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NONNEGATIVE SOLUTIONS TO AN INTEGRAL EQUATION AND ITS APPLICATIONS TO SYSTEMS OF BOUNDARY VALUE PROBLEMS

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Abstract. We study the existence of positive eigenvalues yielding nonnegative solutions to an integral equation. Also we study the positivity of solutions on specific sets. These results are obtained by using a fixed point theorem in cones and are illustrated by application to systems of boundary value problems.

1. Introduction

In this paper we study the existence of positive eigenvalues that yield nonnegative solutions to the integral equation

$$u(t) = \lambda \int_0^1 k_1(t,s)a(s)f\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds, \quad 0 \le t \le 1, \tag{1.1}$$

under the following assumptions:

(A) $f,\, g \in C([0,\infty),[0,\infty))$,
(B) $a, b \in C([0,1],[0,\infty))$, and each does not vanish identically on any subinterval of $[0,1]$,
(C) $k_i(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,2$ are continuous functions and there are points $\xi,\, \eta \in [0,1]$ with $\xi < \eta$ for which $\max_{\xi \le r \le \eta}[\min_{\xi \le t \le \eta} k_i(t,r)] > 0$, $i = 1,2$, and positive numbers γ_i, $i = 1,2$ such that
$$\min_{\xi \le r \le \eta} k_i(r,s) \ge \gamma_i k_i(t,s) \quad \text{for } (t,s) \in [0,1]^2,\ i = 1,2.$$

Throughout this paper we will use the notation
$$\gamma = \min\{\gamma_1, \gamma_2\}.$$
Clearly from (C) we have $\gamma_1, \gamma_2 \in (0,1]$ and so $\gamma \in (0,1]$.

A (nonnegative) *solution* of (1.1) is a function u in $C([0,1],[0,\infty))$ that satisfies (1.1) for all $t \in [0,1]$. A solution u will be called *positive on the set* $J \subseteq [0,1]$ if $u(t) > 0$ for all $t \in J$.

The present work is motivated by some recent results on the existence of positive solutions to systems of boundary value problems (BVP, for short) (see, [2] - [12], [26], [30], [31], [34], [35]). The study on the existence of positive solutions

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to BVP was initiated mainly by the work of Il'in and Moiseev (see, [16]). Since then, existence of positive solutions to boundary value problems have attracted the attention of many researches resulting in the publishing of a considerable number of papers on problems concerning differential equations. For some recent results on BVP for differential equations we refer to [19], [22], [27], [29], [32] (for second order equations), to [15], [21], [23] (for third order equations), to [25] (for fourth order equations), to [1], [13], [18], [20], [33] (for higher order equations), while for some results on BVP concerning equations on time scales we refer to [24] and the references cited therein. However the majority of the results obtained concern mainly BVP refering to a single differential equation along various types of boundary conditions and only very recently this study has been expanded to systems of BVP. In this paper we investigate the existence of positive eigenvalues yielding nonnegative solutions to an integral equations which includes, as special cases, a variety of systems of BVP (see, the applications in Section 4). Thus, we may apply our results to a variety of systems of BVP to obtain generalizations and extensions of several known results as well as to establish new results for systems of BVP which have not yet been considered as, for example, a mixed system considered in Section 4. For some existence results concerning integral equations and which are close to the results of this paper we refer to [17]. The main tool in this investigation is a fixed point theorem in cones and the technique used may be viewed as an extended version of the one developed in [28].

The paper is organized in six sections. Section 2 consists of some preliminary results needed for the proof of the main results of the paper which are given in Section 3. In Section 4 we discuss the positivity of a solution on a specific set (this notion has already been introduced in this section) and make comments concerning the main results of the paper as well as the assumptions posed on the functions involved (1.1). Section 5 is devoted to the application of the main results of the paper to systems of boundary value problems. Some of the results obtained in Section 5 are new while some others extend and generalize already known results. The last section of the paper, Section 6, contains a generalization of the main results of the paper to an integral equation which is more general than (1.1), and an application of these results to a system of n boundary-value problems.

