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EXISTENCE RESULTS AND APPLICATIONS FOR GENERAL VARIATIONAL-HEMIVARIATIONAL INEQUALITIES ON UNBOUNDED DOMAINS

ILDIKÓ-ILONA MEZEI, LIA SĂPLĂCAN

Abstract. In this paper we give an existence result for a class of variational-hemivariational inequality on unbounded domain using the mountain pass theorem and the principle of symmetric criticality for Motreanu-Panagiotopoulos type functionals. Next, we give two applications of the obtained result.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be an unbounded domain with smooth boundary, and $p \in]1, N[$ a real number. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which is locally Lipschitz in the second variable and satisfies some conditions (F1)-(F4), presented in section 3.

We denote by $(X, \|\cdot\|)$ a separable, uniformly convex Banach space with strictly convex topological dual $(X^*, \|\cdot\|_*)$ and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. We assume that X is continuously embedded in $L^p(\Omega)$ for every $p \in]1, N[$ and

- (E) X is compactly embedded in $L^r(\Omega)$, for some $r \in]p, p^*[$, where $p^* = pN/(N-p)$.

We consider the duality mapping $J : X \to X^\star$ corresponding to the weight function $\varphi : [0, +\infty[\to [0, +\infty[$ given by $\varphi(t) = t^{p-1}$. We assume it is also given a functional $\alpha : X \to [0, +\infty]$ which is convex, lower semicontinuous, proper (i.e., $\not\equiv +\infty$) whose effective domain $\mathrm{dom}(\alpha) = \{x \in X : \alpha(x) < +\infty\}$ is a (nonempty, closed, convex) cone in X and satisfies the conditions (A1), (A2) from section 3.

Our aim is to study the following variational-hemivariational inequality

- (P) Find $u \in \mathrm{dom}(\alpha)$ such that
$$\langle Ju, v-u\rangle + \int_\Omega F^0(x, u(x); u(x) - v(x))dx + \alpha(v) - \alpha(u) \geq 0$$
for all $v \in \mathrm{dom}(\alpha)$, where F^0 is the general directional derivative of F (see Definition 2.1).

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The Problem (P), where Ω is a bounded domain of $\mathbb{R}^N$ and α is the indicator function $\psi_{\mathcal{K}}$ associated to the convex, positive cone $\mathcal{K}$; i.e.,

$$\psi_{\mathcal{K}}(x) = \begin{cases} 0, & x \in \mathcal{K} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

has been intensively studied; see e.g. the monographs of Motreanu, Rădulescu [10], Motreanu, Panagiotopoulos [9] and the references therein. The study of hemivariational inequalities, when Ω is an unbounded domain, started with the work by Gazzola, Rădulescu [3], followed by the papers by Kristály [4], [5], Dályai and Varga [2] and Varga [12]. On unbounded domains problems of kind (P) with $\alpha = \psi_{\mathcal{K}}$, were studied by Kristály, Cs. Varga, V. Varga in [6] and Lisei, Varga in [8].

Motreanu, Rădulescu [10, example 9.1] study a hemivariational inequality, when the function α is not the indicator function, but the considered domain is bounded only.

The goal of this article is to extend these results for unbounded domains. In section 3 we prove the main result of the paper, the Theorem 3.5, what ensures the existence of a nontrivial solution of the problem (P). Then using the principle of symmetric criticality proved by Kristály, Cs. Varga, V. Varga in [6], we present in Corollary 3.6 a consequence of our main result, which can be applied in the case, when the embedding condition (E) is not satisfied.

In the last section we present two applications. In the first one, the condition (E) is satisfied, so we can use the Theorem 3.5, but in the second one, (E) is not fulfilled. In this case we use the Corollary 3.6.

