Electronic Journal of Differential Equations, Vol. 2009(2009), No. 47, pp. 1–54.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

REGULARITY FOR A CLAMPED GRID EQUATION $u_{xxxx} + u_{yyyy} = f$ ON A DOMAIN WITH A CORNER

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Abstract. The operator $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ appears in a model for the vertical displacement of a two-dimensional grid that consists of two perpendicular sets of elastic fibers or rods. We are interested in the behaviour of such a grid that is clamped at the boundary and more specifically near a corner of the domain. Kondratiev supplied the appropriate setting in the sense of Sobolev type spaces tailored to find the optimal regularity. Inspired by the Laplacian and the Bilaplacian models one expect, except maybe for some special angles that the optimal regularity improves when angle decreases. For the homogeneous Dirichlet problem with this special non-isotropic fourth order operator such a result does not hold true. We will show the existence of an interval $(\frac{1}{2}\pi, \omega_\star)$, $\omega_\star/\pi \approx 0.528\dots$ (in degrees $\omega_\star \approx 95.1\dots^\circ$), in which the optimal regularity improves with increasing opening angle.

Contents

2000 *Mathematics Subject Classification.* 35J40, 46E35, 35P30.
Key words and phrases. Nonisotropic; fourth order PDE; domain with corner;
clamped grid; weighted Sobolev space; regularity.
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Submitted December 10, 2008. Published April 2, 2009.

1. Introduction

1.1. **The model.** A model for small deformations of a thin isotropic elastic plate is $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f$. Here f is a force density and u is the vertical displacement of a plate; the model neglects the influence of horizontal deviations. Non-isotropic elastic plates are still modeled by fourth order differential equations but the coefficients in front of the derivatives of u may vary. Two interesting extreme cases are $L_1 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and $L_2 = \frac{1}{2}\frac{\partial^4}{\partial x^4} + 3\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{2}\frac{\partial^4}{\partial y^4}$. One may think of these operators as of the operators appearing in the model of an elastic medium consisting of two sets of intertwined (not glued) perpendicular fibers: $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ for fibers running in cartesian directions (Figure 1, left). The differential operator is not rotation invariant. For a diagonal grid the rotation of $\frac{1}{4}\pi$ transforms L_1 into L_2 (Figure 1, right). We will call such medium *a grid*.

We should mention that sets of fibers are connected such that the vertical positions coincide but there is no connection that forces a torsion in the fibers. Such torsion would occur if the fibers are glued or imbedded in a softer medium. For those models see [19]. The appropriate linearized model in that last situation would contain mixed fourth order derivatives.

A first place where operator L_1 appears is J. II. Bernoulli's paper [1]. He assumed that it was the appropriate model for an isotropic plate. It was soon dismissed as a model for such a plate, since indeed it failed to have rotational symmetry.

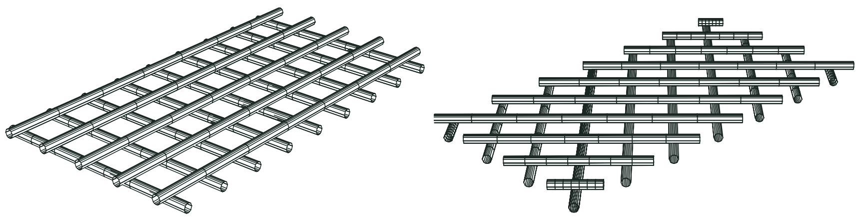

FIGURE 1. A fragment of a rectangular grid with aligned and diagonal fibers.

1.2. **The setting.** We will focus on L_1 supplied with homogeneous Dirichlet boundary conditions. This problem, which we call '*a clamped grid*', is as follows:

$$\begin{aligned} u_{xxxx} + u_{yyyy} &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ is open and bounded, and ν is the unit outward normal vector on $\partial\Omega$. The boundary conditions in (1.1) correspond to the clamped situation meaning that the vertical position and the angle are fixed to be 0 at the boundary.

One verifies directly that the operator $L_1 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ is elliptic in $\overline{\Omega}$. One may also prove, if the normal is well-defined, that the boundary value problem (1.1) is regular elliptic. Indeed, the Dirichlet problem which fixes the zero and first order derivatives at the boundary, is regular elliptic for any fourth order uniformly elliptic operator. Hence, under the assumption that Ω is bounded and $\partial\Omega \in C^\infty$ the full classical regularity result (see e.g. [17]) for problem (1.1) can be used to find for $k \geq 0$ and $p \in 1, \infty)$:

$$\text{if } f \in W^{k,p}(\Omega) \text{ then } u \in W^{k+4,p}(\Omega). \tag{1.2}$$

If Ω in (1.1) has a piecewise smooth boundary $\partial\Omega$ with, say, one angular point, the result (1.2) in general does not apply. Instead, one may use the theory developed by Kondratiev [12]. This theory provides the appropriate treatment of problem (1.1) by employing the weighted Sobolev space $V_\beta^{k,p}(\Omega)$ (see Definition 3.1), where $k \geq 0$ is the differentiability index and $\beta \in \mathbb{R}$ characterizes the powerlike growth of the solution near the angular point. Within the framework of the Kondratiev spaces $V_\beta^{k,p}(\Omega)$ the regularity result "analogous" to (1.2) will then be as follows. There is a countable set of functions $\{u_j\}_{j\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$:

$$\text{if } f \in V_\beta^{k,p}(\Omega) \text{ then } u = w + \sum_{j=1}^{J_k} c_j u_j \text{ with } w \in V_\beta^{k+4,p}(\Omega). \tag{1.3}$$

We will restrict our formulations to $p = 2$.

Partial differential equations on domains with corners have obtained a lot of attention in the literature. After the seminal paper by Kondatiev [12] many authors of which we would like to mention Kozlov, Maz'ya, Rossmann [13, 14], Grisvard [10], Dauge [7], Costabel and Dauge [4], Nazarov and Plamenevsky [18] have contributed. For applications in elasticity theory we refer to Leguillon and Sanchez-Palencia [16], Blum and Rannacher [3]. A recent paper of Kawohl and Sweers [15] concerned the positivity question for the operators L_1 and L_2 in a rectangular domain for hinged boundary conditions.

1.3. **The target.** In this paper, we will focus particularly on the optimal regularity for the boundary value problem which depends on the opening angle of the corner. For the sake of a simple presentation, we will consider (1.1) in a domain $\Omega \subset \mathbb{R}^2$ which has one corner in $0 \in \partial\Omega$ with opening angle $\omega \in (0, 2\pi]$. A more appropriate formulation of the problem should read as:

$$\begin{gathered} u_{xxxx} + u_{yyyy} = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\backslash\{0\}, \end{gathered} \tag{1.4}$$

with prescribed growth behaviour near 0.

To be more precise in the description of a domain Ω, we assume the following condition.

Condition 1.1. The domain Ω has a smooth boundary except at $(x, y) = 0$, and is such that in the vicinity of 0 it locally coincides with a sector. In other words,

(1) $\partial\Omega\backslash\{0\}$ is C^∞,
(2) there exists $\varepsilon > 0, \omega \in (0, 2\pi] : \Omega \cap B_\varepsilon(0) = \mathcal{K}_\omega \cap B_\varepsilon(0)$,

where $B_\varepsilon(0) = \{(x, y) : |(x, y)| < \varepsilon\}$ is the open ball of radius ε centered at $(x, y) = 0$ and $\mathcal{K}_\omega$ an infinite sector with an opening angle ω:

$$\mathcal{K}_\omega = \{(r\cos(\theta), r\sin(\theta)) : \ 0 < r < \infty \text{ and } 0 < \theta < \omega\}\,. \tag{1.5}$$

In Figure 2 some domains Ω which satisfy the condition above are sketched.

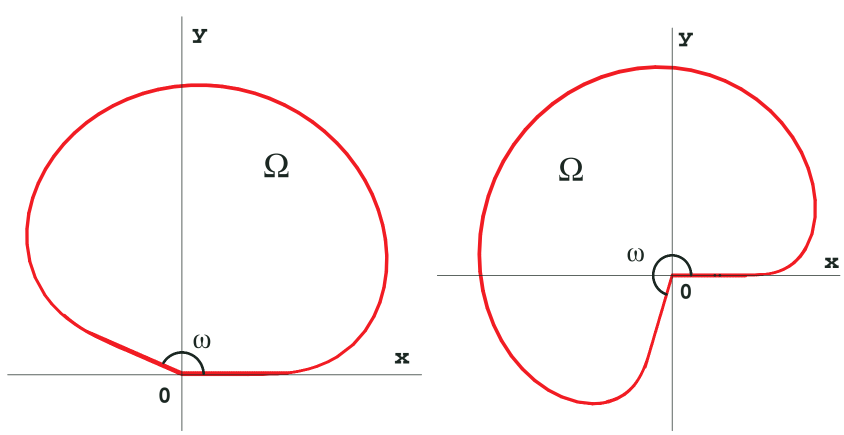

FIGURE 2. Examples for Ω

For the elliptic problem one might roughly distinguish between papers that focus on the general theory and those papers that explicitly study in detail the results for one special model. If one chooses a special fourth order model then it usually has the biharmonic operator in the differential equation. For the biharmonic problem of the type (1.4) the optimal regularity due to the corner of Ω 'improves' when the opening angle ω decreases. In fact Kondratiev in [12, page 210] states that

> " ... and to obtain the theorems about the differential properties of solution. We do this for the number of concrete equations in § 5. In particular, it is derived that the differential properties of the solution are getting better when the cone opening decreases."

One of the peculiar results for the present clamped grid problem is that this does not apply for the whole range 0 to 2π. We will show that there is an interval $(\frac{1}{2}\pi, \omega_\star)$, with $\omega_\star/\pi \approx 0.528\ldots$ (in degrees $\omega_\star \approx 95.1\ldots^\circ$), where the optimal regularity *increases* with increasing ω. This is outlined in the table below. The actual curve that displays the connection between ω and λ, a parameter for the differential properties, is obtained numerically. The discretization is chosen fine enough such that analytical estimates show that the numerical errors are so small that they do not destroy the structure.

operator L in (1.4)	opening angle ω	regularity of the solution u to (1.4) in dependence of ω
Δ^2	$(0, 2\pi]$	decreases
$\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$	$(0, \frac{1}{2}\pi]$, $[\omega_\star, 2\pi]$ $[\frac{1}{2}\pi, \omega_\star]$	decreases increases

Table 1. Optimal regularity of the homogeneous Dirichlet problem for Δ^2 and $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$.

For a graph displaying relation between ω and λ see Figure 3. In Figure 6 one finds a more detailed view. The lowest value of the appearing λ is a measure for the regularity. See Figure 9.

1.4. **The lineup.** The paper is divided into 5 sections and several appendices. We start in Section 2 by recalling the results for existence and uniqueness of a weak solution u to problem (1.4). In Section 3 the weighted Sobolev spaces $V_\beta^{l,2}(\Omega)$ are introduced. Section 4 studies the homogeneous problem (1.4) in the infinite cone $\mathcal{K}_\omega$. We derive (almost explicitly) a countable set of functions $\{u_j\}_{j\in\mathbb{N}}$ solving this problem. They will contribute in Section 5 to the regularity statement for u of type (1.3). We address the Kondratiev theory in order to give the asymptotic representation for the solution u to (1.4) in terms of $\{u_j\}_{j\in\mathbb{N}}$.

The first appendix recalls imbedding results for $W^{k,2}(\Omega)$ and $V_\beta^{l,2}(\Omega)$ based on a Hardy inequality. The other appendices contain computational and numerical results. The elaborate third appendix confirms that indeed the errors in the numerical results are small enough. This appendix also contains an explicit version of the Morse Theorem, which is necessary for an analytical error bound that confirms the numerical results.

2. Existence and uniqueness

For the present so-called clamped boundary conditions existence of an appropriate weak solution can be obtained in a standard way even when the corner is not convex. Let us recall the arguments for the existence of a weak solution to problem (1.4). The function space for these weak solutions is

$$\mathring{W}^{2,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{2,2}(\Omega)}}.$$

where $C_c^\infty(\Omega)$ is the space of infinitely smooth functions with compact support in Ω.

Definition 2.1. A function $\tilde{u} \in \mathring{W}^{2,2}(\Omega)$ is a weak solution of the boundary value problem (1.4) with $f \in L^2(\Omega)$, if

$$\int_\Omega (\tilde{u}_{xx}\varphi_{xx} + \tilde{u}_{yy}\varphi_{yy} - f\varphi)\,dx\,dy = 0 \quad \text{for all } \varphi \in \mathring{W}^{2,2}(\Omega). \tag{2.1}$$

Theorem 2.2. *Suppose $f \in L^2(\Omega)$. Then a weak solution of the boundary value problem* (1.4) *in the sense of Definition 2.1 exists. Moreover, this solution is unique.*

Proof. The proof uses the variational formulation of the problem (1.4), namely,

$$\text{Minimize:} \quad E(u) = \int_\Omega \left(\tfrac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) - fu\right) dx\,dy \quad \text{on } \mathring{W}^{2,2}(\Omega). \tag{2.2}$$

This functional is coercive: For $u \in C_0^\infty(\bar{\Omega})$ it follows from $u = u_x = 0$ on $\partial\Omega$ that one finds by a Poincaré inequality:

$$\int_\Omega u^2 dx\,dy \le C \int_\Omega u_x^2 dx\,dy \le C^2 \int_\Omega u_{xx}^2 dx\,dy \tag{2.3}$$

and a similar result for x replaced by y. For the mixed second derivative the clamped boundary conditions allow an integration by parts such that

$$\int_\Omega u_{xy}^2 dx\,dy = \int_\Omega u_{xx}u_{yy} dx\,dy \le \tfrac{1}{2} \int_\Omega \left(u_{xx}^2 + u_{yy}^2\right) dx\,dy. \tag{2.4}$$

By a density argument (2.3) and (2.4) hold for $u \in \mathring{W}^{2,2}(\Omega)$. Hence $\|u\|_{W^{2,2}(\Omega)} \to \infty$ implies $E(u) \to \infty$. A quadratic functional that is coercive is even strictly convex and hence has at most one minimizer. This minimizer exists since $u \mapsto E(u)$ is weakly lower semicontinuous. The integral form of the Euler-Lagrange equation that the minimizer satisfies, defines this minimizer as a weak solution. Moreover, since a weak solution is a critical point of E defined in (2.2) and since the critical point is unique, so is the weak solution. $\square$

Remark 2.3. For $u \in \mathring{W}^{2,2}(\Omega)$ we have just shown that $\|u\|_{W^{2,2}(\Omega)} \le C \int_\Omega (u_{xx}^2 + u_{yy}^2) dx\,dy$. For the hinged grid, that is $u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ a Poincaré inequality still yields (2.3). Indeed, for $u = 0$ on $\partial\Omega$ there exists on every line $y = c$ that intersects Ω an x_c with $(x_c, c) \in \Omega$ and $u_x(x_c, c) = 0$ and starting from this point one proves the second inequality in (2.3). The real problem is (2.4). Indeed, this estimate does not hold on domains with non-convex corners for $u \in W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$.

3. Kondratiev's weighted Sobolev spaces

Due to Kondratiev [12], one of the appropriate functional spaces for the boundary value problems of the type (1.4) are the weighted Sobolev space $V_\beta^{l,2}$. Such spaces can be defined in different ways: either via the set of the square-integrable weighted weak derivatives in Ω (see [12, 10]), or via the completion of the set of infinitely differentiable on Ω functions with bounded support in Ω, with respect to a certain norm (see [13, 20]).

In our case $\Omega \subset \mathbb{R}^2$ is open, bounded, and has a corner in $0 \in \partial\Omega$. It also holds that $\partial\Omega\backslash\{0\}$ is smooth, and that $\Omega \cap B_\varepsilon(0) = \mathcal{K}_\omega \cap B_\varepsilon(0)$, where $B_\varepsilon(0)$ is a ball of radius $\varepsilon > 0$ and $\mathcal{K}_\omega$ is an infinite sector with an opening angle $\omega \in (0, 2\pi)$. These weighted spaces are as follows:

Definition 3.1. Let $l \in \{0, 1, 2, \dots\}$ and $\beta \in \mathbb{R}$. Then $V_\beta^{l,2}(\Omega)$ is defined as a completion:

$$V_\beta^{l,2}(\Omega) = \overline{C_c^\infty\left(\overline{\Omega}\backslash\{0\}\right)}^{\|\cdot\|} \quad \text{with} \tag{3.1}$$

$$\|u\| := \|u\|_{V_\beta^{l,2}(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega (x^2+y^2)^{\beta-l+|\alpha|} |D^\alpha u|^2 dx\, dy\Big)^{1/2}, \tag{3.2}$$

where

$$C_c^\infty\left(\overline{\Omega}\backslash\{0\}\right) := \left\{u \in C_c^\infty\left(\overline{\Omega}\right) : \text{support}(u) \subset \overline{\Omega}\backslash B_\varepsilon(0)\right\}.$$

The space $V_\beta^{l,2}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ such that for each multiindex $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| \le l$, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$ exists in the weak sense and $r^{\beta-l+|\alpha|} D^\alpha u \in L^2(\Omega)$. Here $r = (x^2+y^2)^{1/2}$.

Straightforward from the definition of the norm the following continuous imbeddings hold (see [13, Section 6.2, lemma 6.2.1]):

$$V_{\beta_2}^{l_2,2}(\Omega) \subset V_{\beta_1}^{l_1,2}(\Omega) \quad \text{if } l_2 \ge l_1 \ge 0,\ \beta_2 - l_2 \le \beta_1 - l_1. \tag{3.3}$$

To have the appropriate space for zero Dirichlet boundary conditions in problem (1.4) we also define the corresponding space.

Definition 3.2. For $l \in \{0, 1, 2, \dots\}$ and $\beta \in \mathbb{R}$, set

$$\mathring{V}_\beta^{l,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}, \tag{3.4}$$

with $\|\cdot\|$ as the norm (3.2) and $C_c^\infty(\Omega) := \left\{u \in C_c^\infty\left(\bar{\Omega}\right) : \text{support}(u) \subset \Omega\right\}$.

Remark 3.3. For $u \in \mathring{V}_\beta^{l,2}(\Omega)$ one finds $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \le \ell - 1$ where $D^\alpha u = 0$ is understood in the sense of traces.

4. Homogeneous problem in an infinite sector, singular solutions

The first step in order to improve the regularity of a weak solution is to consider the homogeneous problem in an infinite cone:

$$\begin{gathered} u_{xxxx} + u_{yyyy} = 0 \quad \text{in } \mathcal{K}_\omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{K}_\omega\backslash\{0\}. \end{gathered} \tag{4.1}$$

Here $\mathcal{K}_\omega$ is as in (1.5). We will derive almost explicit formula's for power type solutions to (4.1).

4.1. **Reduced problem.** The reduced problem for (4.1) is obtained in the following way. By Kondratiev [12] one should consider the power type solutions of (4.1):

$$u = r^{\lambda+1}\Phi(\theta), \tag{4.2}$$

with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Here $\lambda \in \mathbb{C}$ and $\Phi : [0, \omega] \to \mathbb{R}$.

We insert u from (4.2) into problem (4.1) and find

$$\left(\tfrac{\partial^4}{\partial x^4} + \tfrac{\partial^4}{\partial y^4}\right) r^{\lambda+1}\Phi(\theta) = r^{\lambda-3}\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi(\theta),$$

with

$$\begin{aligned}\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right) = {} & \tfrac{3}{4}\left(1+\tfrac{1}{3}\cos(4\theta)\right)\tfrac{d^4}{d\theta^4} + (\lambda-2)\sin(4\theta)\tfrac{d^3}{d\theta^3} + \\ & + \tfrac{3}{2}\left(\lambda^2 - 1 - \left(\lambda^2 - 4\lambda - \tfrac{7}{3}\right)\cos(4\theta)\right)\tfrac{d^2}{d\theta^2} + \\ & + \left(-\lambda^3 + 6\lambda^2 - 7\lambda - 2\right)\sin(4\theta)\tfrac{d}{d\theta} + \\ & + \tfrac{3}{4}\left(\lambda^4 - 2\lambda^2 + 1 + \tfrac{1}{3}\left(\lambda^4 - 8\lambda^3 + 14\lambda^2 + 8\lambda - 15\right)\cos(4\theta)\right).\end{aligned} \tag{4.3}$$

Then we obtain a λ-dependent boundary value problem for Φ:

$$\begin{gathered}\mathcal{L}\left(\theta, \tfrac{d}{d\theta}, \lambda\right)\Phi = 0 \quad \text{in } (0,\omega), \\ \Phi = \tfrac{d\Phi}{d\theta} = 0 \quad \text{on } \partial(0,\omega).\end{gathered} \tag{4.4}$$

Remark 4.1. The nonlinear eigenvalue problem (4.4) appears by a Mellin transformation:

$$\Phi(\theta) = (\mathcal{M}u)(\lambda) = \int_0^\infty r^{-\lambda-2}u(r,\theta)dr.$$

So, the reduced problem for (4.1) we mentioned above is problem (4.4). Before we start analyzing it, let us fix some basic notions.

Definition 4.2. Every number $\lambda_0 \in \mathbb{C}$, such that there exists a nonzero function Φ_0 satisfying (4.4), is said to be an eigenvalue of problem (4.4), while $\Phi_0 \in C^4[0,\omega]$ is called its eigenfunction. Such pairs (λ_0, Φ_0) are called solutions to problem (4.4).

If (λ_0, Φ_0) solves (4.4) and if Φ_1 is a nonzero function that solves

$$\begin{gathered}\mathcal{L}(\lambda_0)\Phi_1 + \mathcal{L}'(\lambda_0)\Phi_0 = 0 \quad \text{in } (0,\omega), \\ \Phi = \tfrac{d\Phi}{d\theta} = 0 \quad \text{on } \partial(0,\omega),\end{gathered} \tag{4.5}$$

then Φ_1 is a generalized eigenfunction (of order 1) for (4.4) with eigenvalue λ_0.

Remark 4.3. *Similarly, one may define generalized eigenfunctions of higher order.*

The following holds for (4.4).

Lemma 4.4. *Let $\theta \in (0,\omega)$, $\omega \le 2\pi$. For every fixed $\lambda \notin \{\pm1, 0\}$ in (4.4), let us set*

$$\begin{aligned}\varphi_1(\theta) &= (\cos(\theta) + \tau_1\sin(\theta))^{\lambda+1}, \quad \varphi_2(\theta) = (\cos(\theta) + \tau_2\sin(\theta))^{\lambda+1}, \\ \varphi_3(\theta) &= (\cos(\theta) - \tau_1\sin(\theta))^{\lambda+1}, \quad \varphi_4(\theta) = (\cos(\theta) - \tau_2\sin(\theta))^{\lambda+1},\end{aligned}$$

where $\tau_1 = \frac{\sqrt{2}}{2}(1+i)$, $\tau_2 = \frac{\sqrt{2}}{2}(1-i)$ and $i = \sqrt{-1}$.

The set $S_\lambda := \{\varphi_m\}_{m=1}^4$ is a fundamental system of solutions to the equation $\mathcal{L}\left(\theta, \frac{\partial}{\partial\theta}, \lambda\right)\Phi = 0$ on $(0,\omega)$.

Proof. The derivation of φ_m, $m = 1, \dots, 4$ in S_λ is rather technical and we refer to Appendix 7. There we also compute the Wronskian:

$$W(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta)) = 16(\lambda+1)^3\lambda^2(\lambda-1)\left(\cos^4(\theta) + \sin^4(\theta)\right)^{\lambda-2}.$$

It is non-zero on $\theta \in (0, 2\pi]$ except for $\lambda \in \{\pm1, 0\}$. Hence, for every fixed $\lambda \notin \{\pm1, 0\}$ the set $\{\varphi_m\}_{m=1}^4$ consists of four linear independent functions on $(0,\omega)$, $\omega \le 2\pi$. $\square$

Lemma 4.5. *In the particular cases $\lambda \in \{\pm 1, 0\}$ in (4.4), one finds the following fundamental systems:*

$$\begin{aligned} S_{-1} &= \{1, \arctan(\cos(2\theta)), \operatorname{arctanh}(\tfrac{\sqrt{2}}{2}\sin(2\theta)), \varphi_4(\theta)\}, \\ S_0 &= \{\sin(\theta),\ \cos(\theta),\ \varphi_3(\theta),\ \varphi_4(\theta)\}, \\ S_1 &= \{1,\ \sin(2\theta),\ \cos(2\theta),\ \varphi_4(\theta)\}, \end{aligned}$$

where the explicit formulas for $\varphi_4 \in S_{-1}$, $\{\varphi_3, \varphi_4\} \in S_0$ and $\varphi_4 \in S_1$ are given in Appendix 7.