2. Preliminaries

For our investigation we consider the set $\mathcal{B} = C([0,1],\mathbb{R})$ equipped with the usual supremum norm $\|\cdot\|$, and its subset $\mathcal{B}^+ = C([0,1],\mathbb{R}^+)$. Furthermore, we define the set $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \big\{x \in \mathcal{B} : x(t) \geq 0 \text{ on } [0,1] \text{ and } \min_{t\in[\xi,\eta]} x(t) \geq \gamma\|x\|\big\}. \tag{2.1}$$

Clearly, $(\mathcal{B},\|\cdot\|)$ is a Banach space and $\mathcal{P}$ is a cone in $\mathcal{B}$. Let $\mathcal{T} : \mathcal{B}^+ \to \mathcal{B}$ be the integral operator defined by

$$\mathcal{T}u(t) := \lambda \int_0^1 k_1(t,s)a(s)f\Big(\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds, \quad u \in \mathcal{P}. \tag{2.2}$$

Now we state a useful observation concerning the image of the operator $\mathcal{T}$.

Lemma 2.1. *Let λ,μ be positive numbers and $\mathcal{P}$ be the cone defined by* (2.1).

(i) If $u \in \mathcal{B}^+$ and $v : [0,1] \to [0,\infty)$ is defined by

$$v(t) = \mu \int_0^1 k_2(t,r)b(r)g(u(r))dr, \quad t \in [0,1], \tag{2.3}$$

then $v \in \mathcal{P}$.

(ii) If $\mathcal{T}$ is the integral operator defined by (2.2), then $\mathcal{T}(\mathcal{B}^+) \subset \mathcal{P}$. In particular, $T(\mathcal{P}) \subset \mathcal{P}$.

Proof. Let μ be a positive number, u be an arbitrary element in $\mathcal{B}^+$ and v be defined by (2.3).

(i) By the nonnegativity of k_2, b and g it follows that $v(t) \geq 0$, $t \in [0,1]$. In view of (A), (B), we have

$$k_2(s,r) \geq \min_{s\in[\xi,\eta]} k_2(s,r), \quad s \in [\xi,\eta], r \in [0,1]$$

and

$$\int_0^1 k_2(s,r)b(r)g(u(r))dr \geq \int_0^1 \min_{s\in[\xi,\eta]} k_2(s,r)b(r)g(u(r))dr, \quad s \in [\xi,\eta]$$

from which we take

$$\min_{s\in[\xi,\eta]} \int_0^1 k_2(s,r)b(r)g(u(r))dr \geq \int_0^1 \min_{s\in[\xi,\eta]} k_2(s,r)b(r)g(u(r))dr.$$

Consequently, employing (C) we have for $t \in [0,1]$ and $s \in [\xi,\eta]$

$$\begin{aligned}\int_0^1 k_2(s,r)b(r)g(u(r))dr &\geq \int_0^1 \min_{s\in[\xi,\eta]} k_2(s,r)b(r)g(u(r))dr \\ &\geq \int_0^1 \gamma_2 k_2(t,r)b(r)g(u(r))dr,\end{aligned}$$

hence, in view of the fact that $\gamma_2 \geq \min\{\gamma_1,\gamma_2\} = \gamma$ and $\mu > 0$ we take

$$\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr \geq \gamma\mu \int_0^1 k_2(t,r)b(r)g(u(r))dr, \tag{2.4}$$

for $s \in [\xi,\eta]$, and $t \in [0,1]$.

Since (2.4) is true for any $s \in [\xi,\eta]$ and any $t \in [0,1]$, it follows that

$$\min_{s\in[\xi,\eta]} v(s) \geq \gamma v(t) \quad t \in [0,1],$$

and so $\min_{s\in[\xi,\eta]} v(s) \geq \gamma\|v\|$, which proves our assertion.