2. Preliminaries

Let Ω, X, F and J be the spaces and functions introduced in the previous section. We denote by $\|\cdot\|_p$ and $\|\cdot\|_r$ the norms on $L^p(\Omega)$ and $L^r(\Omega)$, respectively, i.e.

$$\|u\|_p = \Big(\int_\Omega |u(x)|^p dx \Big)^{1/p}.$$

The symbols $C(p)$ and $C(r)$ stand for the best embedding constants satisfying $\|u\|_p \leq C(p)\|u\|$ and $\|u\|_r \leq C(r)\|u\|$ for all $u \in X$. We recall the following properties of the duality mapping J:

$$\|Ju\|_\star = \varphi(\|u\|), \quad \langle Ju, u\rangle = \|Ju\|_\star \|u\| \quad \text{for all } u \in X,$$

as well as that $J = d\chi$, where $d\chi$ denotes the Gâteaux differential of the convex and Gâteaux differentiable functional $\chi : X \to \mathbb{R}$ defined by $\chi(u) = \frac{1}{p}\|u\|^p$. For these properties of the duality mapping we refer to [1].

Now, we give definitions and basic properties from the theory of generalized differentiation for locally Lipschitz functions; see [1, 9].

Definition 2.1. Let $h : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of h at the point $u \in X$ in the direction $v \in X$ is

$$h^0(u, v) = \limsup_{z \to u,\ t \to 0^+} \frac{h(z + tv) - h(z)}{t}.$$

The generalized gradient of h at $u \in X$ is the set

$$\partial h(u) = \{u^* \in X : \langle u^*, v\rangle \leq h^0(u, v), \quad \forall v \in X\}.$$

Lemma 2.2. *Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then for all $u \in X$ we have*

- (i) $h^0(u, \cdot)$ *is subadditive, positively homogeneous;*
- (ii) $h^0(u, v) = max\{\langle \xi, v\rangle \ : \ \xi \in \partial h(u)\}, \quad \forall v \in X;$
- (iii) $(h + g)^0(u, v) \le h^0(u, v) + g^0(u, v), \quad \forall v \in X;$
- (iv) $(-h)^0(u, v) = h^0(u, -v), \quad \forall v \in X.$

Let $\mathcal{I} : X \to]-\infty, +\infty]$, $\mathcal{I} = h + \psi$, where $h : X \to \mathbb{R}$ is a locally Lipschitz function and $\psi : X \to]-\infty, +\infty]$ is convex, proper and lower semicontinuous. $\mathcal{I}$ is a Motreanu-Panagiotopoulos type functional, see [9, Chapter 3].

Definition 2.3. An element $u \in X$ is a critical point of $\mathcal{I} = h + \psi$, if

$$h^0(u, v - u) + \psi(v) - \psi(u) \ge 0, \forall v \in X.$$

Definition 2.4. Let X be a Banach space and $\mathcal{I} : X \to]-\infty, +\infty]$, $\mathcal{I} = h + \psi$ a Motreanu-Panagiotopoulos type functional. $\mathcal{I}$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ (for short $(PS)_c)$), if every sequence u_n in X satisfying $\mathcal{I}(u_n) \to c$ and

$$h^0(u_n, v - u_n) + \psi(v) - \psi(u_n) \ge -\epsilon_n \|v - u_n\|, \forall v \in X$$

for a sequence ϵ_n in $[0, \infty)$ with $\epsilon_n \to 0$, contains a convergent subsequence.

We recall now the nonsmooth version of the Mountain Pass Theorem, [9, Corollary 3.2].

Theorem 2.5. *Assume that the functional $\mathcal{I} : X \to]-\infty, +\infty]$ defined by $\mathcal{I} = h+\psi$, satisfies (PS), $\mathcal{I}(0) = 0$, and*

- (i) *there exist constants $a > 0$ and $\rho > 0$, such that $\mathcal{I}(u) \ge a$ for all $\|u\| = \rho$;*
- (ii) *there exists $e \in X$, with $\|e\| > \rho$ and $\mathcal{I}(e) \le 0$.*

Then the number

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(f(t)),$$

is a critical value of $\mathcal{I}$ with $c \ge a$, where

$$\Gamma = \{f \in C([0, 1], X) : f(0) = 0,\ f(1) = e\}.$$

In the sequel we introduce the definitions used in the principle of symmetric criticality.

