solution which is weakly almost periodic.

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