(ii) From (i) we have that $v \in \mathcal{P}$, and so, as $k_1, a,\ f$ and v are nonnegative and $\lambda > 0$, following arguments similar to the ones used for the proof of (2.4), one has

$$\min_{s\in[\xi,\eta]} \int_0^1 k_1(s,r)a(r)f(v(r))dr \geq \gamma\Big[\int_0^1 k_1(t,r)a(r)f(v(r))dr\Big], \quad t \in [0,1],$$

that is,

$$\min_{s\in[\xi,\eta]} \mathcal{T}u(s) \geq \gamma\mathcal{T}u(t) \quad \text{for } t \in [0,1],$$

and so,

$$\min_{s\in[\xi,\eta]} \mathcal{T}u(s) \geq \gamma\|\mathcal{T}u\|,$$

which shows that $\mathcal{T}u \in \mathcal{P}$ and completes the proof. $\square$

From Lemma 2.1 and the definition of $\mathcal{T}$ we have immediately the following result.

Lemma 2.2. *A function $u \in C([0,1],[0,\infty))$ is a solution of* (1.1) *if and only if u is a fixed point of the integral operator $\mathcal{T}$ in the cone $\mathcal{P}$.*

Proof. If u is a solution of (1.1), then by the definition of $\mathcal{T}$ we have that $u = \mathcal{T}u$, and by Lemma 2.1 it follows that $\mathcal{T}u \in \mathcal{P}$. □

We close this section by stating the well-known Guo-Krasnosel'skii fixed point theorem [14] which is the basic tool for establishing our results.

Theorem 2.3. *Let B be a Banach space, and let $\mathcal{P} \subset B$ be a cone in B . Assume Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*
$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$
be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Throughout this paper we adopt the notation
$$\begin{gathered}\overline{f_0} = \limsup_{u\to 0+} \frac{f(u)}{u}, \quad \overline{g_0} := \limsup_{u\to 0+} \frac{g(u)}{u}, \\ \underline{f_\infty} = \liminf_{u\to\infty} \frac{f(u)}{u}, \quad \underline{g_\infty} := \liminf_{u\to\infty} \frac{g(u)}{u}\end{gathered} \tag{3.1}$$
and
$$\begin{gathered}\underline{f_0} = \liminf_{u\to 0+} \frac{f(u)}{u}, \quad \underline{g_0} := \liminf_{u\to 0+} \frac{g(u)}{u}, \\ \overline{f_\infty} = \limsup_{u\to\infty} \frac{f(u)}{u}, \quad \overline{g_\infty} := \limsup_{u\to\infty} \frac{g(u)}{u}.\end{gathered} \tag{3.2}$$

Before we state and prove the main results of the paper, we note that by (C) it follows that
$$\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr > 0, \quad \int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr > 0$$
and so $[\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr]^{-1}$ and $[\int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr]^{-1}$ used in stating Theorems 3.1 and 3.2 below are well defined positive real numbers (see, also, the discussion in Section 4).

For our first result, we assume that
$$\overline{f_0}, \overline{g_0} \in [0,\infty) \quad \text{and} \quad \underline{f_\infty}, \underline{g_\infty} \in (0,\infty], \tag{3.3}$$
where $\overline{f_0}$, $\overline{g_0}$, $\underline{f_\infty}$, $\underline{g_\infty}$ are defined by (3.1), and set
$$\begin{gathered}L_1^f := \begin{cases} [\gamma_1 \underline{f_\infty} \int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr]^{-1}, & \text{if } \underline{f_\infty} \in (0,\infty), \\ 0, & \text{if } \underline{f_\infty} = \infty, \end{cases} \\ L_1^g := \begin{cases} [\gamma_2 \underline{g_\infty} \int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr]^{-1}, & \text{if } \underline{g_\infty} \in (0,\infty), \\ 0, & \text{if } \underline{g_\infty} = \infty, \end{cases}\end{gathered} \tag{3.4}$$

and

$$
\begin{aligned}
L_2^f &:= \begin{cases} [\overline{f_0}\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr]^{-1}, & \text{if } \overline{f_0}\in(0,\infty),\\ +\infty, & \text{if } \overline{f_0}=0,\end{cases}\\
L_2^g &:= \begin{cases} [\overline{g_0}\int_0^1 \max_{0\le t\le 1} k_2(t,r)b(r)dr]^{-1}, & \text{if } \overline{g_0}\in(0,\infty),\\ +\infty, & \text{if } \overline{g_0}=0.\end{cases}
\end{aligned}
\tag{3.5}
$$

For our convenience, we will use the notation $I_f = (L_1^f, L_2^f)$ and $I_g = (L_1^g, L_2^g)$.