Proof. The fundamental systems S_{-1}, S_0, S_1 are given in Appendix 7. By straightforward computations one finds that for every above S_λ, $\lambda \in \{\pm 1, 0\}$ the corresponding Wronskian W is proportional to $\left(\cos^4(\theta) + \sin^4(\theta)\right)^{\lambda - 2}$, $\lambda \in \{\pm 1, 0\}$ and hence is nonzero on $\theta \in (0, 2\pi]$. $\square$

In terms of the fundamental systems S we have Φ that solves $\mathcal{L}\left(\theta, \frac{\partial}{\partial\theta}, \lambda\right)\Phi = 0$ as

$$\Phi(\theta) = \sum_{m=1}^{4} b_m \varphi_m(\theta),$$

where $b_m \in \mathbb{C}$. Inserting this expression into the boundary conditions of problem (4.4), we find a homogeneous system of four equations in the unknowns $\{b_m\}_{m=1}^4$ reading as

$$Ab := \begin{pmatrix} \varphi_1(0)) & \varphi_2(0)) & \varphi_3(0)) & \varphi_4(0)) \\ \varphi_1'(0)) & \varphi_2'(0)) & \varphi_3'(0)) & \varphi_4'(0)) \\ \varphi_1(\omega) & \varphi_2(\omega) & \varphi_3(\omega) & \varphi_4(\omega) \\ \varphi_1'(\omega) & \varphi_2'(\omega) & \varphi_3'(\omega) & \varphi_4'(\omega) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 0,$$

where $\omega \in (0, 2\pi]$. It admits non-trivial solutions for $\{b_m\}_{m=1}^4$ if and only if $\det(A) = 0$. Hence, the eigenvalues λ of problem (4.4) in sense of Definition 4.2 will be completely determined by the characteristic equation $\det(A) = 0$.

We deduce the following four cases:

$$\det(A) := \begin{cases} P(\omega, \lambda) & \text{when } \lambda \notin \{\pm 1, 0\}, \\ P_{-1}(\omega) & \text{when } \lambda = -1, \\ P_0(\omega) & \text{when } \lambda = 0, \\ P_1(\omega) & \text{when } \lambda = 1. \end{cases} \tag{4.6}$$

The explicit formulas for P reads as

$$\begin{aligned} P(\omega, \lambda) = &\left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda \\ &+ \left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \Big[2\cos\Big(\lambda\Big(\arctan\Big(\tfrac{\sqrt{2}}{2}\tan(2\omega)\Big) + \ell\pi\Big)\Big) \\ &- 4\cos\left(\lambda \arctan\left(\tan^2(\omega)\right)\right)\Big], \end{aligned} \tag{4.7}$$

where

$$\begin{gathered} \ell = 0 \quad \text{if } \omega \in (0, \frac{1}{4}\pi], \quad \ell = 1 \quad \text{if } \omega \in (\frac{1}{4}\pi, \frac{3}{4}\pi], \\ \ell = 2 \quad \text{if } \omega \in (\frac{3}{4}\pi, \frac{5}{4}\pi], \quad \ell = 3 \quad \text{if } \omega \in (\frac{5}{4}\pi, \frac{7}{4}\pi], \end{gathered}$$

$$\ell = 4 \quad \text{if } \omega \in (\frac{7}{4}\pi, 2\pi].$$

In particular, for $\omega \in \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\}$ in (4.7) we have

$$\begin{gathered}
P(\tfrac{1}{2}\pi, \lambda) = 2 + 2\cos(\pi\lambda) - 4\cos(\tfrac{1}{2}\pi\lambda), \\
P(\pi, \lambda) = -4 + 4\cos^2(\pi\lambda), \\
P(\tfrac{3}{2}\pi, \lambda) = 8\cos^3(\pi\lambda) - 6\cos(\pi\lambda) - 4\cos(\tfrac{1}{2}\pi\lambda) + 2, \\
P(2\pi, \lambda) = 16\cos^4(\pi\lambda) - 16\cos^2(\pi\lambda).
\end{gathered}$$

Formulas for P_{-1}, P_0, P_1 in (4.6) are available in Appendix 7.

4.2. **Analysis of the eigenvalues λ.** To describe the eigenvalues λ of (4.4) for a fixed ω and, what is more important, their behavior in dependence on ω, we analyze the equation $\det(A) = 0$ on the interval $\omega \in (0, 2\pi]$.

First, we find that the equations $P_{-1}(\omega) = 0$ and $P_1(\omega) = 0$ have identical solutions on $(0, 2\pi]$, that are denoted $\omega \in \{\pi, \omega_0, 2\pi\}$. The approximation $\omega_0/\pi \approx 1.424\dots$ (in degrees $\omega_0 \approx 256.25\dots^\circ$) is obtained by the Maple 9.5 package. Equation $P_0(\omega) = 0$ has no solutions on $\omega \in (0, 2\pi]$. Hence, $\lambda \in \{\pm 1\}$ are the eigenvalues of (4.4) for the above values of ω, while $\lambda = 0$ is not an eigenvalue of (4.4).

Now we consider $P(\omega, \lambda) = 0$ on $\omega \in (0, 2\pi]$; here P is given by (4.7). We note that for every $\lambda \in \mathbb{C} \backslash \{\pm 1, 0\}$ it holds that

$$P(\omega, -\lambda) = (\tfrac{3}{4} + \tfrac{1}{4}\cos(4\omega))^{-\lambda} P(\omega, \lambda),$$

that is, the solutions λ of $P(\omega, \lambda) = 0$ are symmetric with respect to the ω-axis. It is immediate that if λ is an eigenvalue then so is $\overline{\lambda}$. It is convenient to introduce the following notation.

Notation 4.6. For every fixed $\omega \in (0, 2\pi]$ we write $\{\lambda_j\}_{j=1}^\infty$ for the collection of the eigenvalues of problem (4.4) in the sense of Definition 4.2, which have positive real part $\mathrm{Re}(\lambda) > 0$ and are ordered by increasing real part.

The complete set of eigenvalues to problem (4.4) will then read as $\{-\lambda_j, \lambda_j\}_{j=1}^\infty$. Now the following lemma can be formulated.

Lemma 4.7. Let $\mathcal{L}$ be the operator given by (4.3).

- For every fixed $\omega \in (0, 2\pi] \backslash \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^\infty$ from Notation 4.6 is given by
$$\{\lambda_j\}_{j=1}^\infty = \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \quad P(\omega, \lambda) = 0\right\}.$$
- For every fixed $\omega \in \{\pi, \omega_0, 2\pi\}$ the set $\{\lambda_j\}_{j=1}^\infty$ from Notation 4.6 is given by
$$\{\lambda_j\}_{j=1}^\infty = \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \in \mathbb{R}^+ \backslash \{1\}, \quad P(\omega, \lambda) = 0\right\} \cup \{1\}.$$

Here ω_0 is a solution of $P_1(\omega) = 0$ on $\omega \in (\pi, 2\pi)$ with the approximation $\omega_0/\pi \approx 1.424\dots$ (in degrees $\omega_0 \approx 256.25\dots^\circ$).

4.3. **Intermezzo: a comparison with Δ^2.** Let the grid-operator $\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}$ in problems (1.4), (4.1) be replaced by the bilaplacian $\Delta^2=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$. We recall some results for that operator, in particular, the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the corresponding reduced problem. We will compare them to those given in Lemma 4.7.

So, for Δ^2 in (4.1) the reduced problem of the type (4.4) has an operator $\mathcal{L}$ reading as (see e.g. [10, page 88]):

$$\mathcal{L}\left(\theta,\tfrac{d}{d\theta},\lambda\right)=\tfrac{d^4}{d\theta^4}+2\left(\lambda^2+1\right)\tfrac{d^2}{d\theta^2}+\left(\lambda^4-2\lambda^2+1\right). \tag{4.8}$$

Proceeding as above one obtains that the corresponding determinants (see [10, page 89] or [3, page 561]) are the following:

$$\det(A):=\begin{cases}\sin^2(\lambda\omega)-\lambda^2\sin^2(\omega) & \text{when } \lambda\notin\{\pm1,0\},\\ \sin^2(\omega)-\omega^2 & \text{when } \lambda=0,\\ \sin(\omega)\left(\sin(\omega)-\omega\cos(\omega)\right) & \text{when } \lambda\in\{\pm1\}.\end{cases} \tag{4.9}$$

Note that for every $\lambda\in\mathbb{C}\backslash\{\pm1,0\}$ the function $\sin^2(\lambda\omega)-\lambda^2\sin^2(\omega)$ is even with respect to ω and hence the Notation 4.6 is applicable here. Analysis of $\det(A)=0$ with $\det(A)$ as in (4.9) enables to formulate the analog of Lemma 4.7. Namely,

Lemma 4.8. *Let $\mathcal{L}$ be the operator given by (4.8).*

- *For every fixed $\omega\in(0,2\pi]\backslash\{\pi,\omega_0,2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 4.6 is given by*
$$\{\lambda_j\}_{j=1}^{\infty}=\left\{\lambda\in\mathbb{C}:\operatorname{Re}(\lambda)\in\mathbb{R}^+\backslash\{1\}:\sin^2(\lambda\omega)-\lambda^2\sin^2(\omega)=0\right\}.$$
- *For every fixed $\omega\in\{\pi,\omega_0,2\pi\}$ the set $\{\lambda_j\}_{j=1}^{\infty}$ from Notation 4.6 is given by*
$$\{\lambda_j\}_{j=1}^{\infty}=\left\{\lambda\in\mathbb{C}:\operatorname{Re}(\lambda)\in\mathbb{R}^+\backslash\{1\}:\sin^2(\lambda\omega)-\lambda^2\sin^2(\omega)=0\right\}\cup\{1\}.$$

Here ω_0 is a solution of $\tan(\omega)=\omega$ on $\omega\in(\pi,2\pi)$ with the approximation $\omega_0/\pi\approx 1.430\dots$ (in degrees $\omega_0\approx 257.45\dots^\circ$).

4.4. **Analysis of the eigenvalues λ (continued).** Let (ω,λ) be the pair that solves the equations of Lemmas 4.7 and 4.8. In Figure 3 we plot the pairs $(\omega,\operatorname{Re}(\lambda))$ inside the region $(\omega,\operatorname{Re}(\lambda))\in(0;2\pi]\times[0,7.200]$.

Remark 4.9. The numerical computations are performed with the Maple 9.5 package in the following way: at a first cycle for every $\omega_n=\frac{21}{180}\pi+\frac{1}{60}\pi n$, $n=0,\dots,113$ we compute the entries of the set $\{\lambda_j\}_{j=1}^{N}$. Here, N is determined by the condition: $\operatorname{Re}(\lambda_N)\le 7.200$ and $\operatorname{Re}(\lambda_{N+1})>7.200$. The points (ω,λ) where λ_j transits from the complex plane to the real one or vice-versa are solutions to the system $P(\omega,\lambda)=0$ and $\frac{\partial P}{\partial\lambda}(\omega,\lambda)=0$ (the justification for the second condition will be discussed in Lemma 4.15).

In Figure 3 one sees the difference in the behavior of the eigenvalues in the corresponding cases. In particular, in the top plot (the case $L=\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}$) there are the loops and the ellipses in the vicinities of $\omega\in\left\{\frac{1}{2}\pi,\frac{3}{2}\pi\right\}$ (we inclose them in the rectangles). The bottom plot (the case $L=\Delta^2$) looks much simpler near the same region. As mentioned, the contribution of the first eigenvalue λ_1 to the regularity of the solution u to our problem (1.4) is the most essential. So, it is important for us to know the dependence of the eigenvalues λ on the opening angle

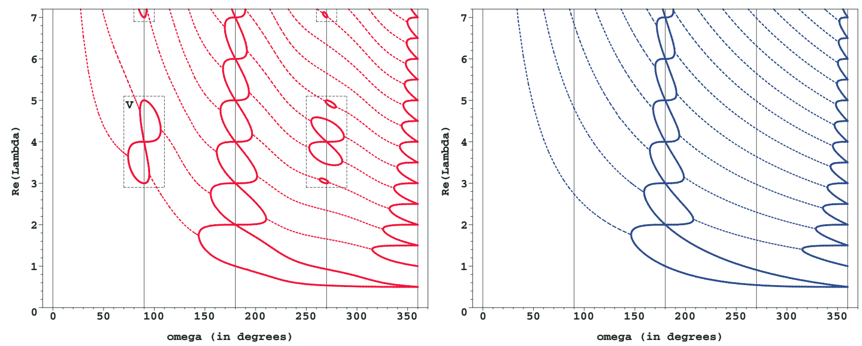

FIGURE 3. Some first eigenvalues λ_j in $(\omega, \mathrm{Re}(\lambda)) \in (0, 2\pi] \times [0, 7.200]$ of problem (4.4), where $\mathcal{L}$ is related respectively to $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ (on the top) and Δ^2 (on the bottom). Dashed lines depict the real part of those $\lambda_j \in \mathbb{C}$, solid lines are for purely real λ_j; the vertical thin lines mark out values $\left\{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\right\}$ on ω-axis.

ω. In this sense, the region $(\omega, \mathrm{Re}(\lambda)) \in V$ (Figure 3, top) seems to be the most interesting part and the model one. One observes that inside V the graph of the implicit function $P(\omega,\lambda) = 0$ looks like a deformed 8-shaped curve. So, if one proves that everywhere in V, $P(\omega,\lambda) = 0$ allows its local parametrization in $\omega \mapsto \lambda = \psi(\omega)$ or $\lambda \mapsto \omega = \varphi(\lambda)$, then the bottom part of this graph is λ_1 and there is a subset of the this bottom part where λ_1 as a function of ω increases with increasing ω.

4.4.1. *Behavior of λ in V.* So let us fix the open rectangular domain $V = \{(\omega, \lambda) : [\frac{70}{180}\pi, \frac{110}{180}\pi] \times [2.900, 5.100]\}$, the function $P \in C^\infty(V, \mathbb{R})$ is given by (4.7) with $\ell = 1$:

$$\begin{aligned} P(\omega, \lambda) = &\Big(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\Big)^{\lambda} + \Big(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\Big)^{\lambda} \\ &+ \big(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\big)^{\frac{1}{2}\lambda}\Big[2\cos\Big(\lambda\Big(\arctan\Big(\tfrac{\sqrt{2}}{2}\tan(2\omega)\Big) + \pi\Big)\Big) \\ &- 4\cos\big(\lambda\arctan\big(\tan^2(\omega)\big)\big)\Big]. \end{aligned} \tag{4.10}$$

and set

$$\Gamma := \{(\omega,\lambda) \in V : P(\omega,\lambda) = 0\}, \tag{4.11}$$

as a zero level set of P in V.

Remark 4.10. To plot the set Γ we perform the computations to $P(\omega,\lambda) = 0$ in V in the spirit of Remark 4.9.

In particular, for $\omega = \frac{1}{2}\pi$ being set in (4.10) we obtain $P\left(\frac{1}{2}\pi,\lambda\right) = 2 + 2\cos(\pi\lambda) - 4\cos\left(\frac{1}{2}\pi\lambda\right)$. The equation $P\left(\frac{1}{2}\pi,\lambda\right) = 0$ admits exact solutions for λ in the interval $(2.900, 5.100)$, namely, $\lambda \in \{3, 4, 5\}$. This yields the points

$$\Big(\frac{1}{2}\pi, 3\Big) =: c_1, \quad \Big(\frac{1}{2}\pi, 4\Big) =: a, \quad \Big(\frac{1}{2}\pi, 5\Big) =: c_4,$$

of Γ. It also holds straightforwardly that $\frac{\partial P}{\partial \omega}(c_1) = \frac{\partial P}{\partial \omega}(c_4) = 0$ and hence one may guess that horizontal tangents to the set Γ exist at those points (in Lemma 4.14

this situation will be discussed in details for the point c_1). For a we find directly that $\frac{\partial P}{\partial \omega}(a) = \frac{\partial P}{\partial \lambda}(a) = 0$ and hence more detailed analysis is required. Additionally to c_1, c_4, we will also specify four other points of the set Γ. Denoted as c_2, c_3, c_5, c_6, they are defined by the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$. The latter condition (we will justify it in Lemma 4.15 for the point c_2) gives us a hint that vertical tangents to Γ exist at those points. The approximations for the coordinates of c_i, $i = 1, \dots, 6$ are listed in the table and we plot the level set Γ in Figure 4.

Point of Γ	Coordinates $(\omega/\pi,\ \lambda)$	ω in degrees	property of Γ at c_k
c_1	$(\frac{1}{2}, 3)$	$90°$	horizontal tangent
c_2	$(0.528\dots, 3.220\dots)$	$\approx 95.1\dots°$	vertical tangent
c_3	$(0.591\dots, 4.291\dots)$	$\approx 106.4\dots°$	vertical tangent
c_4	$(\frac{1}{2}, 5)$	$90°$	horizontal tangent
c_5	$(0.477\dots, 4.746\dots)$	$\approx 85.96\dots°$	vertical tangent
c_6	$(0.412\dots, 3.655\dots)$	$\approx 74.2\dots°$	vertical tangent

TABLE 2. Approximations for the points of the level set Γ.

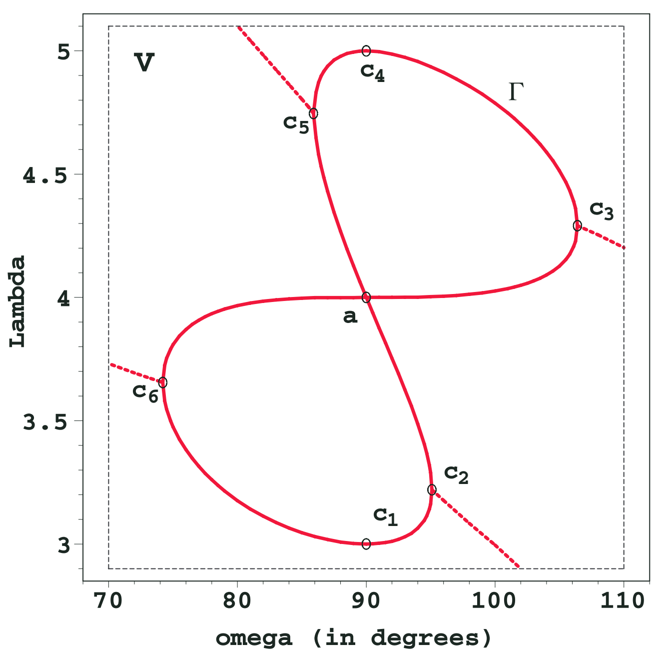

FIGURE 4. The level set Γ (solid line) in V.

As we mention in Remark 4.10, the set Γ as in (4.11) was found by means of numerical computations. In order to show that the plot of Γ is adequate, we study the implicit function $P(\omega,\lambda) = 0$ in V analytically. It is done in several steps.

The first lemma studies $P(\omega,\lambda)=0$ in the vicinity of the point

$$a=(\tfrac{1}{2}\pi,4)\in\Gamma. \tag{4.12}$$

Lemma 4.11. *Let $U=I\times J\subset V$ be the closed rectangle with $I=\left[\frac{88}{180}\pi,\frac{92}{180}\pi\right]$, $J=[3.940,4.060]$ and let point $a\in U$ be as in (4.12). The set Γ given by (4.11) consists of two smooth branches passing through a. Their tangents at a are $\lambda=4$ and $\lambda=-\frac{16\sqrt{2}}{\pi}\omega+4$.*

Proof. Let DP stand for the gradient vector and D^2P is the Hessian matrix. For the given a we already know that $DP(a)=0$. We also find

$$\tfrac{\partial^2 P}{\partial\omega^2}(a)=0,\quad \tfrac{\partial^2 P}{\partial\omega\partial\lambda}(a)=-8\sqrt{2}\pi,\quad \tfrac{\partial^2 P}{\partial\lambda^2}(a)=-\pi^2.$$

That is, $\det D^2P(a)=-128\pi^2$ and by Proposition 8.5 and remark 8.6 (Appendix 8) it holds that

$$P(\omega,\lambda)=-\tfrac{1}{2}h_2(\omega,\lambda)\Big(16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)\Big)\quad\text{on } U, \tag{4.13}$$

where $h_1,h_2\in C^\infty(U,\mathbb{R})$ are given by almost explicit formulas in (8.13), (8.14) in the same lemma. We also have that $h_1(a)=h_2(a)=0$ and

$$\tfrac{\partial h_1}{\partial\omega}(a)=1,\quad \tfrac{\partial h_1}{\partial\lambda}(a)=0, \tag{4.14}$$

$$\tfrac{\partial h_2}{\partial\omega}(a)=0,\quad \tfrac{\partial h_2}{\partial\lambda}(a)=1. \tag{4.15}$$

Due to (4.13) we deduce that in U:

$$P(\omega,\lambda)=0\quad\text{if and only if}\quad h_2(\omega,\lambda)=0\text{ or }16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0. \tag{4.16}$$

By applying the Implicit Function Theorem to the functions $h_2(\omega,\lambda)=0$ and $16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0$ in U one finds a parametrization $\omega\mapsto\lambda=\eta(\omega)$ for each of these implicit functions. Indeed:

(1) For $h_2(\omega,\lambda)=0$ it is shown in Lemma 8.8 (Appendix 8) that

$$\tfrac{\partial h_2}{\partial\lambda}(\omega,\lambda)>0\quad\text{on } U,$$

and hence there exists $\eta_1:I\to J$, $\eta_1\in C^\infty(I)$ such that

$$h_2(\omega,\eta_1(\omega))=0,$$

and

$$\eta_1'(\omega)=-\tfrac{\partial h_2}{\partial\omega}(\omega,\eta_1(\omega))\left[\tfrac{\partial h_2}{\partial\lambda}(\omega,\eta_1(\omega))\right]^{-1},$$

for all $\omega\in I$. We have that $\eta_1(\frac{1}{2}\pi)=4$ and due to (4.15) we find $\eta_1'(\frac{1}{2}\pi)=0$. Hence, there is a smooth branch of Γ in U passing through a, which is given by $\lambda=\eta_1(\omega)$ with the tangent $\lambda=4$.

(2) For $16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)=0$ it is shown in Lemma 8.9 (Appendix 8) that

$$16\sqrt{2}\tfrac{\partial h_1}{\partial\lambda}(\omega,\lambda)+\pi\tfrac{\partial h_2}{\partial\lambda}(\omega,\lambda)>0\quad\text{on } U,$$

and hence there exists $\eta_2:\tilde{I}\to J$, $\eta_2\in C^\infty(\tilde{I})$, where $\tilde{I}\subset I$, such that

$$16\sqrt{2}h_1(\omega,\eta_2(\omega))+\pi h_2(\omega,\eta_2(\omega))=0,$$

and

$$\eta_2'(\omega)=-\frac{16\sqrt{2}\frac{\partial h_1}{\partial\omega}(\omega,\eta_2(\omega))+\pi\frac{\partial h_2}{\partial\omega}(\omega,\eta_2(\omega))}{16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega,\eta_2(\omega))+\pi\frac{\partial h_2}{\partial\lambda}(\omega,\eta_2(\omega))},$$

for all $\omega \in \tilde{I}$. We have that $\eta_2(\frac{1}{2}\pi) = 4$ and due to (4.14) and (4.15) we obtain

$$\eta_2'(\tfrac{1}{2}\pi) = -\tfrac{16\sqrt{2}}{\pi}.$$

Hence, there is another smooth branch of Γ in U passing through a and given by $\lambda = \eta_2(\omega)$. The tangent is $\lambda = -\frac{16\sqrt{2}}{\pi}\omega + 4$. □

The next lemma studies $P(\omega,\lambda) = 0$ locally in V but away from the point a.

Lemma 4.12. *Let*

$$\begin{aligned}
H_1 &= \{(\omega, \lambda) : [\tfrac{84}{180}\pi, \tfrac{90}{180}\pi] \times [4.030, 4.970]\},\\
H_2 &= \{(\omega, \lambda) : [\tfrac{87}{180}\pi, \tfrac{101}{180}\pi] \times [4.750, 5.100]\},\\
H_3 &= \{(\omega, \lambda) : [\tfrac{100}{180}\pi, \tfrac{108}{180}\pi] \times [4.000, 4.850]\},\\
H_4 &= \{(\omega, \lambda) : [\tfrac{91}{180}\pi, \tfrac{102}{180}\pi] \times [3.950, 4.100]\},\\
H_5 &= \{(\omega, \lambda) : [\tfrac{90}{180}\pi, \tfrac{96}{180}\pi] \times [3.030, 3.970]\},\\
H_6 &= \{(\omega, \lambda) : [\tfrac{79}{180}\pi, \tfrac{94}{180}\pi] \times [2.900, 3.230]\},\\
H_7 &= \{(\omega, \lambda) : [\tfrac{72}{180}\pi, \tfrac{80}{180}\pi] \times [3.150, 4.000]\},\\
H_8 &= \{(\omega, \lambda) : [\tfrac{78}{180}\pi, \tfrac{89}{180}\pi] \times [3.900, 4.050]\},
\end{aligned}$$

and U be as in Lemma 4.11. Then $\cup_{j=1}^{8} H_j$ covers the set Γ in V (see Figure 5) and in each H_j the following holds:

Rectangle	Property in H_j	The set Γ in H_j is given by
H_{2k-1}	$\frac{\partial P}{\partial \omega}(\omega, \lambda) \neq 0$	$\omega = \phi_{2k-1}(\lambda) : \phi_{2k-1} \in C^\infty(J_{2k-1})$
H_{2k}	$\frac{\partial P}{\partial \lambda}(\omega, \lambda) \neq 0$	$\lambda = \psi_{2k}(\omega) : \psi_{2k} \in C^\infty(I_{2k})$

Here $k = 1, \dots, 4$.