Let G be a topological group which acts linearly on X, i.e., the action $G \times X \to X : \ [g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle_X = \langle x^*, g^{-1}u\rangle_X$ for every $g \in G$, $u \in X$ and $x^* \in X^*$. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is G-invariant, if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is G-invariant, if $gK = \{gu : u \in K\} \subseteq K$ for every $g \in G$. Let

$$X^G = \{u \in X : gu = u \text{ for every } g \text{ in } G\}$$

be the fixed point set of X under G.

Now, we are in the position to state the Principle of Symmetric Criticality for Motreanu-Panagiotopoulos functionals (see [6, 8]), which is a very useful tool in studying the solutions of variational-hemivariational inequalities.

Theorem 2.6. *Let X be a reflexive Banach space and $\mathcal{I} = h + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Motreanu-Panagiotopoulos type functional. If a compact group G acts linearly on X, and the functionals h and ψ are G-invariant, then every critical point of $\mathcal{I}|_{X^G}$ is also a critical point of $\mathcal{I}$.*

3. Main Result

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which is locally Lipschitz in the second variable and verifies the following conditions:

(F1) $F(x, 0) = 0$ for a.e. $x \in \Omega$ and there exists a constant $c_1 > 0$ such that $|\xi| \leq c_1(|s|^{p-1} + |s|^{r-1})$ whenever $\xi \in \partial F(x, s)$ with $(x, s) \in \Omega \times \mathbb{R}$.
(F2) $\lim_{s\to 0} \max\{|\xi| : \xi \in \partial F(x, s)\}/|s|^{p-1} = 0$ uniformly for $x \in \Omega$.
(F3) There exists a constant $\nu \in]p, p^*[$ such that

$$\nu F(x, s) + F^0(x, s; -s) \leq 0 \quad \text{for all } (x, s) \in \Omega \times \mathbb{R}.$$

(F4) There exists a constant $R > 0$ such that

$$c_R =: \inf\{F(x, s) : (x, |s|) \in \Omega \times [R, +\infty)\} > 0.$$

Here we have denoted by $\partial F(x, s)$ and $F^0(x, s; \cdot)$ the generalized gradient and the generalized directional derivative of $F(x, \cdot)$ at the point $s \in \mathbb{R}$, respectively.

Arguing as in [4, Lemma 4], we have the following result.

Remark 3.1. If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (F1)-(F2), then for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

(i) $|\xi| \leq \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}$ for all $\xi \in \partial F(x, s)$ with $(x, s) \in \Omega \times \mathbb{R}$;
(ii) $|F(x, s)| \leq \varepsilon|s|^p + c(\varepsilon)|s|^r$ for all $(x, s) \in \Omega \times \mathbb{R}$.

For a later use we state an other preliminary result.

Lemma 3.2. *If the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1), (F3), (F4)*, then there exist positive constants c_2, c_3 such that*

$$F(x, s) \geq c_2|s|^\nu - c_3|s|^p \quad \textit{for all } (x, s) \in \Omega \times \mathbb{R}.$$

Proof. For a fixed $(x, s) \in \Omega \times \mathbb{R}$ we consider the locally Lipschitz function $g : [1, +\infty[\to \mathbb{R}$ defined by $g(t) = t^{-\nu} F(x, ts)$. Given any $t > 1$, by applying the Lebourg's mean value theorem for the locally Lipschitz function g on the interval [1,t] and using condition (F3), we obtain the estimate $F(x, ts) \geq t^\nu F(x, s)$. Combining with (F4) we obtain the following inequality

$$F(x, s) \geq \frac{c_R}{R^\nu}|s|^\nu \quad \text{for all } (x, s) \in \Omega \times [R, +\infty[. \tag{3.1}$$

On the other hand, (F1) yields

$$\begin{aligned} F(x, s) &\geq -\frac{c_1}{p}(|s|^p + |s|^r) \\ &\geq -\frac{c_1}{p}(1 + |s|^{r-p})|s|^p + \frac{c_1}{p}|s|^\nu - \frac{c_1}{p}R^{\nu-p}|s|^p \\ &\geq \frac{c_1}{p}|s|^\nu - \frac{c_1}{p}(1 + R^{r-p} + R^{\nu-p})|s|^p \quad \text{for all } (x, s) \in \Omega \times [0, R]. \end{aligned}$$

Taking into account (3.1) we get the desired conclusion. □

Due to the growth condition (F1), the functional $\mathcal{F} : X \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))dx \quad \text{for all } u \in X$$

is locally Lipschitz and the generalized directional derivative $(-\mathcal{F})^0$ of $-\mathcal{F}$ satisfies

$$(-\mathcal{F})^0(u, v) \le \int_\Omega (-F)^0(x, u(x); v(x))dx \quad \text{for all } u, v \in X; \tag{3.2}$$

see [1].