Theorem 3.1. *Assume conditions* (A), (B), (C), (3.3) *are satisfied and define* L_1^f, L_1^g *by* (3.4) *and* L_2^f, L_2^g *by* (3.5). *Then, for* λ, μ *with* $(\lambda,\mu)\in I_f\times I_g$ *there exists a nonnegative solution* u *of* (1.1).

Proof. Let $(\lambda,\mu)\in(L_1^f,L_2^f)\times(L_1^g,L_2^g)$ and consider the integral operator $\mathcal{T}:\mathcal{B}^+\to\mathcal{B}$ defined by (2.2). In view of Lemma 2.2, all we have to prove is that there exists a (nonzero) fixed point of $\mathcal{T}$ in the cone $\mathcal{P}$. We note that by Lemma 2.1, we have $T\mathcal{P}\subset\mathcal{P}$ while, by using standard arguments, it is not difficult to show that the integral operator $\mathcal{T}$ is completely continuous.

By the definition of L_2^f, L_2^g and the choice of λ and μ, we may always consider an $\varepsilon>0$ such that

$$
\lambda \le \Big[(\overline{f_0}+\varepsilon)\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr\Big]^{-1} \tag{3.6}
$$

$$
\mu \le \Big[(\overline{g_0}+\varepsilon)\int_0^1 \max_{0\le t\le 1} k_2(t,r)b(r)dr\Big]^{-1}. \tag{3.7}
$$

We note that the assumption $\overline{f_0}$, $\overline{g_0}\in[0,\infty)$ yields that for the positive number ε considered, there exists an $H_1>0$ such that

$$
0\le \frac{f(x)}{x} < \overline{f_0}+\varepsilon \quad\text{and}\quad 0\le\frac{g(x)}{x}<\overline{g_0}+\varepsilon,\quad \text{for all } x\in(0,H_1]
$$

from which, in view of the continuity of f,g at 0 we find

$$
0\le f(x)\le(\overline{f_0}+\varepsilon)x \quad\text{and}\quad 0\le g(x)\le(\overline{g_0}+\varepsilon)x,\quad \text{for all } x\in[0,H_1].
$$

Consequently, we have

$$
f(t)\le(\overline{f_0}+\varepsilon)t\le(\overline{f_0}+\varepsilon)x \quad \text{for any } t\in[0,x]\subseteq[0,H_1], \tag{3.8}
$$

$$
g(t)\le(\overline{g_0}+\varepsilon)t\le(\overline{g_0}+\varepsilon)x \quad \text{for any } t\in[0,x]\subseteq[0,H_1]. \tag{3.9}
$$

Setting

$$
f^*(x)=\sup_{t\in[0,x]} f(t),\quad x\in[0,\infty),
$$

from (3.8) it follows that

$$
f(x)\le f^*(x)\le(\overline{f_0}+\epsilon)x \quad \text{for } x\in[0,H_1]. \tag{3.10}
$$

Set $\Omega_1=\{x\in\mathcal{P}:\|x\|<H_1\}$, and let u be an (arbitrary) element in $\partial\Omega_1$. Then $u(r)\le\|u\|=H_1$ for any $r\in[0,1]$ and taking into consideration (3.9), (3.7) and

the choice of ε we have for $s \in [0,1]$

$$\begin{aligned}
\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr &\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)g(u(r))dr \\
&\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)(\overline{g_0}+\epsilon)u(r)dr \\
&\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)dr(\overline{g_0}+\epsilon)\|u\| \\
&\le \|u\| = H_1,
\end{aligned}$$

and so

$$\mu \max_{0\le s\le 1} \int_0^1 k_2(s,r)b(r)g(u(r))dr \in [0,H_1], \quad \text{for all } s \in [0,1].$$