Proof. In Claims 8.10 – 8.17 of Appendix 8 we constructed the rectangles $H_j \subset V$, $j = 1, \dots, 8$ such that the results of the second column in a table above hold. In Figure 5 we sketched the covering of the set Γ in V with the rectangles H_j, $j = 1, \dots, 8$.

Due to result of the second column we can apply the Implicit Function Theorem to the function $P(\omega,\lambda) = 0$ in every H_j, $j = 1, \dots, 8$ in order to obtain $\omega = \phi_{2k-1}(\lambda)$ or $\lambda = \psi_{2k}(\omega)$, $k = 1, \dots, 4$. By assumption $P \in C^\infty(V, \mathbb{R})$ and hence ϕ, ψ are C^∞ on the corresponding intervals J, I. □

Based on the results of the two lemmas above, we arrive at the following result.

Proposition 4.13. *The set Γ given by (4.11) is an 8-shaped curve. That is, there exists an open set $\tilde{V} \supset [-1, 1]^2$ and a C^∞-diffeomorphism $S : V \to \tilde{V}$ such that*

$$S(\Gamma) = \{(\sin(2t), \sin(t)),\ 0 \leq t < 2\pi\}.$$

Henceforth, we will call the set Γ a curve (having one self-intersection point) which means that every part of the set Γ is locally parametrizable in ω or λ.

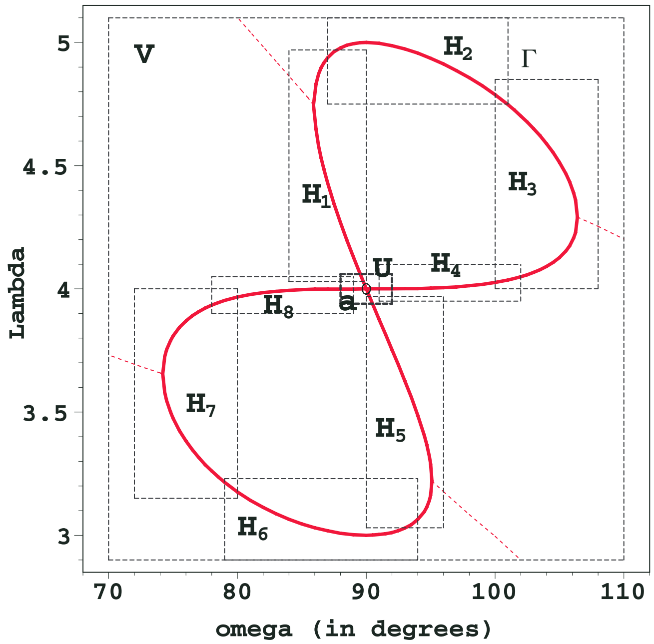

FIGURE 5. For lemma 4.12.

4.4.2. *Eigenvalue* λ_1 *as the bottom part of* Γ. The curve Γ in a rectangle V combines the graphs of the first four eigenvalues $\lambda_1, \dots, \lambda_4$ of the boundary value problem (4.4) as functions of ω as far as they are real. Here we focus on the eigenvalue λ_1 which is a bottom part of Γ (the segment $c_6c_1c_2 \subset \Gamma$ in Figure 4). In particular, we prove that as a function of ω the eigenvalue $\lambda_1 = \lambda_1(\omega)$ increases between the points c_1, c_2 (the approximations for their coordinates are given in Table 2). The situation is illustrated by Figure 6.

To prove this result, we follow the approach used in Lemmas 4.11 and 4.12. To be more precise, we fix two rectangles $\{H_0, H_\star\} \subset V$ such that $H_0 \cap H_\star \neq \emptyset$ and $H_0 \cup H_\star$ covers the part of Γ containing the segment c_1c_2 (see Figure 7). We parameterize Γ in $H_0, H_\star$ as $\omega \mapsto \lambda = \psi(\omega)$ and $\lambda \mapsto \omega = \varphi(\lambda)$, respectively, and study the properties of these parametrizations (convexity-concavity, extremum points, the intervals of increase-decrease). This will enable to gain the information about c_1c_2.

Lemma 4.14. *Let* $H_0 = I_0 \times J_0 \subset V$ *be the closed rectangle with* $I_0 = \left[\frac{84}{180}\pi, \frac{94}{180}\pi\right]$ *and* $J_0 = [2.960, 3.060]$. *It holds that* Γ *in* H_0 *is given by* $\lambda = \psi(\omega)$, $\psi \in C^\infty(\omega_\alpha, \omega_\beta)$, $(\omega_\alpha, \omega_\beta) \subset I_0$ *and is such that it attains its minimum on* $(\omega_\alpha, \omega_\beta)$ *at* $\omega = \omega_0 = \frac{1}{2}\pi$ *and increases monotonically on* (ω_0, ω_β). *Here* $\omega_\alpha, \omega_\beta$ *are the solutions to the equation* $P(\omega, 3.060) = 0$ *on* $\omega \in \left(\frac{84}{180}\pi, \frac{1}{2}\pi\right)$ *and on* $\omega \in \left(\frac{1}{2}\pi, \frac{94}{180}\pi\right)$, *respectively, with* P *given by (4.10).*

Proof. By Lemma 4.12 we know that

$$P(\omega, \lambda) = 0 \quad \text{if and only if } P\left(\omega, \psi(\omega)\right) = 0 \text{ in } H_6, \tag{4.17}$$

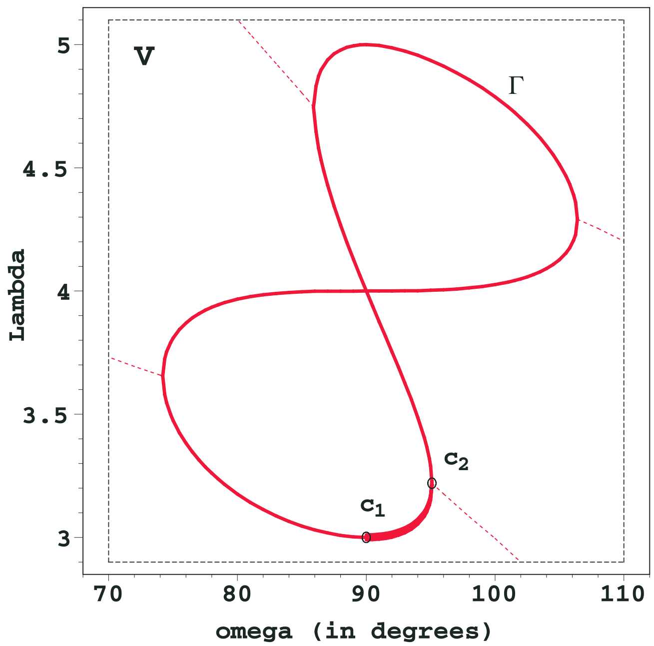

FIGURE 6. Increase of λ_1 between c_1 and c_2

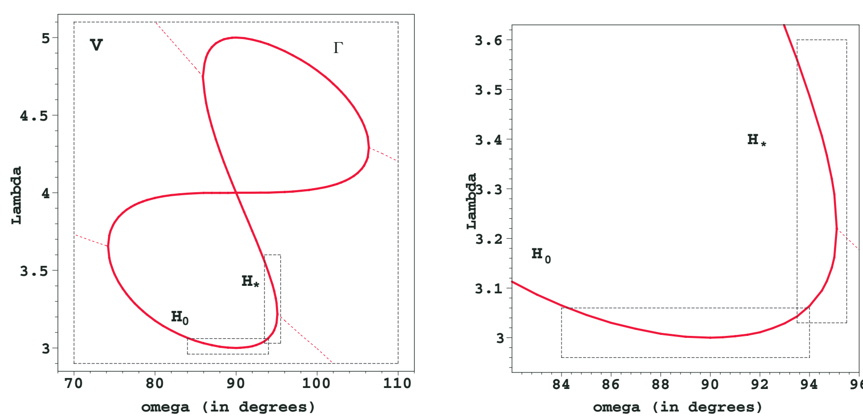

FIGURE 7. The rectangles $H_0, H_\star$ from lemmas 4.14 and 4.15, respectively (on the left); the enlarged view (on the right).

and if we take the rectangle H_0 defined as in lemma above, then due to $H_0 \subset H_6$, (4.17) will also hold in H_0. Moreover, we also set H_0 in such a way that its top boundary intersects Γ at two points, meaning that we find two solutions of $P(\omega, 3.060) = 0$ with P as in (4.10). We name these two solutions $\omega_\alpha, \omega_\beta$.

Hence, we deduce that Γ in H_0 is given by $\lambda = \psi(\omega)$, $\psi \in C^\infty(\omega_\alpha, \omega_\beta)$ and satisfies $\psi(\omega_\alpha) = \psi(\omega_\beta) = 3.060$. Due to condition

$$\psi(\omega_\alpha) = \psi(\omega_\beta),$$

by Rolle's theorem there exists $\omega_0 \in (\omega_\alpha, \omega_\beta)$ such that $\psi'(\omega_0) = 0$.

Since $P(\omega_0, \psi(\omega_0)) = 0$ and due to

$$\psi'(\omega) = -\tfrac{\partial P}{\partial \omega}(\omega, \psi(\omega))[\tfrac{\partial P}{\partial \lambda}(\omega, \psi(\omega))]^{-1},$$

we solve the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \omega}(\omega, \lambda) = 0$ in H_0 in order to find ω_0. Its solution is a point $c_1 = (\frac{1}{2}\pi, 3)$ and hence

$$\omega_0 = \tfrac{1}{2}\pi.$$

We deduce that $\lambda = \psi(\omega)$ attains its local extremum at $\omega = \omega_0$.

Next we show that $\lambda = \psi(\omega)$ has a minimum at $\omega = \omega_0$ on $(\omega_\alpha, \omega_\beta)$. For this purpose we consider a function $G \in C^\infty(H_0, \mathbb{R})$ such that

$$G(\omega, \psi(\omega)) = \psi''(\omega). \tag{4.18}$$

For an explicit formula for G see Appendix 8. In Claim 8.18 of this Appendix we show that

$$G(\omega, \lambda) > 0 \quad \text{on } H_0. \tag{4.19}$$

This condition together with (4.18) yields

$$G(\omega, \psi(\omega)) = \psi''(\omega) > 0 \quad \text{on } (\omega_\alpha, \omega_\beta),$$

meaning that $\lambda = \psi(\omega)$ is convex on $(\omega_\alpha, \omega_\beta)$.

The result is that $\lambda = \psi(\omega)$ attains its minimum on $(\omega_\alpha, \omega_\beta)$ at $\omega = \omega_0 = \frac{1}{2}\pi$ and increases monotonically on the interval $\omega \in (\omega_0, \omega_\beta)$. $\square$

We also have the following result.

Lemma 4.15. *Let $H_\star = I_\star \times J_\star \subset V$ be the closed rectangle with $I_\star = \left[\frac{93.5}{180}\pi, \frac{95.5}{180}\pi\right]$ and $J_\star = [3.030, 3.600]$. It holds that Γ in $H_\star$ is given by $\omega = \varphi(\lambda)$, $\varphi \in C^\infty(\lambda_\gamma, \lambda_\delta)$, $(\lambda_\gamma, \lambda_\delta) \subset J_\star$ and is such that it attains its maximum on $(\lambda_\gamma, \lambda_\delta)$ at $\lambda = \lambda_\star \approx 3.220\dots$ and increases monotonically on the interval $(\lambda_\gamma, \lambda_\star)$. Here $\lambda_\gamma, \lambda_\delta$ are the solutions to the equation $P\left(\frac{93.5}{180}\pi, \lambda\right) = 0$ on $\lambda \in (3.030, 3.100)$ and on $\lambda \in (3.500, 3.600)$, respectively. Also, $\lambda_\star$ is the solution to the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$ on $\lambda \in (\lambda_\gamma, \lambda_\delta)$; P given by (4.10).*

Proof. By Lemma 4.12 we know that

$$P(\omega, \lambda) = 0 \quad \text{if and only if} \quad P(\varphi(\lambda), \lambda) = 0 \text{ in } H_5, \tag{4.20}$$

and if we take the rectangle $H_\star$ defined as in lemma above, then due to $H_\star \subset H_5$, (4.20) will also hold in $H_\star$. Moreover, we also set $H_\star$ in such a way that its left boundary intersects Γ at two points, meaning we find two solutions of $P\left(\frac{93.5}{180}\pi, \lambda\right) = 0$ with P as in (4.10). We name these two solutions $\lambda_\gamma, \lambda_\delta$.

Hence, we deduce that Γ in $H_\star$ is given by $\omega = \varphi(\lambda)$, $\varphi \in C^\infty(\lambda_\gamma, \lambda_\delta)$ and satisfies $\varphi(\lambda_\gamma) = \varphi(\lambda_\delta) = \frac{93.5}{180}\pi$. Due to condition

$$\varphi(\lambda_\gamma) = \varphi(\lambda_\delta),$$

by Rolle's theorem there exists $\lambda_\star \in (\lambda_\gamma, \lambda_\delta)$ such that $\varphi'(\lambda_\star) = 0$.

Since $P(\varphi(\lambda_\star), \lambda_\star) = 0$ and due to

$$\varphi'(\lambda) = -\tfrac{\partial P}{\partial \lambda}(\varphi(\lambda), \lambda)[\tfrac{\partial P}{\partial \omega}(\varphi(\lambda), \lambda)]^{-1},$$

we solve the system $P(\omega, \lambda) = 0$ and $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$ in $H_\star$ in order to find $\lambda_\star$. Its solution is a point $c_2 = \left(\tilde{\omega}, \tilde{\lambda}\right)$, where $\tilde{\omega}/\pi \approx 0.528\dots$ and $\tilde{\lambda} \approx 3.220\dots$. Hence,

$$\lambda_\star \approx 3.220\dots .$$

We deduce that $\omega = \varphi(\lambda)$ attains its local extremum at $\lambda = \lambda_\star$.

Next we show that $\omega = \varphi(\lambda)$ has a maximum at $\lambda = \lambda_\star$ on $(\lambda_\gamma, \lambda_\delta)$. For this purpose we consider a function $F \in C^\infty(H_\star, \mathbb{R})$ such that

$$F(\varphi(\lambda), \lambda) = \varphi''(\lambda). \tag{4.21}$$

For explicit formula for F see Appendix 8. In Claim 8.19 of this Appendix we show that

$$F(\omega, \lambda) < 0 \quad \text{on } H_\star. \tag{4.22}$$

This condition together with (4.21) yields

$$F(\varphi(\lambda), \lambda) = \varphi''(\lambda) < 0 \quad \text{on } (\lambda_\gamma, \lambda_\delta),$$

meaning that $\omega = \varphi(\lambda)$ is concave on $(\lambda_\gamma, \lambda_\delta)$. The result is that $\omega = \varphi(\lambda)$ attains its maximum on $(\lambda_\gamma, \lambda_\delta)$ at $\lambda = \lambda_\star \approx 3.220\dots$ and increases monotonically on the interval $\lambda \in (\lambda_\gamma, \lambda_\star)$. ☐

Theorem 4.16. *As a function of ω the first eigenvalue $\lambda_1 = \lambda_1(\omega)$ of the boundary value problem* (4.4) *increases on $\omega \in \left(\frac{1}{2}\pi, \omega_\star\right)$. Here $\omega_\star/\pi \approx 0.528\dots$ (in degrees $\omega_\star \approx 95.1\dots^\circ$) and $\lambda_\star \approx 3.220\dots$.*

4.5. **The multiplicities of $\{\lambda_j\}_{j=1}^\infty$ and the structure of a singular solution.** Here we proceed with the qualitative analysis of the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of problem (4.4).

Definition 4.17. Let $\omega \in (0, 2\pi]$ be fixed. The eigenvalue λ_j, $j \in \mathbb{N}^+$ of problem (4.4) is said to have an algebraic multiplicity $\kappa^{(j)} \geq 1$, if the following holds:

$$P(\omega, \lambda_j) = 0, \quad \tfrac{dP}{d\lambda}(\omega, \lambda_j) = 0, \quad \dots, \quad \tfrac{d^{\kappa^{(j)}-1}P}{d\lambda^{\kappa^{(j)}-1}}(\omega, \lambda_j) = 0, \quad \tfrac{d^{\kappa^{(j)}}P}{d\lambda^{\kappa^{(j)}}}(\omega, \lambda_j) \neq 0.$$

Based on the numerical approximations for some first eigenvalues λ_j, $j \in \mathbb{N}^+$ depicted in Figure 3 (the top one) and partly by our derivations (namely, the existence of the solution to the system $P(\omega, \lambda) = \frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$ in Lemma 4.15) we believe that the maximal algebraic multiplicity of a certain λ_j of problem (4.4) is at most 2 . Indeed, generically 3 curves never intersect at one point, meaning that geometrically the algebraic multiplicity will always be at most 2.

Definition 4.18. The eigenvalue λ_j, $j \in \mathbb{N}^+$ of problem (4.4) is said to have a geometric multiplicity $I^{(j)} \geq 1$, if the number of linearly independent eigenfunctions Φ equals $I^{(j)}$.

For given λ_j, $j \in \mathbb{N}^+$ of problem (4.4) the three cases occur:

1. $\kappa^{(j)} = I^{(j)} = 1$ one finds a solution $(\lambda_j, \Phi_0^{(j)})$ of (4.4) and then the solution of (4.1) reads:

$$u_0^{(j)} = r^{\lambda_j+1}\Phi_0^{(j)}(\theta); \tag{4.23}$$

2. $\kappa^{(j)} = 2$, $I^{(j)} = 1$ one finds a solution $(\lambda_j, \Phi_0^{(j)})$ of (4.4) and a generalized solution $(\lambda_j, \Phi_1^{(j)})$, with $\Phi_1^{(j)}$ found from the equation

$$\mathcal{L}(\lambda_j)\Phi_1^{(j)} + \mathcal{L}'(\lambda_j)\Phi_0^{(j)} = 0,$$

where $\mathcal{L}(\lambda)$ is given by (4.3) and $\mathcal{L}'(\lambda) = \frac{d}{d\lambda}\mathcal{L}(\lambda)$. Then we have two solutions of (4.1):

$$u_0^{(j)} = r^{\lambda_j+1}\Phi_0^{(j)}(\theta) \quad \text{and} \quad u_1^{(j)} = r^{\lambda_j+1}\left(\Phi_1^{(j)}(\theta) + \log(r)\Phi_0^{(j)}(\theta)\right); \tag{4.24}$$

3. $\kappa^{(j)} = I^{(j)} = 2$ one finds two solutions $(\lambda_j, \Phi_0^{(j)})$, $(\lambda_j, \Phi_1^{(j)})$ of (4.4), where $\Phi_0^{(j)}$ and $\Phi_1^{(j)}$ are linearly independent on $\theta \in (0, \omega)$ and then we again have two solutions of (4.1):

$$u_0^{(j)} = r^{\lambda_j+1}\Phi_0^{(j)}(\theta) \quad \text{and} \quad u_1^{(j)} = r^{\lambda_j+1}\Phi_1^{(j)}(\theta). \tag{4.25}$$

Let us note that for an opening angle $\omega \in \{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\}$ of the sector $\mathcal{K}_\omega$ one can find the eigenvalues $\{\lambda_j\}_j^\infty$ of the problem (4.4) explicitly. Moreover, if $\omega \in \left\{\frac{1}{2}\pi, \pi\right\}$, then for every given λ_j one can compute explicitly the corresponding eigenfunctions $\Phi_q^{(j)}$, $q = 0, \dots, \kappa^{(j)} - 1$, whereas if $\omega \in \left\{\frac{3}{2}\pi, 2\pi\right\}$, the eigenfunctions $\Phi_q^{(j)}$ can be computed explicitly only for some λ_j. Thus, in Appendix 9 we bring the formulas of some first functions $\Phi_q^{(j)}$ (if computable) and the respective solutions $u_q^{(j)}$ to (4.1).

These functions $r^{\lambda+1}\Phi(\theta)$ and $r^{\lambda+1}\log(r)\Phi(\theta)$ determine the bands for the regularity in Kondratiev's theory. Details are found in the next section.

5. Regularity results

In this subsection we will give the regularity result for the boundary value problem (1.4) under consideration. In order to do this we refer to the key theorem of the Kondratiev theory (see e.g. [13, Theorem 6.4.1]). The general version adapted to our problem (1.4) will read as:

Theorem 5.1 (Kondratiev). *Let $u \in V_{\beta_1}^{l_1,2}(\Omega)$ with $l_1 \in \mathbb{N}$, $\beta_1 \in \mathbb{R}$ be a solution of the elliptic boundary value problem* (1.4).

Suppose that $f \in V_{\beta_2}^{l_2,2}(\Omega)$, where $l_2 \in \mathbb{N}$, $\beta_2 \in \mathbb{R}$ and such that $l_1 - \beta_1 < l_2 - \beta_2 + 4$. If no eigenvalue λ_j of problem (4.4) *lies on the lines*

$$\operatorname{Re}(\lambda) = l_1 - \beta_1 - 2, \quad \operatorname{Re}(\lambda) = l_2 - \beta_2 + 2,$$

while the strip

$$l_1 - \beta_1 - 2 < \operatorname{Re}(\lambda) < l_2 - \beta_2 + 2,$$

contains the eigenvalues $\lambda_n, \lambda_{n+1} \dots, \lambda_{n+N}$, then u has the representation

$$u = w + \chi(r) \sum_{j=n}^{n+N} \sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}, \tag{5.1}$$

where $w \in V_{\beta_2}^{l_2+4,2}(\Omega)$, $\chi \in C_0^\infty[0, \varepsilon)$ is a cut-off function such that $\chi(r) = 1$ in the neighborhood of $r = 0$, $\kappa^{(j)} \leq 2$ is the algebraic multiplicity of λ_j and $u_q^{(j)}$ are the solutions of the problem (4.1) in $\mathcal{K}_\omega$ given by formulas (4.23)*,* (4.24)*,* (4.25)*.*

Let us recall that by Theorem 2.2 there exists a unique weak solution $u \in \mathring{W}^{2,2}(\Omega)$ of (1.4) with $f \in L^2(\Omega)$. Since $L^2(\Omega) = V_0^{0,2}(\Omega)$ and by Corollary 6.4 one has $\mathring{W}^{2,2}(\Omega) = \mathring{V}_0^{2,2}(\Omega) \subset V_0^{2,2}(\Omega)$, we conclude that for $f \in V_0^{0,2}(\Omega)$ we have $u \in V_0^{2,2}(\Omega)$.