Let $\alpha : X \to [0, +\infty]$ be a functional which is convex, lower semicontinuous, proper, whose effective domain $\text{dom}(\alpha) = \{x \in X : \alpha(x) < +\infty\}$ is a cone in X. In addition, we assume that the following conditions hold:

- (A1) $\alpha(0) = 0$ and there exist constants σ, a_1, a_2 such that
$$\big(1 + \frac{1}{\nu}\big)\alpha(v) - \frac{1}{\nu}\alpha(2v) \ge -a_1\|v\|^\sigma - a_2 \quad \text{for all } v \in \text{dom}(\alpha),$$
where either $0 \le \sigma < p$ and $a_1, a_2 \in [0, +\infty)$, or $\sigma = p$ and $0 \le a_1 < \frac{1}{p} - \frac{1}{\nu}$, $a_2 \ge 0$.
- (A2) There exists $u_0 \in X \cap L^\nu(\Omega)$ such that
$$\liminf_{t\to\infty} \frac{\alpha(tu_0)}{t^\nu\|u_0\|_\nu^\nu} < c_2.$$

Consider the functional $\mathcal{I} : X \to]-\infty, +\infty]$ given by

$$\mathcal{I}(u) = \frac{1}{p}\|u\|^p - \mathcal{F}(u) + \alpha(u) \quad \forall u \in X.$$

The next lemma points out the relationship between its critical points and the solutions of Problem (P).

Lemma 3.3. *Every critical point of the functional $\mathcal{I}$ is a solution of Problem* (P).

Proof. Let $u \in X$ be a critical point of $\mathcal{I}$, then using the definition 2.3 and the properties of the duality mapping J, we get

$$\langle Ju, v - u\rangle + (-\mathcal{F})^0(u; v - u) + \alpha(v) - \alpha(u) \ge 0, \quad \forall v \in X.$$

In view of (3.2) and using the lemma 2.2(iv), we deduce the result. □

Proposition 3.4. *Assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions* (F1)-(F3) *and $\alpha : X \to [0, +\infty]$ is a convex, lower semicontinuous proper function which fulfills* (A1). *Then $\mathcal{I}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*

Proof. Fix $c \in \mathbb{R}$ and let $(u_n) \subset \text{dom}(\alpha)$ be a sequence such that

$$\mathcal{I}(u_n) = \frac{1}{p}\|u_n\|^p - \mathcal{F}(u_n) + \alpha(u_n) \to c \tag{3.3}$$

and for every $v \in \text{dom}(\alpha)$ holds

$$\langle Ju_n, v-u_n\rangle + \int_\Omega F^0(x, u_n(x); u_n(x)-v(x))dx + \alpha(v) - \alpha(u_n) \ge -\varepsilon_n\|v-u_n\|, \tag{3.4}$$

with a sequence

$$\varepsilon_n \searrow 0, \quad \text{as } n \to \infty. \tag{3.5}$$

Setting $v = 2u_n$ in (3.4), we get

$$\langle Ju_n, u_n\rangle + \int_\Omega F^0(x, u_n(x); -u_n(x))dz + \alpha(2u_n) - \alpha(u_n) \geq -\varepsilon_n\|u_n\|. \tag{3.6}$$