Consequently, in view of (3.6) and (3.10) and employing the nondecreasing character of f^*, we obtain, for $t \in [0,1]$,

$$\begin{aligned}
\mathcal{T}u(t) &= \lambda \int_0^1 k_1(t,s)a(s)f\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds \\
&\le \lambda \int_0^1 k_1(t,s)a(s)f^*\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)f^*\Big(\mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)g(u(r))dr\Big)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)f^*(H_1)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)(\overline{f_0}+\epsilon)H_1ds \\
&= \Big[\lambda \int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)(\overline{f_0}+\epsilon)ds\Big]H_1 \\
&\le H_1 = \|u\|,
\end{aligned}$$

which implies

$$\|\mathcal{T}u\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.11}$$

Now let us note that, in case that $\underline{f_\infty}$ and $\underline{g_\infty}$ are positive real numbers, then by the definition of L_1^f and L_1^g and the choice of λ, μ it follows that there exists a positive number ϵ with $0 < \epsilon < \min\{\underline{f_\infty}, \underline{g_\infty}\}$ such that

$$\begin{gathered}
\Big[\gamma_1 \int_\xi^\eta k_1(\xi,r)a(r)(\underline{f_\infty}-\epsilon)dr\Big]^{-1} \le \lambda, \\
\Big[\gamma_2 \int_\xi^\eta k_2(\xi,r)b(r)(\underline{g_\infty}-\epsilon)dr\Big]^{-1} \le \mu.
\end{gathered}$$

Set

$$\begin{gathered}
\widehat{f_\infty} = \begin{cases} (\underline{f_\infty}-\epsilon), & \text{if } \underline{f_\infty} \in (0,\infty) \\ \big[\lambda\gamma_1 \int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\big]^{-1}, & \text{if } \underline{f_\infty} = \infty, \end{cases} \\
\widehat{g_\infty} = \begin{cases} (\underline{g_\infty}-\epsilon), & \text{if } \underline{g_\infty} \in (0,\infty) \\ \big[\mu\gamma_2 \int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr\big]^{-1}, & \text{if } \underline{g_\infty} = \infty. \end{cases}
\end{gathered}$$

Clearly, $\widehat{f_\infty}$ and $\widehat{g_\infty}$ are well defined and they are both positive real numbers regardless of $\widehat{f_\infty}$ and $\widehat{g_\infty}$ being finite or not. Having in mind the way that ϵ is considered, we observe that

$$\begin{aligned}
&\widehat{f_\infty}\Big[\lambda\gamma_1\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]\\
&=\begin{cases} 1, & \text{if } \underline{f_\infty}=\infty\\ [\lambda\gamma_1\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr](\underline{f_\infty}-\epsilon)\ge 1, & \text{if } \underline{f_\infty}\in(0,\infty).\end{cases}
\end{aligned}$$

Consequently,

$$1\le \lambda\gamma_1\Big[\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]\widehat{f_\infty}, \tag{3.12}$$

and, by similar arguments,

$$1\le \mu\gamma_2\Big[\int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr\Big]\widehat{g_\infty}. \tag{3.13}$$

In view of the definitions of $\underline{f_\infty}$, $\underline{g_\infty}$, and $\widehat{f_\infty}$, $\widehat{g_\infty}$, it follows that we may always find an $\overline{H}_2>2H_1$ such that

$$f(x)\ge \widehat{f_\infty}x \quad \text{for any } x\ge \overline{H}_2, \tag{3.14}$$

$$g(x)\ge \widehat{g_\infty}x \quad \text{for any } x\ge \overline{H}_2. \tag{3.15}$$

Set $H_2=\max\{2H_1,\frac{\overline{H}_2}{\gamma}\}$, and consider an arbitrary $u\in\mathcal{P}$ with $\|u\|=H_2$. Then, by the way that the cone $\mathcal{P}$ is constructed we have

$$u(r)\ge \min_{t\in[\xi,\eta]} u(t)\ge \gamma\|u\|\ge \overline{H}_2 \quad \text{for } r\in[\xi,\eta],$$

and so, by (3.15)

$$g(u(r))\ge \widehat{g_\infty}u(r) \quad \text{for } r\in[\xi,\eta].$$

Using once more the fact that $u\in\mathcal{P}$ implies $u(r)\ge\gamma\|u\|$ for $r\in[\xi,\eta]$, in view of the last inequality we take for $s\in[\xi,\eta]$