Then assuming more regularity for $f \in V_0^{0,2}(\Omega)$ we apply Theorem 5.1 to our problem (1.4). Using Lemma 6.3 we may consider three different cases:

$$f \in \begin{cases} V_\beta^{k,2}(\Omega), & k \geq 0,\ \beta \geq k, \\ \mathring{W}^{k,2}(\Omega), & k \geq 1, \\ W^{k,2}(\Omega), & k \geq 0, \end{cases}$$

and obtain the following result (in order to describe all three cases and also for the convenience we arranged this result as a table):

Theorem 5.2. *Let $f \in L^2(\Omega)$ and let $u \in \mathring{W}^{2,2}(\Omega)$ be a weak solution to* (1.4)*.*

f is in	$V_\beta^{k,2}(\Omega)$, $k \geq 0$, $\beta \geq k$	$\mathring{W}^{k,2}(\Omega)$, $k \geq 1$	$W^{k,2}(\Omega)$, $k \geq 0$
no eigenvalue λ_j of (4.4) lies on the lines	$\begin{cases}\mathrm{Re}(\lambda) = 0, \\ \mathrm{Re}(\lambda) = k - \beta + 2\end{cases}$	$\begin{cases}\mathrm{Re}(\lambda) = 0, \\ \mathrm{Re}(\lambda) = k + 2\end{cases}$	$\begin{cases}\mathrm{Re}(\lambda) = 0, \\ \mathrm{Re}(\lambda) = 2\end{cases}$
while the strip	$0 < \mathrm{Re}(\lambda) < k - \beta + 2$	$0 < \mathrm{Re}(\lambda) < k + 2$	$0 < \mathrm{Re}(\lambda) < 2$
contains	$\lambda_1, \lambda_2, \dots, \lambda_N$		
	$u = w + \chi(r) \sum_{j=1}^N \sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}$		
where w is in	$V_\beta^{k+4,2}(\Omega)$	$W^{k+4,2}(\Omega)$	$W^{k+4,2}(\Omega, \lvert x\rvert^{2k} d\mu)$

and $\chi \in C_0^\infty[0, \varepsilon)$ is a cut-off function such that $\chi = 1$ in the neighborhood of $r = 0$, $\kappa^{(j)} \leq 2$ is the algebraic multiplicity of λ_j and $u_q^{(j)}$ are the solutions of the problem (4.1) in $\mathcal{K}_\omega$ given by formulas (4.23)*,* (4.24)*,* (4.25)*.*

Remark 5.3. The first column gives the optimal regularity in the sense of Kondratiev's spaces. The second and third column shows two corollaries with the more commonly used Sobolev spaces. Away from the corner also these results are optimal. Of course the optimal regularity near a corner can not be stated using just the standard Sobolev spaces $W^{\ell,2}(\Omega)$.

Proof of Theorem Theorem2. The first column is just a representation of the previous theorem in the case that $f \in L^2(\Omega)$ and a weak solution $u \in \mathring{W}^{2,2}(\Omega)$ is known to exist. Additional regularity in the sense of Kondratiev's weighted Sobolev spaces for f implies the representation as stated for the solution u. In the second and third column the most common special cases are listed independently. For the second column one uses the imbeddings

$$\mathring{W}^{k,2}(\Omega) \subset V_0^{k,2}(\Omega) \text{ and } V_0^{k+4,2}(\Omega) \subset W^{k+4,2}(\Omega),$$

and for the third

$$W^{k,2}(\Omega) \subset V_k^{k,2}(\Omega) \text{ and } V_k^{k+4,2}(\Omega) \subset W^{k+4,2}(\Omega, |x|^{2k} d\mu).$$

□

As the last step of our analysis we derive the regularity in Ω for the singular part $\sum_{j=1}^N \sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}$ of the solution u given in Theorem 5.2.

5.1. **Regularity for the singular part of u.** The first term of the summation $\sum_{j=1}^N \sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}$ in the solution u defines the regularity of the whole sum. From formulas (4.23), (4.24), (4.25) we know that depending on the algebraic multiplicity of λ_1 it reads as

$$r^{\lambda_1+1}\Phi(\theta) \quad \text{or} \quad r^{\lambda_1+1}\left(\Psi(\theta) + \log(r)\Phi(\theta)\right),$$

where $\lambda_1 \in \mathbb{C}$ is the first eigenvalue of (4.4) such that $\mathrm{Re}(\lambda_1) > 0$ and $\Phi(\theta), \Psi(\theta) \in C^\infty[0, \omega]$, $0 < \omega < 2\pi$.

Lemma 5.4. *Let $\Phi(\theta) \in C^\infty[0,\omega]$ be nontrivial and $0 < \omega < 2\pi$. Let also $\lambda \in \mathbb{C}\backslash\mathbb{Z}$ with $\operatorname{Re}(\lambda) > 0$. Suppose that $k \in \{0, 1, 2, 3, \dots\}$. Then the following are equivalent:*

1. $r^{\lambda+1}\Phi(\theta) \in W^{k,2}(\Omega)$,
2. $r^{\lambda+1}\log(r)\Phi(\theta) \in W^{k,2}(\Omega)$,
3. $\operatorname{Re}(\lambda) + 1 > k - 1$.

Proof. If λ is not an integer, then the first item and second items are equivalent with an integrability condition for the k^{th} -derivative that reads as $2\operatorname{Re}(\lambda + 1 - k) + 1 > -1$. □

Remark 5.5. To restrict the already heavy technical aspects we have not considered (weighted) Sobolev spaces with non-integer coefficients k and Hölder spaces. A similar result will hold for k is noninteger. Concerning Hölder spaces:

$$r^{\lambda+1}\Phi(\theta) \in C^{k,\gamma}(\bar{\Omega}) \quad \text{for } \operatorname{Re}(\lambda) + 1 \geq k + \gamma \text{ with } k \in \mathbb{N}, \gamma \in [0, 1).$$

For the second function it holds that

$$r^{\lambda+1}\log(r)\Phi(\theta) \in C^{k,\gamma}(\bar{\Omega}) \quad \text{for } \operatorname{Re}(\lambda) + 1 > k + \gamma \text{ with } k \in \mathbb{N}, \gamma \in [0, 1).$$

A useful consequence of the above lemma is that for every fixed $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$ we deduce

$$\left\{r^{\lambda+1}\Phi(\theta), r^{\lambda+1}\log(r)\Phi(\theta)\right\} \in W^{\lceil \operatorname{Re}(\lambda)\rceil+1,2}(\Omega), \tag{5.2}$$

where $\lceil\cdot\rceil$ stands for the ceiling function (defined as $\lceil x\rceil = \min\{n \in \mathbb{Z} : x \leq n\}$).

Remark 5.6. In the particular cases, namely, $\omega = \frac{1}{2}\pi$ and $\omega = \pi$ we know that each term $u_q^{(j)}$ of the singular part $\sum_{j=1}^{N}\sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}$ is a polynomial in x, y of order $\lambda_j + 1$ (see Appendix 9). That is, for every λ_j, $j \in \mathbb{N}^+$ we have

$$r^{\lambda_j+1}\Phi^{(j)}(\theta) = P_{\lambda_j+1}(x, y) \in C^\infty\left(\overline{\Omega}\right). \tag{5.3}$$

For non-polynomials the result in Lemma 5.4 even holds for $\lambda \in \mathbb{N}$.

Now, in order to use the result of Lemma 5.4, we proceed with Figure 8 where we plot the $\operatorname{Re}(\lambda_1)$ as a function of the opening angle ω on the interval $\omega \in (0, 2\pi]$. The two cases are compared: the plots of $\operatorname{Re}(\lambda_1)$ of the boundary value problem (4.4) for $\mathcal{L}$ related to $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ and $L = \Delta^2$. In Figure 9 we split the plot of Figure 8 into two: a plot of $\operatorname{Re}(\lambda_1)$ on $\omega \in (0, \pi]$ and $\omega \in [\pi, 2\pi]$.

Based on the numerical approximations to λ_1 and partly on the analytical estimates for λ_1 we conclude the following.

Claim 5.7. $(0, 2\pi] \ni \omega \mapsto \operatorname{Re}(\lambda_1(\omega))$ is a continuous function and

$$\text{for } \omega \in (0, \omega_1)/\{\frac{1}{2}\pi\} : \operatorname{Re}(\lambda_1) > 3, \quad \text{for } \omega = \frac{1}{2}\pi : \operatorname{Re}(\lambda_1) = 3,$$
$$\text{for } \omega \in [\omega_1, \omega_2) : 3 \geq \operatorname{Re}(\lambda_1) > 2, \quad \text{for } \omega \in [\omega_2, \pi) : 2 \geq \operatorname{Re}(\lambda_1) > 1,$$
$$\text{for } \omega \in [\pi, 2\pi] : 1 \geq \operatorname{Re}(\lambda_1) \geq \frac{1}{2}.$$

Here ω_1, ω_2 are respectively the solutions of $P(\omega, 3 + i\xi) = 0$ on $\omega \in \left(\frac{1}{2}\pi, \frac{120}{180}\pi\right)$ and of $P(\omega, 2 + i\xi) = 0$ on $\omega \in \left(\frac{2}{3}\pi, \frac{3}{4}\pi\right)$, where P as in formula (4.7) for $\ell = 1$. The approximation are $\omega_1/\pi \approx 0.555\dots$ (in degrees $\omega_1 \approx 99.9\dots°$) and $\omega_2/\pi \approx 0.720\dots$ (in degrees $\omega_2 \approx 129.7\dots°$).

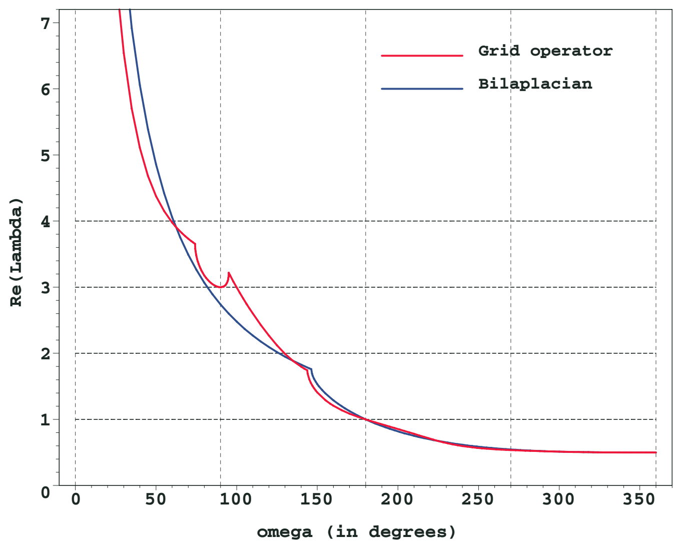

FIGURE 8. The plot of eigenvalue λ_1 in $(\omega, \mathrm{Re}(\lambda)) \in (0, 2\pi] \times [0, 7.200]$ of problem (4.4). For $\mathcal{L}$ related to $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$, λ_1 is represented by the red line and for $\mathcal{L}$ related to Δ^2, by the blue line. Dashed lines depict the real part of $\lambda_1 \in \mathbb{C}$, solid lines are for purely real λ_1; the vertical lines mark out $\left\{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\right\}$ on ω-axis.

5.2. **Consequences.** For the numerical results from Claim 5.7, it holds by Theorem 5.2 that:

Corollary 5.8. *Let $u \in \mathring{W}^{2,2}(\Omega)$ be a weak solution of problem* (1.4) *with $f \in L^2(\Omega)$. Then*

$$\text{for } \omega \in (0, \omega_2) : u \in W^{4,2}(\Omega), \quad \text{for } \omega \in (\omega_2, \pi) : u \in W^{3,2}(\Omega),$$
$$\text{for } \omega = \pi : u \in W^{4,2}(\Omega), \quad \text{for } \omega \in (\pi, 2\pi] : u \in W^{2,2}(\Omega).$$

Here ω_2 is as in Claim 5.7.

Remark 5.9. For the opening angle $\omega = \omega_2$ we have $\mathrm{Re}(\lambda_1) = 2$ and hence Theorem 5.2 does not apply. Nevertheless, assuming $f \in L^2(\Omega)$ to be more regular, e.g. in $V_0^{1,2}(\Omega)$ or $\mathring{W}^{1,2}(\Omega)$, we may show that

$$\text{for } \omega = \omega_2 : u \in W^{3,2}(\Omega).$$

Proof. By Theorem 5.2 if $f \in L^2(\Omega)$, the solution u of problem (1.4) reads as

$$u = w + \chi(r) \sum_{0<\lambda_j<2} \sum_{q=0}^{\kappa^{(j)}-1} c_q^{(j)} u_q^{(j)}, \tag{5.4}$$

with $w \in W^{4,2}(\Omega)$. Due to 5.7 we see that the sum in (5.4) has no terms when $\omega \in (0, \omega_2)$ and hence for $\omega \in (0, \omega_2)$ we have $u \in W^{4,2}(\Omega)$.

For $\omega \in (\omega_2, \pi) \cup (\pi, 2\pi]$ the first term of sum in (5.4), depending on the algebraic multiplicity of λ_1, reads as $u_0^{(1)} = r^{\lambda_1+1}\Phi_0^{(1)}(\theta)$, or as a linear combination of

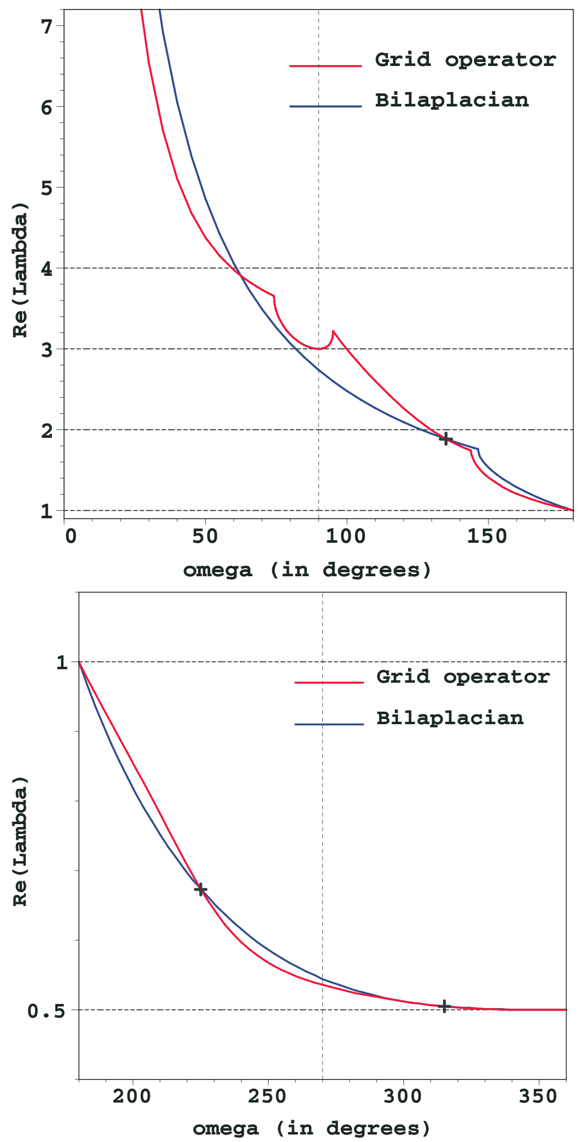

FIGURE 9. The plot of Figure 8 rescaled. The curves for the grid-operator and the bilaplacian intersect four times. Three of these points (we obviously exclude $\omega = \pi$) seem to be special: It looks like the curves intersect at $\omega = \frac{3}{4}\pi$, $\omega = \frac{5}{4}\pi$ and $\omega = \frac{7}{4}\pi$ (the intersection points are marked by cross). The numerical approximation shows however that the first values $\lambda_{1,Grid}$ and $\lambda_{1,Bilaplace}$ at those points only coincide up to three digits.

$u_0^{(1)} = r^{\lambda_1+1}\Phi_0^{(1)}(\theta)$ and $u_1^{(1)} = r^{\lambda_1+1}\left(\Phi_1^{(1)}(\theta) + \log(r)\Phi_0^{(1)}(\theta)\right)$. By (5.2) we have that $\left\{u_0^{(1)}, u_1^{(1)}\right\} \in W^{\lceil \mathrm{Re}(\lambda)\rceil+1,2}(\Omega)$, where due to Claim 5.7 we may deduce that

$\lceil \mathrm{Re}(\lambda_1)\rceil + 1 = 3$, when $\omega \in (\omega_2, \pi)$ and $\lceil \mathrm{Re}(\lambda_1)\rceil + 1 = 2$, when $\omega \in (\pi, 2\pi]$. This results in $u \in W^{3,2}(\Omega)$ for $\omega \in (\omega_2, \pi)$ and $u \in W^{2,2}(\Omega)$ for $\omega \in (\pi, 2\pi]$.

Finally, for $\omega = \pi$ due to (5.3) the singular part is of $C^\infty(\overline{\Omega})$ and hence $u \in W^{4,2}(\Omega)$ in this case. □

6. Comparing (weighted) Sobolev spaces

6.1. **One-dimensional Hardy-type inequalities.**

Lemma 6.1 (A higher order one-dimensional Hardy inequality)**.** *Let w be a function in $C_0^\infty[x_1, x_2]$. For every $k \geq 1$ it holds that*

$$\int_{x_1}^{x_2} \Big(\tfrac{w(x)}{(x-x_1)^k}\Big)^2 dx = \tfrac{4^k}{(2k-1)^2(2k-3)^2 \dots 3^2 1^2} \int_{x_1}^{x_2} \Big(w^{(k)}(x)\Big)^2 dx. \tag{6.1}$$

Proof. It holds straightforwardly that

$$\begin{aligned}
&\int_{x_1}^{x_2} \Big(\tfrac{w(x)}{(x-x_1)^k}\Big)^2 dx \\
&= \frac{1}{1-2k} \Big[(w(x))^2 (x - x_1)^{1-2k}\Big] |_{x_1}^{x_2} + \frac{2}{2k-1} \int_{x_1}^{x_2} w(x)w'(x)(x - x_1)^{1-2k} dx \\
&\leq \frac{2}{2k-1} \Big(\int_{x_1}^{x_2} \Big(\tfrac{w(x)}{(x-x_1)^k}\Big)^2 dx\Big)^{1/2} \Big(\int_{x_1}^{x_2} \Big(\tfrac{w'(x)}{(x-x_1)^{k-1}}\Big)^2 dx\Big)^{1/2}
\end{aligned}$$

and the first step in the proof of (6.1) follows. Repeating the argument for w' and $k - 1$ etc. will give the result. □

Remark 6.2. Since $\mathring{W}^{k,2}(x_1, x_2)$, $k \geq 1$ is the closure of $C_0^\infty[x_1, x_2]$ in the $W^{k,2}$-norm, one can use the results of Lemma 6.1 for every $w \in \mathring{W}^{k,2}(x_1, x_2)$, $k \geq 1$.

6.2. **Imbeddings.** As mentioned e.g. in [12, page 240] or [13, Chapter 7, summary], the family of weighted spaces $V_\beta^{l,2}$ does not contain the ordinary Sobolev spaces without weight. More precisely: $W^{k,2} \notin \Big\{V_\beta^{l,2}\Big\}_{l,\beta}$ for $k \geq 1$. We will prove the imbedding results for bounded Ω that satisfy Condition 1.1.

Lemma 6.3. *Let $\beta \in \mathbb{R}$ and $l \in \{0, 1, 2, \dots\}$. Then the following holds:*

(a) $V_\beta^{l,2}(\Omega) \subset W^{l,2}(\Omega)$ *if and only if* $\beta \leq 0$,
(b) $W^{l,2}(\Omega) \subset V_\beta^{l,2}(\Omega)$ *if and only if* $\beta \geq l$,
(c) $\mathring{V}_\beta^{l,2}(\Omega) \subset \mathring{W}^{l,2}(\Omega)$ *if and only if* $\beta \leq 0$,
(d) $\mathring{W}^{l,2}(\Omega) \subset \mathring{V}_\beta^{l,2}(\Omega)$ *if and only if* $\beta \geq 0$.

Corollary 6.4. *For $l \in \{0, 1, 2, \dots\}$ one has*

$$\mathring{W}^{l,2}(\Omega) = \mathring{V}_0^{l,2}(\Omega).$$

Proof of Lemma 6.3. Let Ω be as in Condition 1.1 and $\Omega \subset B_M(0)$, where $B_M(0)$ is an open ball of radius $M > 0$. The statement in a) goes as follows: for $(x, y) \in \Omega$ one has $0 \leq r \leq M$ and hence $r^{2(\beta-l+|\alpha|)} \geq M^{2(\beta-l+|\alpha|)}$ if and only if $\beta-l+|\alpha| \leq 0$. Since $0 \leq |\alpha| \leq l$, we obtain $\beta \leq 0$. This enables us to have the estimate

$$\|u\|_{V_\beta^{l,2}(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega r^{2(\beta-l+|\alpha|)} |D^\alpha u|^2 dx\, dy\Big)^{1/2}$$

$$
\begin{aligned}
&\geq \Big(\sum_{|\alpha|=0}^{l} \int_\Omega M^{2(\beta-l+|\alpha|)} |D^\alpha u|^2 dx\, dy \Big)^{1/2} \\
&\geq \min(1, M^{\beta-l}) \Big(\sum_{|\alpha|=0}^{l} \int_\Omega |D^\alpha u|^2 dx\, dy \Big)^{1/2} \\
&= \min(1, M^{\beta-l}) \|u\|_{W^{l,2}(\Omega)},
\end{aligned}
$$

which is the result in (a).

To prove the statement in (b) we notice that $r^{2(\beta-l+|\alpha|)} \leq M^{2(\beta-l+|\alpha|)}$ if and only if $\beta - l + |\alpha| \geq 0$. Due to $0 \leq |\alpha| \leq l$, we obtain $\beta \geq l$ and then the estimate holds

$$
\begin{aligned}
\|u\|_{V_\beta^{l,2}(\Omega)} &= \Big(\sum_{|\alpha|=0}^{l} \int_\Omega r^{2(\beta-l+|\alpha|)} |D^\alpha u|^2 dx\, dy \Big)^{1/2} \\
&\leq \Big(\sum_{|\alpha|=0}^{l} \int_\Omega M^{2(\beta-l+|\alpha|)} |D^\alpha u|^2 dx\, dy \Big)^{1/2} \\
&\leq \max(1, M^{\beta-l}) \Big(\sum_{|\alpha|=0}^{l} \int_\Omega |D^\alpha u|^2 dx\, dy \Big)^{1/2} \\
&= \max(1, M^{\beta-l}) \|u\|_{W^{l,2}(\Omega)}.
\end{aligned}
$$

This is the result in (b).

To prove the statements in (c) and (d) we set $\theta = \frac{1}{2}\omega$ where the opening angle $\omega \in (0, 2\pi)$. We also use the fact that for our domain there exists $c > 0$ such that $r > c\rho(x, y)$, where ρ denotes the distance from a point (x, y) on the lines

$$
\ell : y = \tan(\theta)x + \tau,
$$

with $\tau \in \mathbb{R}$ to the point $(x_1, y_1) \in \partial\Omega$. In particular, it holds that

$$
\rho^2 = (x - x_1)^2 (1 + \tan^2(\theta)).
$$

We may integrate along the lines ℓ and use the one-dimensional Hardy-inequality to find that there exist $\tilde{C}_l \in \mathbb{R}^+$ with

$$
\|u\|_{V_0^{l,2}(\Omega)} \leq \tilde{C}_l \|u\|_{W^{l,2}(\Omega)} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{6.2}
$$

On the other hand, using the same trick as in proof of a) we find $C_l \in \mathbb{R}^+$ such that

$$
C_l \|u\|_{W^{l,2}(\Omega)} \leq \|u\|_{V_0^{l,2}(\Omega)} \quad \text{for all } u \in C_c^\infty(\Omega). \tag{6.3}
$$

Estimates (6.2), (6.3) yield

$$
\mathring{W}^{l,2}(\Omega) = \mathring{V}_0^{l,2}(\Omega).
$$

Due to imbedding $\mathring{V}_{\beta_1}^{l,2}(\Omega) \subset \mathring{V}_0^{l,2}(\Omega) \subset \mathring{V}_{\beta_2}^{l,2}(\Omega)$ when $\beta_1 \leq 0 \leq \beta_2$ one obtains the result in (c) and (d). $\square$

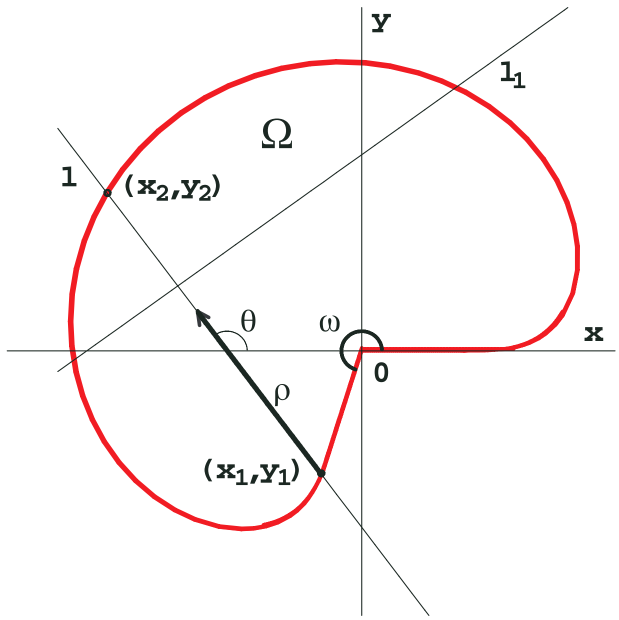

FIGURE 10. Domain Ω with a concave corner ω intersected by ℓ

7. A fundamental system of solutions

7.1. Derivation of system S_λ. Let us find the fundamental set of solutions to equation

$$\mathcal{L}(\theta, \tfrac{d}{d\theta}, \lambda)\Phi = 0, \tag{7.1}$$

with $\mathcal{L}(\theta, \frac{d}{d\theta}, \lambda)$ as in formula (4.3). For this $\mathcal{L}$ it seems to be hard to derive a set of functions solving (7.1) explicitly. The following approach applies in this case.