We infer from (3.3) that for large $n \in \mathbb{N}$ one has

$$c + 1 \geq \frac{1}{p}\|u_n\|^p - \mathcal{F}(u_n) + \alpha(u_n). \tag{3.7}$$

Combining the inequalities (3.6) and (3.7), and using the conditions (F3) and (A1), we get

$$\begin{aligned} c + 1 + \frac{\varepsilon_n}{\nu}\|u_n\| &\geq \big(\frac{1}{p} - \frac{1}{\nu}\big)\|u_n\|^p - \frac{\alpha(2u_n)}{\nu} + \big(1 + \frac{1}{\nu}\big)\alpha(u_n) \\ &\quad - \int_\Omega [F(x, u_n(x)) + \frac{1}{\nu}F^0(x, u_n(x); -u_n(x))]dx \\ &\geq \big(\frac{1}{p} - \frac{1}{\nu}\big)\|u_n\|^p - a_1\|u_n\|^\sigma - a_2. \end{aligned}$$

This estimate ensures that the sequence $\{u_n\}$ is bounded in X. Since X is reflexive, it follows that there exists an element $u \in \mathrm{dom}(\alpha)$ such that $\{u_n\}$ has a subsequence (denoted also by $\{u_n\}$) such that u_n converges weakly to u in X. Using the embedding condition (E), namely that X is compactly embedded in $L^r(\Omega)$, we have that $u_n \to u$ in $L^r(\Omega)$, which is

$$\|u_n - u\|_r \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Replacing $v = u$ in (3.4), we have

$$\langle Ju_n, u - u_n\rangle + \int_\Omega F^0(z, u_n(z); u_n(z) - u(z))dz + \alpha(u) - \alpha(u_n) \geq -\varepsilon_n\|u_n - u\|. \tag{3.9}$$

Using the inequality (3.9), the Remark 3.1 (i), the Hölder's inequality and the continuity of the inclusion $X \hookrightarrow L^p(\Omega)$, for any $\varepsilon > 0$ we derive

$$\begin{aligned} \langle Ju_n, u_n - u\rangle &\leq \varepsilon C(p)\|u_n - u\|\|u_n\|_p^{p-1} + c(\varepsilon)\|u_n - u\|_r\|u_n\|_r^{r-1} \\ &\quad + \alpha(u) - \alpha(u_n) + \varepsilon_n\|u_n - u\|. \end{aligned}$$

Due to the sequentially weakly lower semicontinuity of α, to the arbitrary choice of $\varepsilon > 0$, to the convergence (3.5) and (3.8), we deduce that

$$\limsup_{n\to\infty}\langle Ju_n, u_n - u\rangle \leq 0.$$

Taking into account that the duality operator J has the (S_+) property (see[11, Proposition 2.1]), namely

> If $\{u_n\}$ converges weakly in X to u and $\limsup_{n\to\infty}\langle Ju_n, u_n - u\rangle \leq 0$, then $u_n \to u$ in X,

we conclude that $u_n \to u$ in X, which completes the proof. □

Now we state the main result of this paper.

Theorem 3.5. *Assume that the condition* (E) *holds and the functions* $F : \Omega\times\mathbb{R} \to \mathbb{R}$ *and* $\alpha : X \to [0, +\infty]$ *satisfy* (F1)-(F4), (A1)-(A2). *Then the Problem* (P) *has a nontrivial solution.*

Proof. According to Lemma 3.3 it is sufficient to prove the existence of a critical point of functional $\mathcal{I}$. For this, we check if the functional $\mathcal{I}$ satisfies the conditions of the Mountain Pass Theorem 2.5, in order to apply it.

First, we note that Proposition 3.4 guarantees that $\mathcal{I}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

We claim that there exist constants $a > 0$ and $\rho > 0$ such that $\mathcal{I}(u) \geq a$ for all $\|u\| = \rho$. In order to justify this, let us fix some $\varepsilon \in]0, \frac{1}{pC(p)^p}[$. By Remark 3.1(ii), by the continuity of the embedding $X \hookrightarrow L^p(\Omega)$ and because $\alpha \geq 0$, we have

$$\begin{aligned}
\mathcal{I}(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(x, u(x))dx + \alpha(u_0) \\
&\geq \frac{1}{p}\|u\|^p - \varepsilon \int_\Omega |u(x)|^p dx - c(\varepsilon) \int_\Omega |u(x)|^r dx + \alpha(u_0) \\
&\geq \Big(\frac{1}{p} - \varepsilon C(p)^p\Big)\|u\|^p - c(\varepsilon)C(r)^r\|u\|^r \quad \text{for all } u \in X.
\end{aligned}$$

Using the notation $A = \frac{1}{p} - \varepsilon C(p)^p$ and $B = c(\varepsilon)C(r)^r$, we see that the function $g : \mathbb{R}_+ \to \mathbb{R}$ given by $g(t) = At^p - Bt^r$ attains its global maximum at the point $t_M = \big(\frac{pA}{rB}\big)^{\frac{1}{r-p}}$. The claim follows taking $\rho = t_M$ and any $a \in]0, g(t_M)[$.