$$\begin{aligned}
\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr &\ge \mu\int_\xi^\eta k_2(s,r)b(r)g(u(r))dr\\
&\ge \mu\int_\xi^\eta \min_{\xi\le s\le\eta} k_2(s,r)b(r)\widehat{g_\infty}u(r)dr\\
&\ge \mu\Big[\int_\xi^\eta \min_{\xi\le s\le\eta} k_2(s,r)b(r)dr\Big]\widehat{g_\infty}\gamma\|u\|;
\end{aligned}$$

i.e.,

$$\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr\ge \mu\Big[\int_\xi^\eta \min_{\xi\le s\le\eta} k_2(s,r)b(r)dr\Big]\widehat{g_\infty}\gamma\|u\|, \tag{3.16}$$

for $s\in[\xi,\eta]$, and so, as $\gamma\|u\|\ge\overline{H}_2$, by (3.13), we obtain

$$\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr\ge \overline{H}_2 \quad \text{for } s\in[\xi,\eta]. \tag{3.17}$$

Employing (3.17) and the fact that $H_2\ge\overline{H}_2$, by (3.14) we find that

$$f\Big(\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)\ge \widehat{f_\infty}\Big[\mu\int_0^1 k_2(s,r)b(r)g(u(r))dr\Big] \quad \text{for } s\in[\xi,\eta].$$

In view of this inequality and by (3.16) we have

$$\begin{aligned}
&\mathcal{T}u(\xi)\\
&= \lambda \int_0^1 k_1(\xi,s)a(s)f\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds\\
&\geq \lambda \int_\xi^\eta k_1(\xi,s)a(s)f\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds\\
&\geq \lambda \int_\xi^\eta k_1(\xi,s)a(s)\widehat{f_\infty}\Big[\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big]ds\\
&\geq \lambda \int_\xi^\eta k_1(\xi,s)a(s)\widehat{f_\infty}\Big\{\mu\Big[\int_\xi^\eta \min_{\xi\leq s\leq\eta} k_2(s,r)b(r)dr\Big]\widehat{g_\infty}\gamma\|u\|\Big\}ds\\
&= \Big\{\lambda\gamma_1\Big[\int_\xi^\eta k_1(\xi,s)a(s)ds\Big]\widehat{f_\infty}\Big\}\Big\{\mu\gamma_2\Big[\int_\xi^\eta \min_{\xi\leq s\leq\eta} k_2(s,r)b(r)dr\Big]\widehat{g_\infty}\Big\}\|u\|\\
&\geq \Big\{\lambda\gamma_1\Big[\int_\xi^\eta \min_{\xi\leq t\leq\eta} k_1(t,s)a(s)ds\Big]\widehat{f_\infty}\Big\}\Big\{\mu\gamma_2\Big[\int_\xi^\eta \min_{\xi\leq s\leq\eta} k_2(s,r)b(r)dr\Big]\widehat{g_\infty}\Big\}\|u\|
\end{aligned}$$

which, by (3.12) and (3.13) gives

$$\mathcal{T}u(\xi) \geq \|u\| = H_2.$$

Consequently, we may infer that $\|\mathcal{T}u\| \geq \|u\|$ for $u \in \mathcal{P}$ with $\|u\| = H_2$. Hence, setting $\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\}$, it follows that

$$\|\mathcal{T}u\| \geq \|u\| \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.18}$$

In view of (3.11) and (3.18), from Theorem 2.3 it follows that the operator $\mathcal{T}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ i.e., the integral equation (1.1) has a solution in the cone $\mathcal{P}$. It is clear that this solution u is nontrivial as $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ implies that $0 < H_1 \leq \|u\|$. The proof is complete. $\square$