For $L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ we find $L\left(r^{\lambda+1}\Phi\right) = r^{\lambda-3}\mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right)\Phi$ and hence instead of $\mathcal{L}\left(\theta, \frac{d}{d\theta}, \lambda\right)\Phi = 0$ we may consider the equation

$$L\left(r^{\lambda+1}\Phi\right) = 0. \tag{7.2}$$

Operator L admits the decomposition

$$L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} = \prod_{p=1}^{2}\left(\frac{\partial}{\partial y} - \tau_p \frac{\partial}{\partial x}\right)\prod_{p=1}^{2}\left(\frac{\partial}{\partial y} + \tau_p \frac{\partial}{\partial x}\right),$$

with $\tau_1 = \frac{\sqrt{2}}{2}(1+i)$, $\tau_2 = \frac{\sqrt{2}}{2}(1-i)$ and hence every function of the form $F(x \pm \tau_p y)$ solves (7.2). Therefore, we have that

$$r^{\lambda+1}\Phi(\theta) = \sum_{p=1}^{2} c_p f_p(x + \tau_p y) + c_{p+2} f_{p+2}(x - \tau_p y),$$

and after translation $\{f_p, f_{p+2}\}_{p=1}^{2}$ into polar coordinates we set

$$f_p\left(r\cos(\theta) + \tau_p r\sin(\theta)\right) := \left(r\cos(\theta) + \tau_p r\sin(\theta)\right)^{\lambda+1},$$

$$f_{p+2}\left(r\cos(\theta)-\tau_p r\sin(\theta)\right):=\left(r\cos(\theta)-\tau_p r\sin(\theta)\right)^{\lambda+1},$$

So, the set of functions

$$\varphi_1(\theta)=\left(\cos(\theta)+\tau_1\sin(\theta)\right)^{\lambda+1},\quad \varphi_2(\theta)=\left(\cos(\theta)+\tau_2\sin(\theta)\right)^{\lambda+1}, \tag{7.3}$$

$$\varphi_3(\theta)=\left(\cos(\theta)-\tau_1\sin(\theta)\right)^{\lambda+1},\quad \varphi_4(\theta)=\left(\cos(\theta)-\tau_2\sin(\theta)\right)^{\lambda+1}, \tag{7.4}$$

is a set of solutions to (7.1). The Wronskian for $\{\varphi_m\}_{m=1}^4$ reads as

$$W\left(\varphi_1(\theta),\varphi_2(\theta),\varphi_3(\theta),\varphi_4(\theta)\right)=\det\begin{pmatrix}\varphi_1(\theta) & \varphi_2(\theta) & \varphi_3(\theta) & \varphi_4(\theta)\\ \varphi_1'(\theta) & \varphi_2'(\theta) & \varphi_3'(\theta) & \varphi_4'(\theta)\\ \varphi_1''(\theta) & \varphi_2''(\theta) & \varphi_3''(\theta) & \varphi_4''(\theta)\\ \varphi_1'''(\theta) & \varphi_2'''(\theta) & \varphi_3'''(\theta) & \varphi_4'''(\theta)\end{pmatrix}, \tag{7.5}$$

and by straightforward computations one finds

$$W=16\left(\lambda+1\right)^3\lambda^2\left(\lambda-1\right)\left(\cos^4(\theta)+\sin^4(\theta)\right)^{\lambda-2},$$

which is non-zero on $\theta\in(0,2\pi]$ except for $\lambda\in\{\pm1,0\}$. Hence, except for these values $\{\varphi_m\}_{m=1}^4$ given in (7.3), (7.4) is a fundamental system of solutions to (7.1).

7.2. **Derivation of systems S_{-1}, S_0, S_1.** Here we find the fundamental systems of solutions to equation $\mathcal{L}\left(\theta,\frac{\partial}{\partial\theta},\lambda\right)\Phi=0$ when $\lambda\in\{\pm1,0\}$. We will go into details in solving the corresponding equation for every $\lambda\in\{\pm1,0\}$.

7.2.1. *Case* $\lambda=-1$. For $\lambda=-1$ the equation (7.1) reads as

$$\tfrac14\left(3+\cos(4\theta)\right)\Phi''''-3\sin(4\theta)\Phi'''+\left(3-11\cos(4\theta)\right)\Phi''+12\sin(4\theta)\Phi'=0. \tag{7.6}$$

First we set $\Phi(\theta)=\int F(\theta)d\theta$ and obtain the equation for F:

$$\tfrac14\left(3+\cos(4\theta)\right)F'''-3\sin(4\theta)F''+\left(3-11\cos(4\theta)\right)F'+12\sin(4\theta)F=0.$$

The first integral of the above equation reads as

$$\tfrac14\left(3+\cos(4\theta)\right)F''-2\sin(4\theta)F'+3(1-\cos(4\theta))F=c_0.$$

We use the change of variables $F(\theta)=\left(3+\cos(4\theta)\right)^{-1}G(\theta)$ and get

$$G''+4G=4c_0.$$

Solution of the last equation reads as

$$G(\theta)=c_1\sin(2\theta)+c_2\cos(2\theta)+c_0.$$

and then

$$F(\theta)=c_1\tfrac{\sin(2\theta)}{3+\cos(4\theta)}+c_2\tfrac{\cos(2\theta)}{3+\cos(4\theta)}+c_3\tfrac{1}{3+\cos(4\theta)}.$$

As a result, Φ that solves (7.6) will read as

$$\Phi(\theta)=A_1+A_2\int\tfrac{\sin(2\theta)}{3+\cos(4\theta)}d\theta+A_3\int\tfrac{\cos(2\theta)}{3+\cos(4\theta)}d\theta+A_4\int\tfrac{1}{3+\cos(4\theta)}d\theta,$$

and then the candidates that may form the fundamental system of solutions to (7.6) will be the following:

$$\varphi_1(\theta)=1,$$

$$\varphi_2(\theta)=-4\int\tfrac{\sin(2\theta)}{3+\cos(4\theta)}d\theta=\arctan\left(\cos(2\theta)\right),$$

$$\varphi_3(\theta)=4\sqrt2\int\tfrac{\cos(2\theta)}{3+\cos(4\theta)}d\theta=\operatorname{arctanh}\left(\tfrac{\sqrt2}{2}\sin(2\theta)\right),$$

$$\varphi_4(\theta) = 4\sqrt{2}\int \tfrac{1}{3+\cos(4\theta)} d\theta = \begin{cases} \arctan\Big(\tfrac{\sqrt{2}}{2}\tan(2\theta)\Big) + \ell\pi, & \theta \in \big(\tfrac{2\ell-1}{4}\pi, \tfrac{2\ell+1}{4}\pi\big), \\ 2\theta & \theta = \tfrac{2\ell+1}{4}\pi, \end{cases}$$

with $\ell = 0, \dots, 4$. Formula for φ_4 is given on the interval $\theta \in \big(-\frac{1}{4}\pi, 2\pi + \frac{1}{4}\pi\big)$ in order to have the concise explicit form.

The Wronskian W of $\varphi_1, \dots, \varphi_4$ is proportional to $\big(\cos^4(\theta) + \sin^4(\theta)\big)^{-3}$ and is non-zero on $\theta \in (0, 2\pi]$. Hence, $\{\varphi_m\}_{m=1}^4$ defined as above is a fundamental system of solutions to (7.6).

7.2.2. *Case* $\lambda = 0$. For $\lambda = 0$ the equation (7.1) reads as:

$$\begin{aligned} &\tfrac{1}{4}\left(3 + \cos(4\theta)\right)\Phi'''' - 2\sin(4\theta)\Phi''' + \tfrac{1}{2}\left(3 - 7\cos(4\theta)\right)\Phi'' \\ &- 2\sin(4\theta)\Phi' + \tfrac{3}{4}\left(1 - 5\cos(4\theta)\right)\Phi = 0, \end{aligned} \tag{7.7}$$

and can be split as follows:

$$\Big(\tfrac{d^2}{d\theta^2} + 1\Big)\Big(\tfrac{1}{4}\left(3 + \cos(4\theta)\right)\Big(\tfrac{d^2}{d\theta^2} + 1\Big)\Big)\Phi = 0.$$

So, Φ solves

$$\Phi'' + \Phi = A\tfrac{\sin(\theta)}{3+\cos(4\theta)} + B\tfrac{\cos(\theta)}{3+\cos(4\theta)},$$

and after integrating this equation we obtain

$$\begin{aligned} &\Phi(\theta) \\ &= A_1\sin(\theta) + A_2\cos(\theta) + A_3\Big(\tfrac{1}{2}\sin(\theta)\arctan\left(\cos(2\theta)\right) + 4\cos(\theta)\int_0^\theta \tfrac{\sin^2(y)}{3+\cos(4y)}dy\Big) \\ &\quad + A_4\Big(\tfrac{1}{2}\cos(\theta)\arctan\left(\cos(2\theta)\right) + 4\sin(\theta)\int_0^\theta \tfrac{\cos^2(y)}{3+\cos(4y)}dy\Big). \end{aligned}$$

The candidates that may form the fundamental system of solutions to (7.7) will be the following:

$$\begin{gathered} \varphi_1(\theta) = \sin(\theta), \quad \varphi_2(\theta) = \cos(\theta), \\ \varphi_3(\theta) = \tfrac{1}{2}\sin(\theta)\arctan\left(\cos(2\theta)\right) + 4\cos(\theta)\int_0^\theta \tfrac{\sin^2(y)}{3+\cos(4y)}dy, \\ \varphi_4(\theta) = \tfrac{1}{2}\cos(\theta)\arctan\left(\cos(2\theta)\right) + 4\sin(\theta)\int_0^\theta \tfrac{\cos^2(y)}{3+\cos(4y)}dy. \end{gathered}$$

The Wronskian W of $\varphi_1, \dots, \varphi_4$ is proportional to $\big(\cos^4(\theta) + \sin^4(\theta)\big)^{-2}$ and is non-zero on $\theta \in (0, 2\pi]$. Hence, $\{\varphi_m\}_{m=1}^4$ defined as above is a fundamental system of solutions to (7.7).

7.2.3. *Case* $\lambda = 1$. For $\lambda = 1$ the equation (7.1) reads as:

$$\tfrac{1}{4}\left(3 + \cos(4\theta)\right)\Phi'''' - \sin(4\theta)\Phi''' + \left(3 + \cos(4\theta)\right)\Phi'' - 4\sin(4\theta)\Phi' = 0. \tag{7.8}$$

We set $\Phi(\theta) = \int F(\theta)d\theta$ and obtain the equation for F:

$$\tfrac{1}{4}\left(3 + \cos(4\theta)\right)F''' - \sin(4\theta)F'' + \left(3 + \cos(4\theta)\right)F' - 4\sin(4\theta)F = 0.$$

It holds that

$$\begin{gathered} \left(3 + \cos(4\theta)\right)F''' - 4\sin(4\theta)F'' = -2g(\theta), \\ \left(3 + \cos(4\theta)\right)F' - 4\sin(4\theta)F = \frac{1}{2}g(\theta). \end{gathered}$$

and we obtain, respectively,

$$F''(\theta) = -4\tfrac{\int g(\theta)d\theta + C_1}{3+\cos(4\theta)}, \quad F(\theta) = \tfrac{\int g(\theta)d\theta + C_2}{3+\cos(4\theta)}.$$

Comparing the expressions for $F''(\theta)$ and $F(\theta)$ we deduce that F solves

$$F'' + 4F = \frac{c_0}{3+\cos(4\theta)}. \tag{7.9}$$

The solution of (7.9) reads as

$$\begin{aligned} F(\theta) &= c_1 \sin(2\theta) + c_2\cos(2\theta) + c_0\Big(\tfrac{1}{4}\cos(2\theta)\arctan\left(\cos(2\theta)\right) \\ &\quad + \tfrac{\sqrt{2}}{8}\sin(2\theta)\text{arctanh}\big(\tfrac{\sqrt{2}}{2}\sin(2\theta)\big)\Big), \end{aligned}$$

which being integrated yields

$$\begin{aligned} \Phi(\theta) &= A_1 + A_2\cos(2\theta) + A_3\sin(2\theta) \\ &\quad + A_4\int_0^\theta \Big(\cos(2y)\arctan\left(\cos(2y)\right) + \tfrac{\sqrt{2}}{2}\sin(2y)\text{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2y)\right)\Big)\,dy \end{aligned} \tag{7.10}$$

The candidates that may form the fundamental system of solutions to (7.8) will be the following:

$$\varphi_1(\theta) = 1, \quad \varphi_2(\theta) = \sin(2\theta), \quad \varphi_3(\theta) = \cos(2\theta),$$
$$\varphi_4(\theta) = \int_0^\theta \Big(\cos(2y)\arctan\left(\cos(2y)\right) + \tfrac{\sqrt{2}}{2}\sin(2y)\text{arctanh}\left(\tfrac{\sqrt{2}}{2}\sin(2y)\right)\Big)\,dy.$$

The Wronskian W of $\varphi_1,\dots,\varphi_4$ is proportional to $\left(\cos^4(\theta)+\sin^4(\theta)\right)^{-1}$ and is nonzero on $\theta\in(0,2\pi]$. Hence, $\{\varphi_m\}_{m=1}^4$ defined as above is a fundamental system of solutions to (7.7).

7.3. **The explicit formulas for P_{-1}, P_0, P_1,** For $\lambda\in\{\pm1,0\}$ we obtain:

$$\begin{gathered} P_{-1}(\omega) = -\tfrac{32}{\pi}\sin(2\omega)\int_0^\omega \tfrac{\sin^2(\theta)}{3+\cos(4\theta)}d\theta + (1-\cos(2\omega))\left(1-\tfrac{4}{\pi}\arctan\left(\cos(2\omega)\right)\right), \\ P_0(\omega) = \pi\arctan\left(\cos(2\omega)\right) - 2\arctan^2\left(\cos(2\omega)\right) \\ + 128\int_0^\omega \tfrac{\sin^2(\theta)}{3+\cos(4\theta)}d\theta\int_0^\omega \tfrac{\cos^2(\theta)}{3+\cos(4\theta)}d\theta, \\ P_1(\omega) = -\sin(2\omega)\left(\sin(2\omega)-\tfrac{8}{\pi}\varphi_4(\omega)\right) + (1-\cos(2\omega))\left(\cos(2\omega)-\tfrac{4}{\pi}\varphi_4'(\omega)\right), \end{gathered}$$

where $\varphi_4(\theta)$ is given by corresponding formula from the case $\lambda = 1$.

8. Analytical tools for the numerical computation

8.1. **Implicit function and discretization.** Consider a rectangle $U = [a,b]\times[c,d]$. For $n,m\in\mathbb{N}^+$ and $i=0,\dots,n$, $j=0,\dots,m$ we set

$$x_i = a + i\Delta x, \quad y_j = c + j\Delta y,$$

where $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{m}$.

Let $F\in C^1(U,\mathbb{R})$ such that $F(x_i,y_j)>0$ for all $i=0,\dots,n$ and $j=0,\dots,m$. The question to resolve is how fine should we take the discretization of U in order to be sure that $F>0$ on U. The following result holds.

Lemma 8.1. *Suppose*$\min_{(x_i,y_j)\in U} F(x_i,y_j) > 0$. *If*

$$\max\{\Delta x, \Delta y\} \le \sqrt{2}\frac{\min_{(x_i,y_j)\in U} F(x_i,y_j)}{\sup_U |DF(x,y)|}, \tag{8.1}$$

then F is strictly positive on U.

Proof. For every $(x,y)\in U$ there is (x_i,y_j) with $|x-x_i|\le \frac{1}{2}\Delta x$ and $|y-y_j|\le \frac{1}{2}\Delta y$. By the mean value theorem there exists $(\xi_1,\xi_2)\in[(x,y),(x_i,y_j)]$ such that

$$F(x,y) = F(x_i,y_j) + DF(\xi_1,\xi_2)\cdot(x-x_i, y-y_j).$$

The following chain of estimates then holds

$$\begin{aligned} F(x,y) &= F(x_i,y_j) + DF(\xi_1,\xi_2)\cdot(x-x_i,y-y_j) \\ &\ge \min_{(x_i,y_j)\in U} F(x_i,y_j) - \sup_U |DF(x,y)||(x-x_i,y-y_j)| \\ &\ge \min_{(x_i,y_j)\in U} F(x_i,y_j) - \frac{\sqrt{2}}{2}\sup_U |DF(x,y)| \max\{\Delta x,\Delta y\}. \end{aligned}$$

This last expression is positive if (8.1) holds. □

8.2. **A version of the Morse theorem.** Let $V\subset\mathbb{R}^2$ be open and bounded, $F\in C^\infty(V,\mathbb{R})$. For the gradient of F we use DF and D^2F is the Hessian matrix.

Definition 8.2. A point $a\in V$ is said to be a critical point of F if $DF(a)=0$. Moreover, the critical point $a\in V$ is said to be non-degenerate if $\det D^2F(a)\neq 0$.

To study the level set Γ defined in subsection 4.4.1 we need the Morse theorem. The original version of the theorem reads as (see [21]):

> Let V be a Banach space, O a convex neighborhood of the origin in V and $f: O\to\mathbb{R}$ a C^{k+2} function ($k\ge 1$) having the origin as a non-degenerate critical point, with $f(0)=0$. Then there is a neighborhood U of the origin and a C^k diffeomorphism $\phi: U\to O$ with $\phi(0)=0$ and $D\phi(0)=I$, the identity map of V, such that for $x\in U$, $f(\phi(x)) = \frac{1}{2}(D^2f(0)x,x)$.

Below we give our formulation of the theorem. This formulation is more convenient for our purposes. We will give a constructive proof that allows us to find an explicit neighborhood of a critical point where the diffeomorphism exists.

Theorem 8.3. *Let $V\subset\mathbb{R}^2$ be open and bounded, $F\in C^\infty(V,\mathbb{R})$. Suppose $a=(a_1,a_2)\in V$ is a non-degenerate critical point of F. There exists a neighborhood $W_a\subset V$ of a and a C^∞-diffeomorphism $h: W_a\to U_0$, where $U_0\subset\mathbb{R}^2$ is a neighborhood of 0, such that F in W_a is representable as:*

$$F(x) = F(a) + h(x)\left(\tfrac{1}{2}D^2F(a)\right)h(x)^T, \tag{8.2}$$

where T stands for a transposition. Moreover, the neighborhood W_a is fixed by

$$W_a\subset\left\{x\in V: \det B(x)\ge 0 \quad \text{and} \quad b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0\right\}.$$

where

$$B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x)\end{pmatrix} = \left(\tfrac{1}{2}D^2F(a)\right)^{-1}\int_0^1\int_0^1 sD^2F\left(a+ts\left(x-a\right)\right)dt\,ds.$$

Proof. Let $a \in V$ be such that $DF(a) = 0$ and $\det D^2F(a) \neq 0$. First, for every $x \in V$ we have

$$F(x) - F(a) = F\left(a + s(x-a)\right)|_0^1 = \int_0^1 \tfrac{d}{ds} F\left(a + s(x-a)\right) ds,$$

and due to $\frac{d}{ds}F\left(a + s(x-a)\right) = DF\left(a + s(x-a)\right)(x-a)^T$ it will follow that

$$F(x) = F(a) + \int_0^1 DF\left(a + s(x-a)\right) ds (x-a)^T.$$

Analogously, we obtain

$$DF(x) = DF(a) + \int_0^1 D^2F\left(a + t(x-a)\right) dt (x-a)^T,$$

where by assumption $DF(a) = 0$. As a result, for every $x \in V$, F is representable in terms of D^2F as:

$$F(x) = F(a) + (x-a) \int_0^1 \int_0^1 sD^2F\left(a + ts(x-a)\right) dt\, ds (x-a)^T,$$

or shortly

$$F(x) = F(a) + (x-a)K(x)(x-a)^T. \tag{8.3}$$

Here $K(x) = \int_0^1 \int_0^1 sD^2F\left(a + ts(x-a)\right) dt\, ds$ is a symmetric matrix. With this definition, $K(a) = \frac{1}{2}D^2F(a)$ is symmetric and invertible ($\det D^2F(a) \neq 0$ by assumption).

Let us bring some intermediate results.

(1) For every $x \in V$, there exists matrix B such that

$$K(x) = K(a)B(x). \tag{8.4}$$

Indeed, since $K(a)$ is invertible, the matrix $B(x) = K(a)^{-1}K(x)$ is well-defined. We write

$$B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix},$$

Since $F \in C^\infty(V, \mathbb{R})$, so are b_{ij}, $i, j = 1, 2$. Note that $B(a) = I$.

(2) Since $x \mapsto B(x)$ is C^∞ in a neighborhood of a and $B(a) = I$, $B(x)$ is positive definite in a neighborhood of a, and hence allows a square root. In particular, it holds that

$$C(x) = \sqrt{B(x)} := \tfrac{1}{2\pi i} \oint_\gamma \sqrt{z}\left(Iz - B(x)\right)^{-1} dz, \tag{8.5}$$

where γ is a Jordan curve in $\mathbb{C}$ which goes around the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of $B(x)$ and does not intersect $\mathrm{Re}(z) \leq 0$, $\mathrm{Im}(z) = 0$.

One may check that C, defined as follows

$$C(x) = \begin{pmatrix} \frac{b_{11}(x) + \sqrt{\det B(x)}}{\sqrt{b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)}} & \frac{b_{12}(x)}{\sqrt{b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)}} \\ \frac{b_{21}(x)}{\sqrt{b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)}} & \frac{b_{22}(x) + \sqrt{\det B(x)}}{\sqrt{b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)}} \end{pmatrix},$$

is indeed such that

$$C(x)^2 = B(x). \tag{8.6}$$

With this definition, $C(a) = I$ and $C(a)^2 = I = B(a)$ as required. Also, matrix C is well defined when

$$\det B(x) \geq 0, \quad b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0. \tag{8.7}$$

(3) For those x one finds

$$\begin{aligned} K(a)B(x) &= K(x) = K(x)K(a)^{-1}K(a) \\ &= K(x)^T \left(K(a)^{-1}\right)^T K(a) \\ &= \left(K(a)^{-1}K(x)\right)^T K(a) = B^T(x)K(a). \end{aligned}$$

Due to this equality, we deduce the following

$$(Iz - B(x))\, K(a)^{-1} = K(a)^{-1}\, (Iz - B(x))^T\,,$$

and hence

$$\begin{aligned} K(a)\,(Iz - B(x))^{-1} &= \left((Iz - B(x))\, K(a)^{-1}\right)^{-1} = \left(K(a)^{-1}\,(Iz - B(x))^T\right)^{-1} \\ &= \left((Iz - B(x))^T\right)^{-1} K(a) = \left((Iz - B(x))^{-1}\right)^T K(a). \end{aligned}$$

Applying the integration (8.5) to the last identity we find

$$K(a)C(x) = C(x)^T K(a). \tag{8.8}$$

Combining (8.4), (8.6) and (8.8) we have

$$K(x) = K(a)B(x) = K(a)C(x)^2 = C(x)^T K(a)C(x),$$

and therefore (8.3) for those x results in

$$F(x) = F(a) + F(a) + h(x)K(a)h(x)^T, \tag{8.9}$$

where $K(a) = \frac{1}{2}D^2F(a)$ and

$$h(x)^T = C(x)(x - a)^T.$$

Note that by (8.7) the representation for F in (8.9) holds on a set $W_a \subset V$ which is star-shaped with respect to a and such that

$$W_a \subset \left\{x \in V : \det B(x) \geq 0 \text{ and } b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0\right\}. \tag{8.10}$$

□

Remark 8.4. For each pair (F, a) one can obtain an explicit estimate for W_a in (8.10). We will do this in the next subsection for the pair we are interested in.

8.3. **The Morse Theorem applied.** Let P be the function given by formula (4.10) and which is defined on

$$V = \left\{(\omega, \lambda) : \left[\tfrac{70}{180}\pi, \tfrac{110}{180}\pi\right] \times [2.900, 5.100]\right\}.$$

Let us recall it here:

$$\begin{aligned} P(\omega, \lambda) = &\left(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda + \left(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\right)^\lambda \\ &+ \left(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\right)^{\frac{1}{2}\lambda} \Big[2\cos\Big(\lambda\Big(\arctan\Big(\tfrac{\sqrt{2}}{2}\tan(2\omega)\Big) + \pi\Big)\Big) \\ &- 4\cos\left(\lambda \arctan\left(\tan^2(\omega)\right)\right)\Big]. \end{aligned} \tag{8.11}$$

The point $a=(\frac{1}{2}\pi,4)$ is such that $P(a)=0$ and $DP(a)=0$. Theorem 8.3 gives us the tool to study P in the vicinity of a. In particular, the following holds.