Now, we have to check if there exists an $e \in X$ with $\|e\| > \rho$ and $\mathcal{I}(e) \leq 0$. Using the element $u_0 \in dom(\alpha)$ supplied by assumption $(A2)$, from Lemma 3.2 it turns out that for every $t > 0$ we may write

$$\begin{aligned}
\mathcal{I}(tu_0) &= \frac{1}{p}\|u_0 t\|^p - \mathcal{F}(tu_0) + \alpha(u_0) \\
&= \frac{1}{p}\|u_0\|^p|t|^p - \int_\Omega F(x, tu_0(x))dx + \alpha(u_0) \\
&\leq \frac{1}{p}\|u_0\|^p|t|^p - \int_\Omega \left(c_2|tu_0(x)|^\nu - c_3|tu_0(x)|^p\right) dx + \alpha(u_0) \\
&= \frac{1}{p}\|u_0\|^p|t|^p - c_2|t|^\nu\|u_0\|_\nu^\nu + c_3|t|^p\|u_0\|_p^p + \alpha(u_0).
\end{aligned}$$

Now, using the continuity of the embedding $X \hookrightarrow L^p(\Omega)$, we obtain

$$\mathcal{I}(tu_0) \leq t^\nu\Big(\big(\frac{1}{p} + c_3C(p)^p\big)\|u_0\|^p t^{p-\nu} + \frac{\alpha(tu_0)}{t^\nu} - c_2\|u_0\|_\nu^\nu\Big) \tag{3.10}$$

It is known from (A2), that a sequence $\{t_n\}$ can be found, such that $t_n \to +\infty$ and

$$\lim_{n\to\infty} \frac{1}{t_n^\nu}\alpha(t_n u_0) < c_2\|u_0\|_\nu^\nu. \tag{3.11}$$

Now, setting $e = t_n u_0$ for any n sufficiently large, we obtain that $\mathcal{I}(e) \leq 0$, since the inequalities (3.10), (3.11) hold and $\nu > p$.

The above claims ensure that Theorem 2.5 can be applied to the functional $\mathcal{I}$, and so the number

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(f(t)),$$

is a critical value of $\mathcal{I}$ with $c \geq a$, where $\Gamma = \{f \in C([0,1], X) : f(0) = 0,\ f(1) = e\}$.

The desired conclusion is thus established. □

In many situations the condition (E) is not satisfied, but if G is a compact topological group which acts linearly on X, then the following embedding is compact:

(EG) $X^G \hookrightarrow L^r(\Omega)$ for some $r \in]p, p^\star[$.

Now, let G be the compact topological group $O(N)$ or a subgroup of $O(N)$. Suppose that G acts continuously and linearly on the Banach space X and the following conditions hold:

(FG) $F(gx, s) = F(x, s)$ for every $g \in G, x \in \Omega$ and $s \in \mathbb{R}$.
(AG) $\alpha(gu) = \alpha(u)$ for every $g \in G, u \in X$.

By combining Theorems 3.5 and 2.6 we obtain the following result.

Corollary 3.6. *Let G be the compact topological group $O(N)$ or a subgroup of $O(N)$. We suppose that G acts continuously and linearly on the separable and reflexive Banach space X. If the conditions* (EG), (F1)-(F4), (A1)-(A2), (FG), (AG) *hold, then the problem* (P) *has a nontrivial radial solution.*

4. Applications

In this section we give two applications. In the first one, we state an existence result for a general variational-hemivariational inequality using weighted Sobolev spaces and the Theorem 3.5. In the second application we apply the Corollary 3.6 for a variational-hemivariational inequality on strip-like domains.