For our second result, we assume that

$$\underline{f_0}, \underline{g_0} \in (0,\infty] \quad \text{and} \quad \overline{f_\infty}, \overline{g_\infty} \in [0,\infty), \tag{3.19}$$

where $\underline{f_0}, \underline{g_0}, \overline{f_\infty}, \overline{g_\infty}$ are defined by (3.2) and set

$$\begin{gathered}
L_3^f := \begin{cases} \big[\gamma_1 \int_\xi^\eta \min_{\xi\leq t\leq\eta} k_1(t,r)a(r)\underline{f_0}dr\big]^{-1}, & \text{if } \underline{f_0} \in (0,\infty),\\ 0, & \text{if } \underline{f_0} = \infty, \end{cases}\\
L_3^g := \begin{cases} \big[\gamma_2 \int_\xi^\eta \min_{\xi\leq t\leq\eta} k_2(t,r)b(r)g_0dr\big]^{-1}, & \text{if } g_0 \in (0,\infty),\\ 0 & \text{if } \underline{g_0} = \infty, \end{cases}
\end{gathered} \tag{3.20}$$

and

$$\begin{gathered}
L_4^f := \begin{cases} \big[\int_0^1 \max_{0\leq t\leq 1} k_1(t,r)a(r)\overline{f_\infty}dr\big]^{-1}, & \text{if } \overline{f_\infty} \in (0,\infty),\\ +\infty, & \text{if } \overline{f_\infty} = 0\,, \end{cases}\\
L_4^g := \begin{cases} \big[\int_0^1 \max_{0\leq t\leq 1} k_2(t,r)b(r)\overline{g_\infty}dr\big]^{-1}, & \text{if } \overline{g_\infty} \in (0,\infty),\\ +\infty, & \text{if } \overline{g_\infty} = 0. \end{cases}
\end{gathered} \tag{3.21}$$

Theorem 3.2. *Assume conditions* (A), (B), (C), (3.19) *are satisfied and define* L_3^f, L_3^g *by* (3.20) *and* L_4^f, L_4^g *by* (3.21). *Moreover, assume that* $g(0) = 0$. *Then, for*

λ, μ with $(\lambda,\mu) \in (L_3^f, L_4^f) \times (L_3^g, L_4^g)$ the integral equation (1.1) *has a nonnegative solution.*

Proof. Let $(\lambda,\mu) \in (L_3^f, L_4^f) \times (L_3^g, L_4^g)$ and $\mathcal{T}$ be the integral operator defined by (2.2). By Lemma 2.2 it suffices to prove that $\mathcal{T}$ has a fixed point in the cone $\mathcal{P}$. We note that completely continuity of the operator $\mathcal{T}$ follows by standard arguments while by Lemma 2.1 we have $\mathcal{T}\mathcal{P} \subset \mathcal{P}$.

We observe that if $\underline{f_0}, \underline{g_0}$ are positive real numbers then by the definition of L_3^f and L_3^g and the choice of λ, μ it follows that there exists a positive number ε such that $0 < \varepsilon < \min\{\underline{f_0}, \underline{g_0}\}$ and

$$\Big[\gamma_1 \int_\xi^\eta \min_{\xi \le t \le \eta} k_1(t,r)a(r)(\underline{f_0} - \varepsilon)dr\Big]^{-1} \le \lambda, \tag{3.22}$$

$$\Big[\gamma_2 \int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r)b(r)(\underline{g_0} - \varepsilon)dr\Big]^{-1} \le \mu. \tag{3.23}$$

Set

$$\widetilde{f_0} = \begin{cases} (\underline{f_0} - \varepsilon), & \text{if } \underline{f_0} \in (0,\infty), \\ [\lambda\gamma_1 \int_\xi^\eta \min_{\xi \le t \le \eta} k_1(t,r)a(r)dr]^{-1}, & \text{if } \underline{f_0} = \infty, \end{cases}$$

$$\widetilde{g_0} = \begin{cases} (\underline{g_0} - \varepsilon), & \text{if } \underline{g_0} \in (0,\infty), \\ \big[\mu\gamma_2 \int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r)b(r)dr\big]^{-1}, & \text{if } \underline{g_0} = \infty, \end{cases}$$

and note that $\widetilde{f_0}$ and $\widetilde{g_0}$ are positive real numbers regardless if some (or none) of $\underline{f_0}$, $\underline{g_0}$ are finite or not. In view of (3.22) and (3.23) and by arguments similar to the ones used in Theorem 3.1, one may see that

$$1 \le \lambda\gamma_1 \Big[\int_\xi^\eta \min_{\xi \le t \le \eta} k_1(t,r)a(r)dr\Big]\widetilde{f_0}, \tag{3.24}$$

$$1 \le \mu\gamma_2 \Big[\int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r)b(r)dr\Big]\widetilde{g_0}. \tag{3.25}$$