Proposition 8.5. *Let a be as above. There is a closed ball $W_R(a)\subset V$ of a radius R centered at a such that on $W_R(a)$ we have:*

$$P(\omega,\lambda)=-\tfrac{1}{2}h_2(\omega,\lambda)\left(16\sqrt{2}h_1(\omega,\lambda)+\pi h_2(\omega,\lambda)\right). \tag{8.12}$$

Here $h_1,h_2\in C^\infty\left(W_R(a),\mathbb{R}\right)$ are given by:

$$h_1(\omega,\lambda)=\left(\omega-\tfrac{1}{2}\pi\right)c_{11}(\omega,\lambda)+(\lambda-4)\,c_{12}(\omega,\lambda), \tag{8.13}$$

$$h_2(\omega,\lambda)=\left(\omega-\tfrac{1}{2}\pi\right)c_{21}(\omega,\lambda)+(\lambda-4)\,c_{22}(\omega,\lambda), \tag{8.14}$$

with $c_{ij}\in C^\infty\left(W_R(a),\mathbb{R}\right)$, $i,j=1,2$ are the entries of matrix C:

$$C(\omega,\lambda)=\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}(\omega,\lambda)=\begin{pmatrix}\frac{b_{11}+\sqrt{\det B}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}}&\frac{b_{12}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}}\\\frac{b_{21}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}}&\frac{b_{22}+\sqrt{\det B}}{\sqrt{b_{11}+2\sqrt{\det B}+b_{22}}}\end{pmatrix}(\omega,\lambda), \tag{8.15}$$

while $b_{ij}\in C^\infty(V,\mathbb{R})$, $i,j=1,2$ are as follows

$$\begin{aligned}B(\omega,\lambda)&=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}(\omega,\lambda)\\&=\left(\tfrac{1}{2}D^2P(a)\right)^{-1}\int_0^1\int_0^1 sD^2P\left(a+ts\left((\omega,\lambda)-a\right)\right)dt\,ds.\end{aligned} \tag{8.16}$$

Note that $B(a)=I$ and $C(a)=I$. The ball $W_R(a)$ is fixed by

$$W_R(a):=\left\{(\omega,\lambda)\in V:\left|\left(\omega-\tfrac{1}{2}\pi,\lambda-4\right)\right|\le R\right\}, \tag{8.17}$$

with $R=-\frac{1}{120}\pi^2-\frac{\sqrt{2}}{18}\pi+\frac{1}{120}\pi\sqrt{\pi^2+\frac{40}{3}\sqrt{2}\pi+\frac{1568}{9}}\approx 0.078\dots$ (In ω-direction we have that $R\approx 4.5\dots^\circ$).

8.3.1. *Computational results I.* It is straightforward for $a=\left(\frac{1}{2}\pi,4\right)$ that

$$\tfrac{\partial^2P}{\partial\omega^2}(a)=0,\quad \tfrac{\partial^2P}{\partial\omega\partial\lambda}(a)=-8\sqrt{2}\pi,\quad \tfrac{\partial^2P}{\partial\lambda^2}(a)=-\pi^2, \tag{8.18}$$

and hence

$$\det D^2P(a)=-128\pi^2. \tag{8.19}$$

To simplify the notation, we use x instead of (ω,λ) when (ω,λ) stands for a argument. Now let us bring two alternatives representations for the entries of matrix B given by (8.16), which we will use later on.

Representation I. We will need the explicit formula for the coefficients b_{ij}, $i,j=1,2$. Let us find them in a straightforward way from (8.16). We write down the integral term $\int_0^1\int_0^1 sD^2P\left(a+ts\left(x-a\right)\right)dt\,ds$ in (8.16) as follows

$$\int_0^1\int_0^1 sD^2P\left(a+ts\left(x-a\right)\right)dt\,ds=\begin{pmatrix}r_1(x)&r_2(x)\\r_2(x)&r_3(x)\end{pmatrix}, \tag{8.20}$$

where r_j, $j=1,2,3$ read as:

$$r_1(x)=\int_0^1\int_0^1 s\tfrac{\partial^2P}{\partial\omega^2}(a+ts(x-a))dt\,ds, \tag{8.21}$$

$$r_2(x)=\int_0^1\int_0^1 s\tfrac{\partial^2P}{\partial\omega\partial\lambda}(a+ts(x-a))dt\,ds, \tag{8.22}$$

$$r_3(x) = \int_0^1 \int_0^1 s \frac{\partial^2 P}{\partial \lambda^2}(a + ts(x-a))dt\, ds. \tag{8.23}$$

Then the entries b_{ij}, $i, j = 1, 2$ of matrix B in terms of r_j, $j = 1, 2, 3$ and due to (8.18), (8.19) will read:

$$b_{11}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \lambda^2}(a) r_1(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_2(x) \Big) = \frac{1}{64} r_1(x) - \frac{\sqrt{2}}{8\pi} r_2(x), \tag{8.24}$$

$$b_{12}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \lambda^2}(a) r_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_3(x) \Big) = \frac{1}{64} r_2(x) - \frac{\sqrt{2}}{8\pi} r_3(x), \tag{8.25}$$

$$b_{21}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \omega^2}(a) r_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_1(x) \Big) = -\frac{\sqrt{2}}{8\pi} r_1(x), \tag{8.26}$$

$$b_{22}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \omega^2}(a) r_3(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) r_2(x) \Big) = -\frac{\sqrt{2}}{8\pi} r_2(x). \tag{8.27}$$

Representation II. On the other hand, let us obtain for r_j, $j = 1, 2, 3$ the following representations:

$$r_1(x) = \frac{1}{2}\frac{\partial^2 P}{\partial \omega^2}(a) + q_1(x), \quad r_2(x) = \frac{1}{2}\frac{\partial^2 P}{\partial \omega \partial \lambda}(a) + q_2(x), \quad r_3(x) = \frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}(a) + q_3(x),$$

where

$$\begin{aligned} q_1(x) = \int_0^1 \int_0^1 \int_0^1 & ts^2 \Big(\frac{\partial^3 P}{\partial \omega^3}(a + ts\sigma(x-a)), \frac{\partial^3 P}{\partial \omega^2 \partial \lambda}(a + ts\sigma(x-a)) \Big) \\ & \times (x-a)^T d\sigma dt\, ds, \end{aligned} \tag{8.28}$$

$$\begin{aligned} q_2(x) = \int_0^1 \int_0^1 \int_0^1 & ts^2 \Big(\frac{\partial^3 P}{\partial \omega^2 \partial \lambda}(a + ts\sigma(x-a)), \frac{\partial^3 P}{\partial \omega \partial \lambda^2}(a + ts\sigma(x-a)) \Big) \\ & \times (x-a)^T d\sigma dt\, ds, \end{aligned} \tag{8.29}$$

$$\begin{aligned} q_3(x) = \int_0^1 \int_0^1 \int_0^1 & ts^2 \Big(\frac{\partial^3 P}{\partial \omega \partial \lambda^2}(a + ts\sigma(x-a)), \frac{\partial^3 P}{\partial \lambda^3}(a + ts\sigma(x-a)) \Big) \\ & \times (x-a)^T d\sigma dt\, ds. \end{aligned} \tag{8.30}$$

This will yield another representation formulas for b_{ij}, $i, j = 1, 2$ of matrix B, namely,

$$b_{11}(x) = 1 + \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \lambda^2}(a) q_1(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_2(x) \Big) = 1 + \frac{1}{64} q_1(x) - \frac{\sqrt{2}}{8\pi} q_2(x), \tag{8.31}$$

$$b_{12}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \lambda^2}(a) q_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_3(x) \Big) = \frac{1}{64} q_2(x) - \frac{\sqrt{2}}{8\pi} q_3(x), \tag{8.32}$$

$$b_{21}(x) = \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \omega^2}(a) q_2(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_1(x) \Big) = -\frac{\sqrt{2}}{8\pi} q_1(x), \tag{8.33}$$

$$b_{22}(x) = 1 + \frac{2}{\det D^2P(a)} \Big(\frac{\partial^2 P}{\partial \omega^2}(a) q_3(x) - \frac{\partial^2 P}{\partial \omega \partial \lambda}(a) q_2(x) \Big) = 1 - \frac{\sqrt{2}}{8\pi} q_2(x). \tag{8.34}$$

This representation, together with the estimates from above for $|q_j|$, $j = 1, 2, 3$ on V given below, will be useful in the proof of Proposition 8.5.

Estimates for $|q_j|$, $j = 1, 2, 3$ on V. Let q_j, $j = 1, 2, 3$ be given by (8.28) – (8.30). We will need the estimates for $|q_j|$, $j = 1, 2, 3$ on V. Let us also use the notations $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}, \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}$ for the corresponding differentiations in each q_j, $j = 1, 2, 3$. For example for q_1 it will be $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}} = \frac{\partial^3 P}{\partial \omega^3}$ and $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}} = \frac{\partial^3 P}{\partial \omega^2 \partial \lambda}$, etc. For every q_j,

$j = 1, 2, 3$ we then in general have:

$$\begin{aligned}
|q_j(x)| &= \Big| \int_0^1 \int_0^1 \int_0^1 ts^2 \Big(\tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(a + ts\sigma(x-a)), \tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(a + ts\sigma(x-a)) \Big) \\
&\quad \times (x-a)^T d\sigma dt\, ds \Big| \\
&\le \int_0^1 \int_0^1 \int_0^1 ts^2 \sup_{x \in V} \Big| \Big(\tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \Big) \Big| d\sigma dt\, ds |x-a| \\
&= \tfrac{1}{6} \sup_{x \in V} \Big| \Big(\tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \tfrac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \Big) \Big| |x-a| \le M_j |x-a|.
\end{aligned} \tag{8.35}$$

where M_j is an upper bound for function $\frac{1}{6}\Big| \Big(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x) \Big) \Big|$ on V. In particular, one shows by explicit and tedious computations that for $M_1 = 180$, $M_2 = 75$, $M_3 = 30$, the estimate (8.35) for the corresponding $|q_j|$, $j = 1, 2, 3$ holds true. We depict the results in Table 3.

$\lvert q_1(x)\rvert < 180\lvert x-a\rvert$	$\lvert q_2(x)\rvert < 75\lvert x-a\rvert$	$\lvert q_3(x)\rvert < 30\lvert x-a\rvert$

Table 3. Estimates from above for $|q_j|$, $j = 1, 2, 3$ on V

Now we proceed with a proof of Proposition 8.5.

8.3.2. *Checking the range for Morse.*

Proof of Proposition 8.5. Representation (8.12) is a consequence of Theorem 8.3. It is straightforward for $a = \left(\frac{1}{2}\pi, 4\right)$ that $P(a) = 0$. We also find that $DP(a) = 0$, meaning a is a critical point of P. Due to (8.19) we conclude that a is a non-degenerate critical point of P.

By Theorem 8.3 in a vicinity $W_a \subset V$ of a which is defined in (8.10), we obtain

$$\begin{aligned}
P(x) &= (h_1(x), h_2(x)) \cdot \tfrac{1}{2} D^2 P(a) \cdot (h_1(x), h_2(x))^T \\
&= -\tfrac{1}{2} h_2(x) \left(16\sqrt{2} h_1(x) + \pi h_2(x) \right),
\end{aligned} \tag{8.36}$$

where $h_1, h_2 \in C^\infty(W_a, \mathbb{R})$. Their explicit formulas read as (8.13), (8.14).

We show that W_a in our case can be taken as a closed ball $W_R(a)$ centered at a of radius R and the numerical approximation for its range is given by (8.17). We will do this in two steps.

(1) Let us solve the inequality $\det B(x) \ge 0$ on V. Due to (8.31) – (8.34) we will get

$$\begin{aligned}
\det B(x) &= b_{11}(x) b_{22}(x) - b_{12}(x) b_{21}(x) \\
&= 1 - \tfrac{\sqrt{2}}{4\pi} q_2(x) + \tfrac{1}{64} q_1(x) - \tfrac{1}{32\pi^2} q_1(x) q_3(x) + \tfrac{1}{32\pi^2} q_2^2(x) \\
&\ge 1 - \tfrac{\sqrt{2}}{4\pi} |q_2(x)| - \tfrac{1}{64} |q_1(x)| - \tfrac{1}{32\pi^2} |q_1(x)||q_3(x)| \ge \dots
\end{aligned}$$

we use estimates for $|q_1(x)|$, $|q_2(x)|$ and $|q_3(x)|$ from Table 3 to get

$$\dots \ge 1 - \tfrac{75\sqrt{2}}{4\pi} |x-a| - \tfrac{180}{64} |x-a| - \tfrac{5400}{32\pi^2} |x-a|^2.$$

The above expression is positive for all $x \in V$ such that $|x-a| \le R_1$, with

$$R_1 = -\tfrac{1}{120}\pi^2 - \tfrac{\sqrt{2}}{18}\pi + \tfrac{1}{120}\pi \sqrt{\pi^2 + \tfrac{40}{3}\sqrt{2}\pi + \tfrac{1568}{9}}.$$

The numerical approximation is $R_1 \approx 0.078\dots$. Hence, the first estimate for a range of W_a is $|x-a| \le R_1$.

(2) Let us solve $b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) > 0$ on V. We have

$$b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x) \ge b_{11}(x) + b_{22}(x) = \dots$$

due to formulas (8.31), (8.34) we obtain

$$2 - \tfrac{\sqrt{2}}{4\pi} q_2(x) + \tfrac{1}{64} q_1(x) \ge 2 - \tfrac{\sqrt{2}}{4\pi}|q_2(x)| - \tfrac{1}{64}|q_1(x)| \ge \dots$$

we use the estimates for $|q_1(x)|$ and $|q_2(x)|$ from Table 3 to get

$$\dots \ge 2 - \tfrac{75\sqrt{2}}{4\pi}|x-a| - \tfrac{180}{64}|x-a|.$$

The above expression is strictly positive for all $x \in V$ such that

$$|x-a| < R_2 \quad \text{with} \quad R_2 = \tfrac{32\pi}{\left(300\sqrt{2}+45\pi\right)},$$

and this is the second estimate for W_a.

Comparing the approximations to $R_1 \approx 0.078\dots$ and $R_2 \approx 0.178\dots$ we set $R := R_1$ and $W_a := W_R(a) = \{x \in V : |x-a| \le R\}$. Result (8.17) follows. $\square$

Remark 8.6. Let the rectangle $U \subset W_R(a)$ containing the point $a = \left(\frac{1}{2}\pi, 4\right)$ be defined as follows:

$$U := \left\{(\omega,\lambda) : \left[\tfrac{1}{2}\pi - \tfrac{2}{180}\pi, \tfrac{1}{2}\pi + \tfrac{2}{180}\pi\right] \times [4-0.060, 4+0.060]\right\}. \tag{8.37}$$

Proposition 8.5 holds true for the given U.

8.4. **On the insecting curves from Morse.** Here we consider $h_2(\omega,\lambda) = 0$ and $16\sqrt{2}h_1(\omega,\lambda) + \pi h_2(\omega,\lambda) = 0$ in U with h_i, $i = 1,2$ as in Proposition 8.5. Consider in U given by (8.37) the two implicit functions:

$$h_2(\omega,\lambda) = 0, \tag{8.38}$$

$$16\sqrt{2}h_1(\omega,\lambda) + \pi h_2(\omega,\lambda) = 0. \tag{8.39}$$

At $(\omega,\lambda) = a$ it holds that $h_1(a) = h_2(a) = 0$; that is,

$$h_2(a) = 0,$$

$$16\sqrt{2}h_1(a) + \pi h_2(a) = 0.$$

Below, by means of Lemma 8.1 and some numerical computations, we will show that the following holds on U:

$$\tfrac{\partial h_2}{\partial \lambda}(\omega,\lambda) > 0,$$
$$16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega,\lambda) + \pi\tfrac{\partial h_2}{\partial \lambda}(\omega,\lambda) > 0,$$

and hence we can apply the Implicit Function Theorem proving that every function (8.38) and (8.39) allows its local parametrization $\omega \mapsto \lambda(\omega)$ in U. This fact is used in Lemma 4.11. Now some preparatory technical steps are required.

8.4.1. *Computational results II. Upper bounds for* $|r_j|$, $j = 1,2,3$ *on* U. Let r_j, $j = 1,2,3$ be given by (8.21)–(8.23). We will find the upper bounds for $|r_j|$, $j = 1,2,3$ on U. Setting again $\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}$ for the corresponding differentiations in each r_j, $j = 1,2,3$, we in general deduce that

$$\begin{aligned}|r_j(x)| &= |\int_0^1\int_0^1 s\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(a+ts(x-a))dt\,ds| \\ &\le \int_0^1\int_0^1 s\sup_{x\in U}|\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)|dt\,ds \\ &= \frac{1}{2}\sup_{x\in U}|\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)| \le Q_j,\end{aligned}$$

where Q_j, $j = 1,2,3$ is an upper bound for the function $\frac{1}{2}|\frac{\partial^2 P}{\partial(\omega,\lambda)^{\alpha_j}}(x)|$ on U.

In an analogous way we find the upper bounds for $|\frac{\partial r_j}{\partial\omega}|, |\frac{\partial r_j}{\partial\lambda}|$ and $|\frac{\partial^2 r_j}{\partial\omega\partial\lambda}|, |\frac{\partial^2 r_j}{\partial\lambda^2}|$, $j = 1,2,3$ on U, we will need later on. Explicit bounds are given in Table 4. Note that we skip the derivatives $\frac{\partial^2 r_j}{\partial\omega^2}$ since there will be no need for them.

$\vert r_1(x)\vert < 5$	$\vert\frac{\partial r_1}{\partial\omega}(x)\vert < 43$, $\vert\frac{\partial r_1}{\partial\lambda}(x)\vert < 25$	$\vert\frac{\partial^2 r_1}{\partial\omega\partial\lambda}(x)\vert < 175$, $\vert\frac{\partial^2 r_1}{\partial\lambda^2}(x)\vert < 65$
$\vert r_2(x)\vert < 19$	$\vert\frac{\partial r_2}{\partial\omega}(x)\vert < 25$, $\vert\frac{\partial r_2}{\partial\lambda}(x)\vert < 6$	$\vert\frac{\partial^2 r_2}{\partial\omega\partial\lambda}(x)\vert < 65$, $\vert\frac{\partial^2 r_2}{\partial\lambda^2}(x)\vert < 33$
$\vert r_3(x)\vert < 6$	$\vert\frac{\partial r_3}{\partial\omega}(x)\vert < 6$, $\vert\frac{\partial r_3}{\partial\lambda}(x)\vert < 4$	$\vert\frac{\partial^2 r_3}{\partial\omega\partial\lambda}(x)\vert < 33$, $\vert\frac{\partial^2 r_3}{\partial\lambda^2}(x)\vert < 16$

TABLE 4. Estimates from above for the absolute value of r_j, $j = 1,2,3$ and some higher order derivatives on U

Upper bounds for $|q_j|$, $j = 1,2,3$ *on* U. Let q_j, $j = 1,2,3$ by given by (8.28) – (8.30). Earlier we found the estimates for $|q_j|$, $j = 1,2,3$ on V of the type $|q_j| \le M_j|x-a|$, $j = 1,2,3$ (see Table 3). Here, we will obtain the constants which are the upper bounds for $|q_j|$, $j = 1,2,3$ on U.

Setting $\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}, \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}$ for the corresponding differentiations in each q_j, $j = 1,2,3$, analogously to (8.35), we have that

$$\begin{aligned}|q_j(x)| &= |\int_0^1\int_0^1\int_0^1 ts^2\Big(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(a+ts\sigma(x-a)), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(a+ts\sigma(x-a))\Big) \\ &\quad\times (x-a)^T d\sigma dt\,ds| \\ &\le \frac{1}{6}\sup_{x\in U}|\Big(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x)\Big)|\max_U|x-a| \le M_j\max_U|x-a|,\end{aligned} \tag{8.40}$$

where M_j is an upper bound for the function $\frac{1}{6}|\Big(\frac{\partial^3 P}{\partial(\omega,\lambda)^{\alpha_j}}(x), \frac{\partial^3 P}{\partial(\omega,\lambda)^{\beta_j}}(x)\Big)|$ on U. In particular, it holds that

$$\begin{aligned}\max_U|x-a| &= |x-a|\big|_{\partial U} \\ &= \sqrt{\left(\omega-\tfrac{1}{2}\pi\right)^2+(\lambda-4)^2}\Big|_{(\omega,\lambda)=(\frac{1}{2}\pi-\frac{2}{180}\pi,4-0.060)} \\ &= \sqrt{(\tfrac{1}{90}\pi)^2+0.060^2},\end{aligned} \tag{8.41}$$

and
$$M_1 = 44, \quad M_2 = 27, \quad M_3 = 11.$$
The explicit upper bounds are given in Table 5.

$\lvert q_1(x)\rvert < 3.1$	$\lvert q_2(x)\rvert < 1.9$	$\lvert q_3(x)\rvert < 0.8$

TABLE 5. Estimates from above for $|q_j|$, $j = 1, 2, 3$ on U

Lower bounds for $F(x) = \det B(x)$ and $G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)$ on U. Let us set
$$\begin{gathered} F(x) = \det B(x), \\ G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x), \end{gathered} \tag{8.42}$$
b_{ij}, $i, j = 1, 2$ are given in (8.31)–(8.34) and find the lower bounds for F and G on U. By construction of the ball $W_R(a) \supset U$ from Proposition 8.5 we know that $F \geq 0$ and $G > 0$ on $W_R(a)$. More precisely, for every $x \in W_R(a)$ (and hence for every $x \in U$) it holds that
$$\begin{gathered} F(x) \geq 1 - \tfrac{\sqrt{2}}{4\pi}|q_2(x)| - \tfrac{1}{64}|q_1(x)| - \tfrac{1}{32\pi^2}|q_1(x)||q_3(x)|, \\ G(x) \geq 2 - \tfrac{\sqrt{2}}{4\pi}|q_2(x)| - \tfrac{1}{64}|q_1(x)|. \end{gathered}$$
Due to results of Table 5 we finally obtain that on U,
$$\begin{gathered} F(x) \geq 0.730\dots > 0.7, \\ G(x) \geq 1.738\dots > 1.7. \end{gathered} \tag{8.43}$$
Upper bounds for $|b_{ij}|$, $i, j = 1, 2$ on U. Let b_{ij}, $i, j = 1, 2$ be given by (8.24) – (8.27), namely,
$$\begin{gathered} b_{11}(x) = \tfrac{1}{64} r_1(x) - \tfrac{\sqrt{2}}{8\pi} r_2(x), \\ b_{12}(x) = \tfrac{1}{64} r_2(x) - \tfrac{\sqrt{2}}{8\pi} r_3(x), \\ b_{21}(x) = -\tfrac{\sqrt{2}}{8\pi} r_1(x), \quad b_{22}(x) = -\tfrac{\sqrt{2}}{8\pi} r_2(x), \end{gathered}$$
with r_j, $j = 1, 2, 3$ as in (8.21)–(8.23).

Using the results of Table 4 we will find the following upper bounds for the absolute values of b_{ij}, $i, j = 1, 2$ and some higher order derivatives on U (see Table 6).

$\lvert b_{11}(x)\rvert < 1.2$	$\lvert\frac{\partial b_{11}}{\partial\omega}(x)\rvert < 2.1$, $\lvert\frac{\partial b_{11}}{\partial\lambda}(x)\rvert < 0.8$	$\lvert\frac{\partial^2 b_{11}}{\partial\omega\partial\lambda}(x)\rvert < 6.4$, $\lvert\frac{\partial^2 b_{11}}{\partial\lambda^2}(x)\rvert < 2.9$
$\lvert b_{12}(x)\rvert < 0.7$	$\lvert\frac{\partial b_{12}}{\partial\omega}(x)\rvert < 0.8$, $\lvert\frac{\partial b_{12}}{\partial\lambda}(x)\rvert < 0.4$	$\lvert\frac{\partial^2 b_{12}}{\partial\omega\partial\lambda}(x)\rvert < 2.9$, $\lvert\frac{\partial^2 b_{12}}{\partial\lambda^2}(x)\rvert < 1.5$
$\lvert b_{21}(x)\rvert < 0.3$	$\lvert\frac{\partial b_{21}}{\partial\omega}(x)\rvert < 2.5$, $\lvert\frac{\partial b_{21}}{\partial\lambda}(x)\rvert < 1.5$	$\lvert\frac{\partial^2 b_{21}}{\partial\omega\partial\lambda}(x)\rvert < 9.9$, $\lvert\frac{\partial^2 b_{21}}{\partial\lambda^2}(x)\rvert < 3.7$
$\lvert b_{22}(x)\rvert < 1.1$	$\lvert\frac{\partial b_{22}}{\partial\omega}(x)\rvert < 1.5$, $\lvert\frac{\partial b_{22}}{\partial\lambda}(x)\rvert < 0.4$	$\lvert\frac{\partial^2 b_{22}}{\partial\omega\partial\lambda}(x)\rvert < 3.7$, $\lvert\frac{\partial^2 b_{22}}{\partial\lambda^2}(x)\rvert < 1.9$

TABLE 6. Estimates from above for the absolute value of b_{ij}, $i, j = 1, 2$ and some higher order derivatives on U

Upper bounds for $F(x) = \det B(x)$ and $G(x) = b_{11}(x) + 2\sqrt{\det B(x)} + b_{22}(x)$ on U. Recall that F and G are given by (8.42). They are positive functions on U

with lower bounds as in (8.43). Here we find their upper bounds together with the upper bounds for some higher order derivatives.