4.1. **Weighted Sobolev spaces.** Let Ω be $\mathbb{R}^N$ or an unbounded domain in $\mathbb{R}^N$ ($N \geq 2$) with C^1 boundary, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function, which is locally Lipschitz in the second variable and satisfy the conditions (F1)-(F4).

Let $V : \Omega \to \mathbb{R}$ be a continuous potential satisfying the following conditions:

(V1) $\inf_\Omega V > 0$;
(V2) for every $M > 0$ the set $\{x \in \Omega : V(x) \leq M\}$ has finite Lebesgue measure

(note that in particular, condition (V2) is fulfilled whenever V is coercive).

For $p \in]1, N[$ we introduce the space

$$X = \big\{u \in W^{1,p}(\Omega) : \int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx < \infty\big\}$$

endowed with the norm

$$\|u\| = \Big(\int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx\Big)^{1/p}.$$

We denote by $\mathcal{K} = \{u \in X : u \geq 0\}$ and let $\alpha : X \to [0, +\infty]$,

$$\alpha(v) = \begin{cases} \|v\|^p, & v \in \mathcal{K} \\ +\infty, & v \notin \mathcal{K} \end{cases} \tag{4.1}$$

be a function, which is convex, lower semicontinuous and $\mathrm{dom}(\alpha) = \mathcal{K}$. With the definitions above, our Problem (P) becomes

(P1) Find $u \in \mathrm{dom}(\alpha)$ such that

$$\begin{aligned} &\int_\Omega \big(|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x) - u(x)) + V(x)|u(x)|^{p-1}(v(x) - u(x))\big)\,dx \\ &+ \int_\Omega F^0(x, u(x); u(x) - v(x))dx + \alpha(v) - \alpha(u) \geq 0, \quad \forall v \in \mathrm{dom}(\alpha). \end{aligned}$$

The conditions (V1), (V2) guarantee that the space X is compactly embedded in $L^r(\Omega)$ for every $r \in]p, p^\star[$.

It is easy to check, that if $N < p(2^p-1)/(2^p-p-1)$, then the number ν from the condition (F1) is less than p^*. If we suppose in addition that $\nu > 2^p - 1$, then α satisfies the conditions (A1), (A2). So using Theorem 3.5, we can state the following existence result:

Corollary 4.1. *If $\nu > 2^p - 1$, $N < p(2^p-1)/(2^p-p-1)$ and the conditions* (V1), (V2), (F1)-(F4) *are fulfilled, then the problem* (P1) *has a nontrivial nonnegative solution.*

4.2. **Variational-hemivariational inequality on strip-like domains.** In this section we consider stripe-like domains of the form $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega \subset \mathbb{R}^m (m \geq 1)$ is an open bounded set, and $N - m \geq 2$. For $p > 2$ we consider the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm $\|u\| = \big(\int_\Omega |\nabla u|^p\big)^{1/p}$ and the cone $\mathcal{K}$ of positive functions, i.e. $\mathcal{K} = \{u \in W_0^{1,p}(\Omega) : u \geq 0\}$.

Let $\alpha : X \to [0, +\infty]$ be the function defined by relation (4.1). α is convex, lower semicontinuous and $\operatorname{dom}(\alpha) = \mathcal{K}$.

Now, the problem (P) becomes

(P2) Find $u \in \operatorname{dom}(\alpha)$ such that

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x))dx + \int_\Omega F^0(x,u(x);u(x)-v(x))dx\\ &+ \alpha(v) - \alpha(u) \geq 0, \quad \forall\, v \in \operatorname{dom}(\alpha),\end{aligned}$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is the Carathéodory function from in section 3.