In particular, in order to obtain the estimates for F, $|\frac{\partial F}{\partial \omega}|, |\frac{\partial F}{\partial \lambda}|$ and $|\frac{\partial^2 F}{\partial \omega \partial \lambda}|, |\frac{\partial^2 F}{\partial \lambda^2}|$ on U, we use the results of Table 6. The estimates found are presented in the first row of Table 7.

To find the estimates for $|\frac{\partial G}{\partial \omega}|, |\frac{\partial G}{\partial \lambda}|$ and $|\frac{\partial^2 G}{\partial \omega \partial \lambda}|, |\frac{\partial^2 G}{\partial \lambda^2}|$ we use

- the lower bound for Fon U, namely, $F(x) > 0.7$,
- the results of Table 6 and
- the results from the first row of Table 7.

E.g., for $|\frac{\partial G}{\partial \omega}|$ we will have that on U

$$\begin{aligned}
|\tfrac{\partial G}{\partial \omega}(x)| &= |\tfrac{\partial b_{11}}{\partial \omega}(x) + \frac{\frac{\partial F}{\partial \omega}(x)}{\sqrt{F(x)}} + \tfrac{\partial b_{22}}{\partial \omega}(x)| \\
&\le \sup_U |\tfrac{\partial b_{11}}{\partial \omega}(x)| + \frac{\sup_U |\frac{\partial F}{\partial \omega}(x)|}{\inf_U \sqrt{F(x)}} + \sup_U |\tfrac{\partial b_{22}}{\partial \omega}(x)| \\
&< 2.1 + \tfrac{6.1}{\sqrt{0.7}} + 1.5 \approx 10.891 \dots < 11.
\end{aligned}$$

The other estimates on U for the derivatives of G listed above are obtained in an analogous way and presented in Table 7.

$F(x) < 1.53$	$\vert\frac{\partial F}{\partial \omega}(x)\vert < 6.10$, $\vert\frac{\partial F}{\partial \lambda}(x)\vert < 2.53$	$\vert\frac{\partial^2 F}{\partial \omega \partial \lambda}(x)\vert < 23.52$, $\vert\frac{\partial^2 F}{\partial \lambda^2}(x)\vert < 10.35$
–	$\vert\frac{\partial G}{\partial \omega}(x)\vert < 11$, $\vert\frac{\partial G}{\partial \lambda}(x)\vert < 4.3$	$\vert\frac{\partial^2 G}{\partial \omega \partial \lambda}(x)\vert < 51.4$, $\vert\frac{\partial^2 G}{\partial \lambda^2}(x)\vert < 22.7$

TABLE 7. Estimates from above for F and G on U

Upper bounds for $|c_{ij}|$, $i,j = 1,2$ on U. Let c_{ij}, $i,j = 1,2$ be given by given by (8.15). It is convenient for further computations to set

$$c_{ij}(x) = \frac{b_{ij}(x) + A\sqrt{F(x)}}{\sqrt{G(x)}}, \tag{8.44}$$

where F and G are as in (8.42) and

$$A = \begin{cases} 1 & \text{if } (i,j) = \{(1,1),\ (2,2)\}, \\ 0 & \text{if } (i,j) = \{(1,2),\ (2,1)\}. \end{cases}$$

To obtain the estimates for $|\frac{\partial c_{ij}}{\partial \omega}|, |\frac{\partial c_{ij}}{\partial \lambda}|$ and $|\frac{\partial^2 c_{ij}}{\partial \omega \partial \lambda}|, |\frac{\partial^2 c_{ij}}{\partial \lambda^2}|$, $i,j = 1,2$ on U we use

- the lower bound for F and G on U, defined by formula (8.43),
- the results of Table 6 and
- the results of Table 7.

E.g., the estimate for $|\frac{\partial c_{ij}}{\partial \lambda}|$ on U is found in the following way:

$$\begin{aligned}
|\tfrac{\partial c_{ij}}{\partial \lambda}(x)| &= |\frac{\frac{\partial b_{ij}}{\partial \lambda}(x)}{\sqrt{G(x)}} + \tfrac{1}{2} A \frac{\frac{\partial F}{\partial \lambda}(x)}{\sqrt{G(x)}\sqrt{F(x)}} - \tfrac{1}{2} \frac{\frac{\partial G}{\partial \lambda}(x)}{G^{3/2}(x)} \left(b_{ij}(x) + A\sqrt{F(x)}\right)| \\
&\le |\frac{\sup_U |\frac{\partial b_{ij}}{\partial \lambda}(x)|}{\inf_U \sqrt{G(x)}} + \tfrac{1}{2} A \frac{\sup_U |\frac{\partial F}{\partial \lambda}(x)|}{\inf_U \left(\sqrt{G(x)}\sqrt{F(x)}\right)}
\end{aligned}$$

$$+ \frac{1}{2} \frac{\sup_U |\frac{\partial G}{\partial \lambda}(x)|}{\inf_U G^{3/2}(x)} \Big(\sup_U |b_{ij}(x)| + A \sup_U \sqrt{F(x)} \Big) |.$$

The other estimates on U for the derivatives of c_{ij}, $i, j = 1, 2$ listed are obtained in an analogous way and listed in Table 8 below.

$\vert\frac{\partial c_{11}}{\partial \lambda}(x)\vert < 4.2$	$\vert\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x)\vert < 99.4$ $\vert\frac{\partial^2 c_{11}}{\partial \lambda^2}(x)\vert < 35$
$\vert\frac{\partial c_{12}}{\partial \omega}(x)\vert < 2.4$ $\vert\frac{\partial c_{12}}{\partial \lambda}(x)\vert < 1$	$\vert\frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x)\vert < 18.7$ $\vert\frac{\partial^2 c_{12}}{\partial \lambda^2}(x)\vert < 8.1$
$\vert\frac{\partial c_{21}}{\partial \lambda}(x)\vert < 1.5$	$\vert\frac{\partial^2 c_{21}}{\partial \omega \partial \lambda}(x)\vert < 20.1$ $\vert\frac{\partial^2 c_{21}}{\partial \lambda^2}(x)\vert < 8.4$
$\vert\frac{\partial c_{22}}{\partial \omega}(x)\vert < 9.8$ $\vert\frac{\partial c_{22}}{\partial \lambda}(x)\vert < 3.8$	$\vert\frac{\partial^2 c_{22}}{\partial \omega \partial \lambda}(x)\vert < 94.4$ $\vert\frac{\partial^2 c_{22}}{\partial \lambda^2}(x)\vert < 38.7$

Table 8. Estimates from above for the absolute value of some higher order derivatives of c_{ij}, $i, j = 1, 2$ on U

Upper bounds for $|h_i|$, $i = 1, 2$ *on* U. Let h_i, $i = 1, 2$ be given by formulas (8.13), (8.14). First we compute the following derivatives of h_i, $i = 1, 2$ we will need below:

$$\frac{\partial h_1}{\partial \lambda}(x) = c_{12}(x) + \big(\frac{\partial c_{11}}{\partial \lambda}(x), \frac{\partial c_{12}}{\partial \lambda}(x)\big)(x-a)^T, \tag{8.45}$$

$$\frac{\partial^2 h_1}{\partial \omega \partial \lambda}(x) = \frac{\partial c_{12}}{\partial \omega}(x) + \frac{\partial c_{11}}{\partial \lambda}(x) + \Big(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x)\Big)(x-a)^T, \tag{8.46}$$

$$\frac{\partial^2 h_1}{\partial \lambda^2}(x) = 2\frac{\partial c_{12}}{\partial \lambda}(x) + \Big(\frac{\partial^2 c_{11}}{\partial \lambda^2}(x), \frac{\partial^2 c_{12}}{\partial \lambda^2}(x)\Big)(x-a)^T, \tag{8.47}$$

$$\frac{\partial h_2}{\partial \lambda}(x) = c_{22}(x) + \big(\frac{\partial c_{21}}{\partial \lambda}(x), \frac{\partial c_{22}}{\partial \lambda}(x)\big)(x-a)^T, \tag{8.48}$$

$$\frac{\partial^2 h_2}{\partial \omega \partial \lambda}(x) = \frac{\partial c_{22}}{\partial \omega}(x) + \frac{\partial c_{21}}{\partial \lambda}(x) + \Big(\frac{\partial^2 c_{21}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{22}}{\partial \omega \partial \lambda}(x)\Big)(x-a)^T, \tag{8.49}$$

$$\frac{\partial^2 h_2}{\partial \lambda^2}(x) = 2\frac{\partial c_{22}}{\partial \lambda}(x) + \Big(\frac{\partial^2 c_{21}}{\partial \lambda^2}(x), \frac{\partial^2 c_{22}}{\partial \lambda^2}(x)\Big)(x-a)^T, \tag{8.50}$$

where c_{ij}, $i, j = 1, 2$ are as in (8.44) and $a = \left(\frac{1}{2}\pi, 4\right)$.

Using the results of Table 8 we find the estimates for $|\frac{\partial^2 h_i}{\partial \omega \partial \lambda}|, |\frac{\partial^2 h_i}{\partial \lambda^2}|$, $i = 1, 2$ on U. E.g., for $|\frac{\partial^2 h_1}{\partial \omega \partial \lambda}|$ it holds that on U:

$$\begin{aligned} |\frac{\partial^2 h_1}{\partial \omega \partial \lambda}(x)| &= |\frac{\partial c_{12}}{\partial \omega}(x) + \frac{\partial c_{11}}{\partial \lambda}(x) + \Big(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x)\Big)(x-a)^T | \\ &\le \sup_U |\frac{\partial c_{12}}{\partial \omega}(x)| + \sup_U |\frac{\partial c_{11}}{\partial \lambda}(x)| \\ &\quad + \sup_U |\Big(\frac{\partial^2 c_{11}}{\partial \omega \partial \lambda}(x), \frac{\partial^2 c_{12}}{\partial \omega \partial \lambda}(x)\Big)| \max_U |x-a| < \dots \end{aligned}$$

for $\max_U |x-a|$ see formula (8.41) and then

$$\dots < 2.4 + 4.2 + \sqrt{99.4^2 + 18.7^2}\sqrt{\left(\tfrac{1}{90}\pi\right)^2 + 0.060^2} \approx 13.621 \dots < 13.7.$$

Analogously, the other estimates on U are obtained (see Table 9).

$\lvert\frac{\partial^2 h_1}{\partial\omega\partial\lambda}(x)\rvert < 13.7$	$\lvert\frac{\partial^2 h_2}{\partial\omega\partial\lambda}(x)\rvert < 18$
$\lvert\frac{\partial^2 h_1}{\partial\lambda^2}(x)\rvert < 4.5$	$\lvert\frac{\partial^2 h_2}{\partial\lambda^2}(x)\rvert < 10.4$

TABLE 9. Estimates from above for the absolute value of some higher order derivatives of h_i, $i = 1, 2$ on U

8.4.2. *Strict positivity of the functions* $\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ *and* $16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega,\lambda) + \pi\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ *on* U*.* Let us fix the notations which are common for two lemmas.

Notation 8.7. Let U be as in (8.37), namely,

$$U := \left\{(\omega,\lambda) : \left[\tfrac{88}{180}\pi, \tfrac{92}{180}\pi\right] \times [3.940, 4.060]\right\}.$$

By $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,n\\ j=0,\dots,m}}$ we mean a discretization of U which is defined as follows

$$\omega_i = \tfrac{88}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.94 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{4}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.12}{m}.$$

Then we deduce the following two results.

Lemma 8.8. *It holds that*

$$\frac{\partial h_2}{\partial\lambda}(\omega,\lambda) > 0 \quad \textit{on } U. \tag{8.51}$$

Proof. Fix $n = m = 2$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$ of U given by Notation 8.7. By straightforward computations it holds that

$$\tfrac{\partial h_2}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{8.52}$$

for all $i = 0,\dots,2$, $j = 0,\dots,2$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in U} \tfrac{\partial h_2}{\partial\lambda}(\omega_i,\lambda_j) = \tfrac{\partial h_2}{\partial\lambda}(\omega_0,\lambda_0) \approx 0.952\dots. \tag{8.53}$$

Next to this, we compute

$$a_1 = \max\left\{\Delta\omega,\Delta\lambda\right\} = \max\left\{\frac{\frac{4}{180}\pi}{n}, \frac{0.12}{m}\right\} = \frac{0.12}{2} = 0.060,$$

and, by taking into account the results of Table 9 and (8.53), we also find

$$a_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in U} \frac{\partial h_2}{\partial\lambda}(\omega_i,\lambda_j)}{\sup_U \left|\left(\frac{\partial^2 h_2}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 h_2}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.065\dots.$$

Since $a_1 < a_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,2\\ j=0,\dots,2}}$ of U to be appropriate in the sense that condition (8.52) yields a strict positivity of $\frac{\partial h_2}{\partial\lambda}(\omega,\lambda)$ on U. □

Lemma 8.9. *Let U be as in (8.37). It holds that*

$$16\sqrt{2}\frac{\partial h_1}{\partial\lambda}(\omega,\lambda) + \pi\frac{\partial h_2}{\partial\lambda}(\omega,\lambda) > 0 \quad \textit{on } U. \tag{8.54}$$

Proof. Fix $n = 7$, $m = 12$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{i,j}$ of U given by Notation 8.7. By straightforward computations it holds that

$$16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_i, \lambda_j) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j) > 0, \tag{8.55}$$

for all $i = 0, \dots, 7$, $j = 0, \dots, 12$ and, moreover,

$$\begin{aligned} &\min_{(\omega_i,\lambda_j)\in U} \Big(16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_i, \lambda_j) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j)\Big) \\ &= 16\sqrt{2}\tfrac{\partial h_1}{\partial \lambda}(\omega_0, \lambda_0) + \pi \tfrac{\partial h_2}{\partial \lambda}(\omega_0, \lambda_0) \approx 2.936\dots . \end{aligned} \tag{8.56}$$

Next to this, we compute

$$b_1 = \max\big\{\Delta\omega, \Delta\lambda\big\} = \max\big\{\tfrac{\frac{4}{180}\pi}{n}, \tfrac{0.12}{m}\big\} = \tfrac{0.12}{12} = 0.010,$$

and, by taking into account the results of Table 9 and (8.56), we also find

$$\begin{aligned} b_2 &= \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in U}\big(16\sqrt{2}\frac{\partial h_1}{\partial \lambda}(\omega_i, \lambda_j) + \pi \frac{\partial h_2}{\partial \lambda}(\omega_i, \lambda_j)\big)}{\sup_U |\Big(16\sqrt{2}\frac{\partial^2 h_1}{\partial\omega\partial\lambda}(\omega, \lambda) + \pi\frac{\partial^2 h_2}{\partial\omega\partial\lambda}(\omega, \lambda), 16\sqrt{2}\frac{\partial^2 h_1}{\partial\lambda^2}(\omega, \lambda) + \pi\frac{\partial^2 h_2}{\partial\lambda^2}(\omega, \lambda)\Big)|} \\ &\approx 0.011\dots . \end{aligned}$$

Since $b_1 < b_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{i,j}$ of U to be appropriate in the sense that condition (8.55) yields a strict positivity of $16\sqrt{2}\frac{\partial h_1}{\partial \lambda}(\omega, \lambda) + \pi \frac{\partial h_2}{\partial \lambda}(\omega, \lambda)$ on U. □

8.5. **On $P(\omega, \lambda) = 0$ in V away from $a = (\frac{1}{2}\pi, 4)$.** Recall here function P defined on $V = \big\{(\omega, \lambda) : [\frac{70}{180}\pi, \frac{110}{180}\pi] \times [2.900, 5.100]\big\}$:

$$\begin{aligned} P(\omega, \lambda) &= \Big(1 - \tfrac{\sqrt{2}}{2}\sin(2\omega)\Big)^{\lambda} + \Big(1 + \tfrac{\sqrt{2}}{2}\sin(2\omega)\Big)^{\lambda} \\ &\quad + \big(\tfrac{1}{2} + \tfrac{1}{2}\cos^2(2\omega)\big)^{\frac{1}{2}\lambda}\Big[2\cos\Big(\lambda\Big(\arctan\Big(\tfrac{\sqrt{2}}{2}\tan(2\omega)\Big) + \pi\Big)\Big) \\ &\quad - 4\cos\big(\lambda\arctan\big(\tan^2(\omega)\big)\big)\Big]. \end{aligned} \tag{8.57}$$

8.5.1. *Set of Claims I.* In a set of claims below we describe some properties of the first derivatives $\frac{\partial P}{\partial \omega}$ and $\frac{\partial P}{\partial \lambda}$ on V away from the point $a = \big(\frac{1}{2}\pi, 4\big)$ which are used in Lemma 4.11.

Claim 8.10. Let $H_1 \subset V$ be $H_1 = \big\{(\omega, \lambda) : \big[\frac{84}{180}\pi, \frac{90}{180}\pi\big] \times [4.030, 4.970]\big\}$. It holds that $\frac{\partial P}{\partial \omega}(\omega, \lambda) < 0$ on H_1.

Proof. First we find the following estimates:

$$|\frac{\partial^2 P}{\partial \omega^2}(\omega, \lambda)| < 162, \quad |\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega, \lambda)| < 45 \quad \text{on } H_1. \tag{8.58}$$

Then we fix $n = 14$, $m = 120$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{i,j}$ of H_1 such that

$$\omega_i = \tfrac{84}{180}\pi + i\Delta\omega, \quad \lambda_j = 4.030 + j\Delta\lambda,$$

with

$$\Delta\omega = \tfrac{\frac{6}{180}\pi}{n}, \quad \Delta\lambda = \tfrac{0.94}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial \omega}(\omega_i, \lambda_j) > 0, \tag{8.59}$$

for all $i = 0, \dots, 14$, $j = 0, \dots, 120$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_1} \left(-\tfrac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right) = -\tfrac{\partial P}{\partial \omega}(\omega_9,\lambda_0) \approx 1.022\dots. \tag{8.60}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega, \Delta\lambda\} = \max\left\{\tfrac{\frac{6}{180}\pi}{n}, \tfrac{0.94}{m}\right\} = \tfrac{0.94}{120} \approx 0.00783\dots,$$

and, by taking into account (8.58) and (8.60), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_1}\left(-\frac{\partial P}{\partial \omega}(\omega_i,\lambda_j)\right)}{\sup_{H_1}\left|\left(\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right)\right|} \approx 0.00860\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,14\\ j=0,\dots,120}}$ of H_1 to be appropriate in the sense that condition (8.59) yields a strict positivity of $-\frac{\partial P}{\partial\omega}(\omega,\lambda)$ on H_1, or, in other words, $\frac{\partial P}{\partial\omega}(\omega,\lambda) < 0$ on H_1. □

Claim 8.11. Let $H_2 \subset V$ be $H_2 = \left\{(\omega,\lambda) : \left[\frac{87}{180}\pi, \frac{101}{180}\pi\right] \times [4.750, 5.100]\right\}$. It holds that $\frac{\partial P}{\partial\lambda}(\omega,\lambda) > 0$ on H_2.

Proof. First we find the following estimates:

$$\left|\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\right| < 48, \quad \left|\tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right| < 25 \quad \text{on } H_2. \tag{8.61}$$

Then we fix $n = 40$, $m = 55$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0, \dots, 40$, $j = 0, \dots, 55$ of H_2 such that

$$\omega_i = \tfrac{87}{180}\pi + i\Delta\omega, \quad \lambda_j = 4.750 + j\Delta\lambda,$$

with

$$\Delta\omega = \tfrac{\frac{14}{180}\pi}{n}, \quad \Delta\lambda = \tfrac{0.35}{m}.$$

By straightforward computations it holds that

$$\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{8.62}$$

for all $i = 0, \dots, 40$, $j = 0, \dots, 55$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_2} \tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) = \tfrac{\partial P}{\partial\lambda}(\omega_0,\lambda_0) \approx 0.245\dots. \tag{8.63}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega, \Delta\lambda\} = \max\left\{\tfrac{\frac{14}{180}\pi}{n}, \tfrac{0.35}{m}\right\} = \tfrac{0.35}{55} \approx 0.00636\dots,$$

and, by taking into account (8.61) and (8.63), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_2}\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)}{\sup_{H_2}\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00641\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,55}}$ of H_2 to be appropriate in the sense that condition (8.62) yields a strict positivity of $\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_2. □

Claim 8.12. Let $H_3 \subset V$ be $H_3 = \left\{(\omega,\lambda) : \left[\frac{100}{180}\pi, \frac{108}{180}\pi\right] \times [4.000, 4.850]\right\}$. It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) > 0$ on H_3.

Proof. First we find the following estimates:

$$|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)| < 166, \quad |\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)| < 37 \quad \text{on } H_3. \tag{8.64}$$

Then we fix $n = 10$, $m = 60$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$ of H_3 such that

$$\omega_i = \tfrac{100}{180}\pi + i\Delta\omega, \quad \lambda_j = 4.000 + j\Delta\lambda,$$

with

$$\Delta\omega = \tfrac{\frac{8}{180}\pi}{n}, \quad \Delta\lambda = \tfrac{0.85}{m}.$$

By straightforward computations it holds that

$$\tfrac{\partial P}{\partial\omega}(\omega_i,\lambda_j) > 0, \tag{8.65}$$

for all $i = 0,\dots,10$, $j = 0,\dots,60$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_3} \tfrac{\partial P}{\partial\omega}(\omega_i,\lambda_j) = \tfrac{\partial P}{\partial\omega}(\omega_0,\lambda_{11}) \approx 1.885\dots. \tag{8.66}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\big\{\tfrac{\frac{8}{180}\pi}{n}, \tfrac{0.85}{m}\big\} = \tfrac{0.85}{60} \approx 0.0142\dots,$$

and, by taking into account (8.64) and (8.66), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_3}\frac{\partial P}{\partial\omega}(\omega_i,\lambda_j)}{\sup_{H_3}\big|\big(\frac{\partial^2 P}{\partial\omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\big)\big|} \approx 0.0157\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,10\\ j=0,\dots,60}}$ of H_3 to be appropriate in the sense that condition (8.65) yields a strict positivity of $\frac{\partial P}{\partial\omega}(\omega,\lambda)$ on H_3. □

Claim 8.13. Let $H_4 \subset V$ be $H_4 = \{(\omega,\lambda) : [\frac{91}{180}\pi, \frac{102}{180}\pi] \times [3.950, 4.100]\}$. It holds that $\frac{\partial P}{\partial\lambda}(\omega,\lambda) < 0$ on H_4.