In this case the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in [p, p^\star]$ is only continuous. So, Theorem 3.5 cannot be applied. For this, we consider the compact topological group $G = \mathrm{id}^m \times O(N-m)$, where $O(N-m)$ is the orthogonal group of $\mathbb{R}^{N-m}$. An element $u \in W_0^{1,p}(\Omega)$ is axially symmetric if $u(x, gy) = u(x, y)$ for all $x \in \omega, y \in \mathbb{R}^{N-m}$ and $g \in O(N-m)$. We introduce the action of $G = \mathrm{id}^m \times O(N-m)$ on $W_0^{1,p}(\Omega)$ as

$$gu(x,y) = u(x, g^{-1}y)$$

for all $(x,y) \in \Omega, g = \mathrm{id}^m \times g_0 \in G$ and $u \in W_0^{1,p}(\Omega)$. Moreover, the action G on $W_0^{1,p}(\Omega)$ is isometric, that is $\|gu\| = \|u\|$ for all $g \in G, u \in W_0^{1,p}(\Omega)$. Let us denote by

$$W_{0,G}^{1,p}(\Omega) = \{u \in W_0^{1,p}(\Omega) : gu = u \text{ for all } g \in G\}.$$

The embeddings $W_{0,G}^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $r \in]p, p^\star[$ are compact (see [7]), therefore the condition (EG) is satisfied. The function α is G-invariant, so the condition (AG) is fulfilled.

If we suppose in addition that $N < p(2^p-1)/(2^p-p-1)$ and $\nu > 2^p - 1$, then the conditions (A1), (A2) are satisfied too. Hence using the Corollary 3.6 we have the following result:

Corollary 4.2. *If $\nu > 2^p - 1$, $N < p(2^p-1)/(2^p-p-1)$ and the conditions* (F1)-(F4), (FG) *are fulfilled, then the problem* (P2) *has a nontrivial nonnegative solution.*

We remark that the corollaries 4.1, 4.2 are the first applications for variational-hemivariational inequalities defined on unbounded domain, where the function α is different from the indicator function of the convex cone $\mathcal{K}$.

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis.* Wiley, 1983.

[2] Zs. Dályai, Cs. Varga, *An existence result for hemivariational inequalities*, Electronic J. Differential Equations, **37**(2004), 1-17.

[3] F. Gazzola, V. Rădulescu, *A nonsmooth critical point theory approach to some nonlinear elliptic equations in* $\mathbb{R}^n$, Differential Integral Equations, **13**(2000), 47-60.

[4] A. Kristály, *Infinitely many radial and non-radial solutions for a class of hemivariational inequalities*, Rocky Mountain J. Math., **35**(2005), 1173-1190.

[5] A. Kristály, *Multiplicity results for an eigenvalue problem for hemivariational inequalities in strip-like domains*, Set-Valued Analysis, **13**(2005), 85-103.

[6] A. Kristály, Cs. Varga, V. Varga, *A nonsmooth principle of symmetric criticality and variational-hemivariational inequalities*, Journal of Mathematical Analysis and Applications, **325**(2007), no. 2, 975-986.

[7] P.-L. Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal., **49**(1982), 312-334.

[8] H. Lisei, Cs. Varga, *Some Applications to Variational-Hemivariational Inequalities of the Principle of Symmetric Criticality for Motreanu-Panagiotopoulos Type Functionals*, Journal of Global Optimization, **36** (2006), No. 2, pp. 283-305.

[9] D. Motreanu, P. D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities,* Kluwer Academic Publishers, Dordrecht, 1999.

[10] D. Motreanu, V. Rădulescu, *Variational and Non-Variational Methods in Nonlinear Analysis and Boundary Value Problems*, Kluwer Academic Publishers, Boston/Dordrecht/London, 2003.

[11] P. De Nápoli, M.C. Mariani, *Mountain pass solutions to equations of p-Laplacian type*, Nonlinear Analysis , **54**(2003), 1205-1219.

[12] Cs. Varga, *Existence and infinitely many solutions for an abstract class of hemivariational inequalities*, J. Inequal. Appl., **2** (2005), 89-105.

Ildikó-Ilona Mezei

Faculty of Mathematics and Computer Science, University of Babeş-Bolyai, str. M. Kogalniceanu 1, 400084 Cluj Napoca, Romania

E-mail address: ildiko.mezei@math.ubbcluj.ro, mezeiildi@yahoo.com

Lia Săplăcan

Petru Rareş High School, str. Obor 83, 425100 Beclean, Romania

E-mail address: liasaplacan@yahoo.com