Proof. First we find the following estimates:

$$|\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)| < 38, \quad |\tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)| < 11 \quad \text{on } H_4. \tag{8.67}$$

Then we fix $n = 50$, $m = 36$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,50$, $j = 0,\dots,36$ of H_4 such that

$$\omega_i = \tfrac{91}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.950 + j\Delta\lambda,$$

with

$$\Delta\omega = \tfrac{\frac{11}{180}\pi}{n}, \quad \Delta\lambda = \tfrac{0.15}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{8.68}$$

for all $i = 0,\dots,50$, $j = 0,\dots,36$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_4}\big(-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\big) = -\tfrac{\partial P}{\partial\lambda}(\omega_0,\lambda_0) \approx 0.118\dots. \tag{8.69}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\big\{\tfrac{\frac{11}{180}\pi}{n}, \tfrac{0.15}{m}\big\} = \tfrac{0.15}{36} = 0.00416\dots,$$

and, by taking into account (8.67) and (8.69), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_4}\big(-\frac{\partial P}{\partial \lambda}(\omega_i,\lambda_j)\big)}{\sup_{H_4}\big|\big(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\big)\big|} \approx 0.00423\dots .$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,50\\ j=0,\dots,36}}$ of H_4 to be appropriate in the sense that condition (8.68) yields a strict positivity of $-\frac{\partial P}{\partial \lambda}(\omega,\lambda)$ on H_4, or, in other words $\frac{\partial P}{\partial \lambda}(\omega,\lambda) < 0$ on H_4. □

Claim 8.14. *Let $H_5 \subset V$ be $H_5 = \big\{(\omega,\lambda) : \big[\frac{90}{180}\pi, \frac{96}{180}\pi\big] \times [3.030, 3.970]\big\}$. It holds that $\frac{\partial P}{\partial \omega}(\omega,\lambda) > 0$ on H_5.*

Proof. First we find the following estimates:

$$|\frac{\partial^2 P}{\partial \omega^2}(\omega,\lambda)| < 105, \quad |\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)| < 37 \quad \text{on } H_5. \tag{8.70}$$

Then we fix $n = 11$, $m = 94$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,11$, $j = 0,\dots,94$ of H_5 such that

$$\omega_i = \tfrac{90}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.030 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{6}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.94}{m}.$$

By straightforward computations it holds that

$$\tfrac{\partial P}{\partial \omega}(\omega_i,\lambda_j) > 0, \tag{8.71}$$

for all $i = 0,\dots,11$, $j = 0,\dots,94$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_5} \tfrac{\partial P}{\partial \omega}(\omega_i,\lambda_j) = \tfrac{\partial P}{\partial \omega}(\omega_0,\lambda_0) \approx 0.807\dots . \tag{8.72}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\Big\{\frac{\frac{6}{180}\pi}{n}, \frac{0.94}{m}\Big\} = \frac{0.94}{94} \approx 0.0100,$$

and, by taking into account (8.70) and (8.72), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_5}\frac{\partial P}{\partial \omega}(\omega_i,\lambda_j)}{\sup_{H_5}\big|\big(\frac{\partial^2 P}{\partial\omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\big)\big|} \approx 0.0102\dots .$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,11\\ j=0,\dots,94}}$ of H_5 to be appropriate in the sense that condition (8.71) yields a strict positivity of $\frac{\partial P}{\partial \omega}(\omega,\lambda)$ on H_5. □

Claim 8.15. Let $H_6 \subset V$ be $H_6 = \big\{(\omega,\lambda) : \big[\frac{79}{180}\pi, \frac{94}{180}\pi\big] \times [2.900, 3.230]\big\}$.

It holds that $\frac{\partial P}{\partial \lambda}(\omega,\lambda) < 0$ on H_6.

Proof. First we find the following estimates:

$$|\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)| < 33, \quad |\tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)| < 21 \quad \text{on } H_6. \tag{8.73}$$

Then we fix $n = 33$, $m = 41$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,33$, $j = 0,\dots,41$ of H_6 such that

$$\omega_i = \tfrac{79}{180}\pi + i\Delta\omega, \quad \lambda_j = 2.900 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{15}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.33}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{8.74}$$

for all $i = 0,\dots,33$, $j = 0,\dots,41$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_6}\big(-\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\big) = -\tfrac{\partial P}{\partial\lambda}(\omega_{33},\lambda_{41}) \approx 0.227\dots. \tag{8.75}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\big\{\tfrac{\frac{15}{180}\pi}{n}, \tfrac{0.33}{m}\big\} = \tfrac{0.33}{41} = 0.00805\dots,$$

and, by taking into account (8.73) and (8.75), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_6}\big(-\frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)\big)}{\sup_{H_6}\big|\big(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\big)\big|} \approx 0.00820\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,33\\ j=0,\dots,41}}$ of H_6 to be appropriate in the sense that condition (8.74) yields a strict positivity of $-\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_6, or, in other words $\frac{\partial P}{\partial\lambda}(\omega,\lambda) < 0$ on H_6. □

Claim 8.16. Let $H_7 \subset V$ be $H_7 = \big\{(\omega,\lambda) : \big[\frac{72}{180}\pi, \frac{80}{180}\pi\big] \times [3.150, 4.000]\big\}$. It holds that $\frac{\partial P}{\partial\omega}(\omega,\lambda) < 0$ on H_7.

Proof. First we find the following estimates:

$$\big|\frac{\partial^2 P}{\partial\omega^2}(\omega,\lambda)\big| < 115, \quad \big|\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\big| < 21 \quad \text{on } H_7. \tag{8.76}$$

Then we fix $n = 5$, $m = 30$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,5$, $j = 0,\dots,30$ of H_7 such that

$$\omega_i = \tfrac{72}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.150 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{8}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.85}{m}.$$

By straightforward computations it holds that

$$-\tfrac{\partial P}{\partial\omega}(\omega_i,\lambda_j) > 0, \tag{8.77}$$

for all $i = 0,\dots,5$, $j = 0,\dots,30$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_7}\big(-\tfrac{\partial P}{\partial\omega}(\omega_i,\lambda_j)\big) = -\tfrac{\partial P}{\partial\omega}(\omega_5,\lambda_{27}) \approx 2.663\dots. \tag{8.78}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\big\{\tfrac{\frac{8}{180}\pi}{n}, \tfrac{0.85}{m}\big\} = \tfrac{0.85}{30} \approx 0.0283\dots,$$

and, by taking into account (8.76) and (8.78), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_7}\big(-\frac{\partial P}{\partial\omega}(\omega_i,\lambda_j)\big)}{\sup_{H_7}\big|\big(\frac{\partial^2 P}{\partial\omega^2}(\omega,\lambda), \frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\big)\big|} \approx 0.0322\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,5\\ j=0,\dots,30}}$ of H_7 to be appropriate in the sense that condition (8.77) yields a strict positivity of $-\frac{\partial P}{\partial\omega}(\omega,\lambda)$ on H_7, or, in other words, $\frac{\partial P}{\partial\omega}(\omega,\lambda) < 0$ on H_7. □

Claim 8.17. *Let $H_8 \subset V$ be $H_8 = \left\{(\omega,\lambda) : \left[\frac{78}{180}\pi, \frac{89}{180}\pi\right] \times [3.900, 4.050]\right\}$. It holds that $\frac{\partial P}{\partial \lambda}(\omega,\lambda) > 0$ on H_8.*

Proof. First we find the following estimates:

$$|\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)| < 36, \quad |\tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)| < 10 \quad \text{on } H_8. \tag{8.79}$$

Then we fix $n = 40$, $m = 30$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,40$, $j = 0,\dots,30$ of H_8 such that

$$\omega_i = \tfrac{78}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.900 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{11}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.15}{m}.$$

By straightforward computations it holds that

$$\tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) > 0, \tag{8.80}$$

for all $i = 0,\dots,40$, $j = 0,\dots,30$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_8} \tfrac{\partial P}{\partial\lambda}(\omega_i,\lambda_j) = \tfrac{\partial P}{\partial\lambda}(\omega_{40},\lambda_{30}) \approx 0.134\dots. \tag{8.81}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega,\Delta\lambda\} = \max\left\{\frac{\frac{11}{180}\pi}{n}, \frac{0.15}{m}\right\} = \frac{0.15}{30} = 0.00500,$$

and, by taking into account (8.79) and (8.81), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_8} \frac{\partial P}{\partial\lambda}(\omega_i,\lambda_j)}{\sup_{H_8}\sup\left|\left(\frac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda), \frac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\right)\right|} \approx 0.00508\dots.$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i,\lambda_j)\}_{\substack{i=0,\dots,40\\ j=0,\dots,30}}$ of H_8 to be appropriate in the sense that condition (8.80) yields a strict positivity of $\frac{\partial P}{\partial\lambda}(\omega,\lambda)$ on H_8. □

8.5.2. *Set of Claims II.* Let $H_0 \subset V$ be as follows

$$H_0 = \left\{(\omega,\lambda) : \left[\tfrac{84}{180}\pi, \tfrac{94}{180}\pi\right] \times [2.960, 3.060]\right\}.$$

Consider a function $G : H_0 \to \mathbb{R}$ given by

$$\begin{aligned} G(\omega,\lambda) = &-\tfrac{\partial^2 P}{\partial\omega^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\right]^{-1} + 2\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\right]^{-2} \\ &- \tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^2\left[\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\right]^{-3}, \end{aligned}$$

where P is as in (8.57).

Claim 8.18. It holds that $G(\omega,\lambda) > 0$ on H_0.

Proof. First we find the following estimates:

$$|\tfrac{\partial G}{\partial\omega}(\omega,\lambda)| < 25000, \quad |\tfrac{\partial G}{\partial\lambda}(\omega,\lambda)| < 14000 \quad \text{on } H_0. \tag{8.82}$$

Then we fix $n = 600$, $m = 300$ and consider the discretization $\{(\omega_i,\lambda_j)\}_{i,j}$, $i = 0,\dots,600$, $j = 0,\dots,300$ of H_0 such that

$$\omega_i = \tfrac{84}{180}\pi + i\Delta\omega, \quad \lambda_j = 2.960 + j\Delta\lambda,$$

with

$$\Delta\omega = \frac{\frac{10}{180}\pi}{n}, \quad \Delta\lambda = \frac{0.1}{m}.$$

By straightforward computations it holds that

$$G(\omega_i, \lambda_j) > 0, \tag{8.83}$$

for all $i = 0, \dots, 600$, $j = 0, \dots, 300$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_0} G(\omega_i, \lambda_j) = G(\omega_0, \lambda_0) \approx 8.380\dots . \tag{8.84}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega, \Delta\lambda\} = \max\left\{\tfrac{\frac{10}{180}\pi}{n}, \tfrac{0.1}{m}\right\} = \tfrac{0.1}{300} \approx 0.000(3),$$

and, by taking into account (8.82) and (8.84), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_0} G(\omega_i,\lambda_j)}{\sup_{H_0}\left|\left(\frac{\partial G}{\partial \omega}(\omega,\lambda), \frac{\partial G}{\partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.000414\dots .$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{i,j}$ of H_0 to be appropriate in the sense that condition (8.83) yields a strict positivity of $G(\omega, \lambda)$ on H_0. □

Let $H_\star \subset V$ be as follows

$$H_\star = \left\{(\omega,\lambda) : \left[\tfrac{93.5}{180}\pi, \tfrac{95.5}{180}\pi\right] \times [3.030, 3.600]\right\}.$$

Consider a function $F : H_\star \to \mathbb{R}$ given by

$$\begin{aligned} F(\omega,\lambda) = &-\tfrac{\partial^2 P}{\partial\lambda^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-1} + 2\tfrac{\partial^2 P}{\partial\omega\partial\lambda}(\omega,\lambda)\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-2} \\ &- \tfrac{\partial^2 P}{\partial\omega^2}(\omega,\lambda)\left[\tfrac{\partial P}{\partial\lambda}(\omega,\lambda)\right]^{2}\left[\tfrac{\partial P}{\partial\omega}(\omega,\lambda)\right]^{-3}, \end{aligned}$$

where P is as in (8.57).

Claim 8.19. It holds that $F(\omega, \lambda) < 0$ on $H_\star$.

Proof. First we find the following estimates:

$$\left|\tfrac{\partial F}{\partial\omega}(\omega,\lambda)\right| < 180, \quad \left|\tfrac{\partial F}{\partial\lambda}(\omega,\lambda)\right| < 80 \quad \text{on } H_\star. \tag{8.85}$$

Then we fix $n = 70$, $m = 1100$ and consider the discretization $\{(\omega_i, \lambda_j)\}_{i,j}$, $i = 0, \dots, 70$, $j = 0, \dots, 1100$ of $H_\star$ such that

$$\omega_i = \tfrac{93.5}{180}\pi + i\Delta\omega, \quad \lambda_j = 3.030 + j\Delta\lambda,$$

with

$$\Delta\omega = \tfrac{\frac{2}{180}\pi}{n}, \quad \Delta\lambda = \tfrac{0.57}{m}.$$

By straightforward computations it holds that

$$-F(\omega_i, \lambda_j) > 0, \tag{8.86}$$

for all $i = 0, \dots, 70$, $j = 0, \dots, 1100$ and, moreover,

$$\min_{(\omega_i,\lambda_j)\in H_\star} (-F(\omega_i, \lambda_j)) = -F(\omega_{70}, \lambda_{1100}) \approx 0.0773\dots . \tag{8.87}$$

Next to this, we compute

$$c_1 = \max\{\Delta\omega, \Delta\lambda\} = \max\left\{\tfrac{\frac{2}{180}\pi}{n}, \tfrac{0.57}{m}\right\} = \tfrac{0.57}{1100} = 0.000518\dots,$$

and, by taking into account (8.85) and (8.87), we also find

$$c_2 = \sqrt{2}\frac{\min_{(\omega_i,\lambda_j)\in H_\star} (-F(\omega_i,\lambda_j))}{\sup_{H_\star}\left|\left(\frac{\partial F}{\partial \omega}(\omega,\lambda), \frac{\partial F}{\partial \lambda}(\omega,\lambda)\right)\right|} \approx 0.000555\dots .$$

Since $c_1 < c_2$, by Lemma 8.1 we conclude that the constructed discretization $\{(\omega_i, \lambda_j)\}_{i,j}$ of $H_\star$ to be appropriate in the sense that condition (8.74) yields a strict positivity of $-F(\omega,\lambda)$ on $H_\star$, or, in other words $F(\omega,\lambda) < 0$ on $H_\star$. □

9. Explicit formulas to the homogeneous problem in the cone when $\omega \in \left\{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\right\}$

Here we give the explicit solutions to the homogeneous problem (4.1) in $\mathcal{K}_\omega$ when $\omega \in \left\{\frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi\right\}$.

9.1. Case $\omega = \frac{1}{2}\pi$. The eigenvalues λ in this case are determined by the characteristic equation (see subsection 4.1):

$$2 + 2\cos(\pi\lambda) - 4\cos\left(\tfrac{1}{2}\pi\lambda\right) = 0, \quad \lambda \notin \{0, \pm 1\}.$$

The set of positive solutions of the above equation reads as:

$$\{\lambda_j\}_{j=1}^{\infty} = \{\lambda_{3j-2},\ \lambda_{3j-1},\ \lambda_{3j}\}_{j=1}^{\infty} = \{-1+4j,\ 4j,\ 1+4j\}_{j=1}^{\infty}.$$

Here every values $\lambda_{3j-2}, \lambda_{3j}$ has algebraic and geometric multiplicity 1, while λ_{3j-1} has algebraic and geometric multiplicity 2.

j	λ_j	$\kappa^{(j)}$	$u_q^{(j)} = r^{\lambda_j+1}\Phi_q^{(j)}(\theta)$
1	3	1	x^2y^2
2-3	4	2	$\begin{cases} x^3y^2 \\ x^2y^3 \end{cases}$
4	5	1	x^3y^3
5	7	1	$x^6y^2 - x^2y^6$
6-7	8	2	$\begin{cases} x^7y^2 - \frac{7}{3}x^3y^6 \\ x^2y^7 - \frac{7}{3}x^6y \end{cases}$
8	9	1	$x^7y^3 - x^3y^7$
9	11	1	$x^{10}y^2 - 14x^6y^6 + x^2y^{10}$
10-11	12	2	$\begin{cases} x^{11}y^2 - 22x^7y^6 + \frac{11}{3}x^3y^{10} \\ x^2y^{11} - 22x^6y^7 + \frac{11}{3}x^{10}y^3 \end{cases}$
12	13	1	$x^{11}y^3 - \frac{66}{7}x^7y^7 + x^3y^{11}$
etc.			

Table 10. The first three groups of solutions $u_q^{(j)}$, $q = 0, \dots, \kappa^{(j)} - 1$ of problem (4.1) in $\mathcal{K}_\omega$ of measure $\omega = \frac{1}{2}\pi$.

In the table above we see that the functions that solve the homogeneous problem (4.1) in $\mathcal{K}_\omega$ of measure $\omega = \frac{1}{2}\pi$ are given by polynomials in x and y, which makes a difference to the case when the operator of the problem is the bilaplacian Δ^2.

9.2. Case $\omega = \pi$. The eigenvalues λ in this case are determined by the characteristic equation (see subsection 4.1):

$$\sin^2(\pi\lambda) = 0, \quad \lambda \neq \pm 1,$$

plus the values $\lambda = \pm 1$, which are determined by the conditions $P_{-1}(\pi) = P_1(\pi) = 0$. Hence, the set of positive solutions reads as

$$\{\lambda_j\}_{j=1}^{\infty} = j,$$

where $\lambda_1 = 1$ has algebraic and geometric multiplicity 1, while λ_j for $j \geq 2$ has algebraic and geometric multiplicity 2.

j	λ_j	$\kappa^{(j)}$	$u_q^{(j)} = r^{\lambda_j+1}\Phi_q^{(j)}(\theta)$
1	1	1	y^2
$2-3$	2	2	xy^2 y^3
$4-5$	3	2	x^2y^2 xy^3
$6-7$	4	2	x^3y^2 x^2y^3
etc.			

TABLE 11. Some first solutions $u_q^{(j)}$, $q = 0, \dots, \kappa^{(j)} - 1$ of problem (4.1) in $\mathcal{K}_\omega$ of measure $\omega = \pi$.

Here, when the opening angle ω of $\mathcal{K}_\omega$ is π, as well as in the case $\omega = \frac{1}{2}\pi$, the solutions (4.1) in $\mathcal{K}_\omega$ are given by polynomials in x and y. We can not achieve this when the operator L of the problem is the bilaplacian Δ^2.

9.3. **Case $\omega = \frac{3}{2}\pi$.** The characteristic equation $P\left(\frac{3}{2}\pi, \lambda\right) = 0$, $\lambda \neq \pm 1$, where P is given in subsection 4.1, can be factorized and the eigenvalues λ are determined by the system:

$$\cos\left(\tfrac{1}{2}\pi\lambda\right) = 0, \quad \lambda \neq \pm 1,$$
$$\cos\left(\tfrac{1}{2}\pi\lambda\right) = 1, \quad \lambda \neq \pm 1,$$
$$\cos^4\left(\tfrac{1}{2}\pi\lambda\right) + \cos^3\left(\tfrac{1}{2}\pi\lambda\right) - \tfrac{1}{2}\cos^2\left(\tfrac{1}{2}\pi\lambda\right) - \tfrac{1}{2}\cos\left(\tfrac{1}{2}\pi\lambda\right) + \tfrac{1}{16} = 0, \quad \lambda \neq \pm 1.$$

The solutions with positive real part of the system above read, respectively,

$$\{\lambda_{n_1}\}_{n_1=1}^{\infty} = \{1 + 2n_1\}_{n_1=1}^{\infty}, \quad \{\lambda_{n_2}\}_{n_2=1}^{\infty} = \{4n_2\}_{n_2=1}^{\infty},$$
$$\{\lambda_{n_3}\}_{n_3=1}^{\infty} = \{-1 + 2n_3 + (-1)^{n_3}(1-\mu_1)\}_{n_3=1}^{\infty},$$
$$\{\lambda_{n_4}\}_{n_4=1}^{\infty} = \{-1 + 2n_4 + (-1)^{n_4}(1-\mu_2)\}_{n_4=1}^{\infty},$$
$$\{\lambda_{n_5}\}_{n_5=1}^{\infty} = \left\{-1 + 2n_5 + (-1)^{n_5+1}(1-\gamma_1) \pm i\gamma_2\right\}_{n_5=1}^{\infty},$$

where μ_1, μ_2 are the first two positive solutions of the equation

$$s^4 + s^3 - \tfrac{1}{2}s^2 - \tfrac{1}{2}s + \tfrac{1}{16} = 0, \tag{9.1}$$

with $s = \cos\left(\frac{1}{2}\pi\mu\right)$, while (γ_1, γ_2) is the first positive solution of (9.1) with $s = -\cos\left(\frac{1}{2}\pi\gamma_1\right)\cosh\left(\frac{1}{2}\pi\gamma_2\right) + i\sin\left(\frac{1}{2}\pi\gamma_1\right)\sinh\left(\frac{1}{2}\pi\gamma_2\right)$. The numerical approximations (up to three digits) are the following: $\mu_1 \approx 0.536\dots$, $\mu_2 \approx 0.926\dots$ and $\gamma_1 \approx 0.345\dots$, $\gamma_2 \approx 0.179\dots$.

Note also that every $\lambda_{n_2} = 4n_2$, $n_2 = 1, 2.3, \dots$ has algebraic and geometric multiplicity 2, while every λ_{n_k}, for each $k = 1, 3, 4, 5$ has algebraic and geometric multiplicity 1. The set $\{\lambda_j\}_{j=1}^{\infty}$ is the combination of the found sets above.

j	λ_j	$\kappa^{(j)}$	$\Phi_q^{(j)}(\theta)$	$u_q^{(j)} = r^{\lambda_j+1}\Phi_q^{(j)}(\theta)$
1,2,3-4,5-6	$\approx 0.536\dots, \approx 0.926\dots, \approx 1.655\dots \pm i0.179\dots, \approx 2.345\dots \pm i0.179\dots$			
7	3	1	$\sin^2(\theta)\cos^2(\theta)$	x^2y^2
8, 9	$\approx 3.074\dots, \approx 3.464\dots$			
10-11	4	2	$\begin{cases}\sin^2(\theta)\cos^3(\theta)\\ \cos^2(\theta)\sin^3(\theta)\end{cases}$	$\begin{cases}x^3y^2\\ x^2y^3\end{cases}$
12, 13	$\approx 4.536\dots, \approx 4.926\dots,$			
14	5	1	$\sin^3(\theta)\cos^3(\theta)$	x^3y^3
15-16,17-18	$\approx 5.655\dots \pm i0.179\dots, \approx 6.345\dots \pm i0.179\dots,$			
19	7	1	$\sin^2(\theta)\cos^2(\theta)\left(\cos^4(\theta) - \sin^4(\theta)\right)$	$x^6y^2 - x^2y^6$
20, 21	$\approx 7.074\dots, \approx 7.464\dots$			
$22 - 23$	8	2	$\begin{cases}\sin^2(\theta)\cos^3(\theta)\left(\cos^4(\theta) - \frac{7}{3}\sin^4(\theta)\right)\\ \cos^2(\theta)\sin^3(\theta)\left(\sin^4(\theta) - \frac{7}{3}\cos^4(\theta)\right)\end{cases}$	$\begin{cases}x^7y^2 - \frac{7}{3}x^3y^6\\ x^2y^7 - \frac{7}{3}x^6y\end{cases}$
etc.				

TABLE 12. The first solutions $(\lambda_j, \Phi_q^{(j)})$, $q = 0, \dots, \kappa^{(j)} - 1$ of (4.4) and the solutions $u_q^{(j)}$ of (4.1) in $\mathcal{K}_\omega$ of measure $\omega = \frac{3}{2}\pi$. The situation without explicit formula is marked by " $\dots$".

9.4. **Case $\omega = 2\pi$.** The eigenvalues λ in this case are determined by the characteristic equation (see subsection 4.1):

$$\cos^4(\pi\lambda) - \cos^2(\pi\lambda) = 0, \quad \lambda \neq \pm 1,$$

plus the values $\lambda = \pm 1$, which are determined by the conditions $P_{-1}(2\pi) = P_1(2\pi) = 0$. The positive solutions are given by the set

$$\{\lambda_j\}_{j=1}^{\infty} = \tfrac{1}{2}j,$$

where $\lambda_2 = 1$ has algebraic and geometric multiplicity 1, while λ_j for $j = 1$ and $j \geq 3$ has algebraic and geometric multiplicity 2.

Acknowledgements. We thank the anonymous referee for his careful reading of the manuscript.

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j	λ_j	$\kappa^{(j)}$	$\Phi_q^{(j)}(\theta)$	$u_q^{(j)} = r^{\lambda_j+1}\Phi_q^{(j)}(\theta)$
1	$\frac{1}{2}$	2	...	...
2	1	1	$\sin^2(\theta)$	y^2
3	$\frac{3}{2}$	2	...	...
4-5	2	2	$\begin{cases}\sin^2(\theta)\cos(\theta)\\ \sin^3(\theta)\end{cases}$	xy^2 y^3
6	$\frac{5}{2}$	2	...	...
7-8	3	2	$\begin{cases}\sin^2(\theta)\cos^2(\theta)\\ \sin^3(\theta)\cos(\theta)\end{cases}$	x^2y^2 xy^3
9	$\frac{7}{2}$	2	...	...
10-11	4	2	$\begin{cases}\sin^2(\theta)\cos^3(\theta)\\ \sin^3(\theta)\cos^2(\theta)\end{cases}$	x^3y^2 x^2y^3
etc.				

TABLE 13. Some first solutions $(\lambda_j, \Phi_q^{(j)})$, $q = 0, \dots, \kappa^{(j)} - 1$ of (4.4) and the solutions $u_q^{(j)}$ of (4.1) in $\mathcal{K}_\omega$ of measure $\omega = 2\pi$. The situation when the explicit formula unavailable is marked by "...".

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