By assumption (3.19) it follows that for the ε chosen we can always find an $\overline{H}_3 > 0$ such that for any $x \le \overline{H}_3$ it holds

$$\frac{f(x)}{x} \ge \begin{cases} (\underline{f_0} - \varepsilon), & \text{if } \underline{f_0} \in (0,\infty) \\ \big[\lambda\gamma_1 \int_\xi^\eta \min_{\xi \le t \le \eta} k_1(t,r)a(r)dr\big]^{-1} & \text{if } \underline{f_0} = \infty, \end{cases}$$

$$\frac{g(x)}{x} \ge \begin{cases} (\underline{g_0} - \varepsilon), & \text{if } \underline{g_0} \in (0,\infty) \\ \big[\mu\gamma_2 \int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r)b(r)dr\big]^{-1} & \text{if } \underline{g_0} = \infty, \end{cases}$$

hence, in view of the definitions of the positive numbers $\widetilde{f_0}$ and $\widetilde{g_0}$ we have

$$f(x) \ge \widetilde{f_0}x \quad \text{for any } x \in [0, \overline{H}_3], \tag{3.26}$$

$$g(x) \ge \widetilde{g_0}x \quad \text{for any } x \in [0, \overline{H}_3]. \tag{3.27}$$

As g is continuous at zero with $g(0) = 0$, it follows that there exists an $H_3 \le \overline{H}_3$ such that

$$g(x) \le \frac{\overline{H}_3}{\mu \int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r)b(r)dr} \quad \text{for all } x \in [0, H_3]. \tag{3.28}$$

Let $u \in \mathcal{P}$ with $\|u\| = H_3$. Clearly, $u(r) \le \|u\| = H_3$ for all $r \in [0,1]$ and so by (3.27) we take

$$g(u(r)) \ge \widetilde{g_0} u(r), \quad r \in [0,1], \tag{3.29}$$

while, by (3.28) it holds

$$g(u(r)) \le \frac{\overline{H}_3}{\mu \int_\xi^\eta \min_{\xi \le t \le \eta} k_2(t,r) b(r) dr} \quad \text{for all } r \in [0,1]. \tag{3.30}$$

Consequently, for $s \in [0,1]$, we have

$$\mu \int_0^1 k_2(s,r) b(r) g(u(r)) dr \le \mu \int_0^1 k_2(s,r) b(r) \frac{\overline{H}_3}{\mu \int_0^1 k_2(s,w) b(w) dw} dr = \overline{H}_3,$$

which, in view of (3.26) implies

$$f\Big(\mu \int_0^1 k_2(s,r) b(r) g(u(r)) dr\Big) \ge \widetilde{f_0}\Big[\mu \int_0^1 k_2(s,r) b(r) g(u(r)) dr\Big], \tag{3.31}$$

for $s \in [0,1]$. Hence, taking into consideration (3.31), (3.29), and the facts that $\gamma_1\gamma_2 \le \gamma \le 1$ and $u \in \mathcal{P}$, we have

$$\begin{aligned}
\mathcal{T}u(\xi) &= \lambda \int_0^1 k_1(\xi,s) a(s) f\Big(\mu \int_0^1 k_2(s,r) b(r) g(u(r)) dr\Big) ds \\
&\ge \lambda \int_0^1 k_1(\xi,s) a(s) \widetilde{f_0}\Big[\mu \int_0^1 k_2(s,r) b(r) g(u(r)) dr\Big] ds \\
&\ge \lambda \int_\xi^\eta k_1(\xi,s) a(s) \widetilde{f_0}\Big[\mu \int_\xi^\eta k_2(s,r) b(r) \widetilde{g_0} u(r) dr\Big] ds \\
&\ge \lambda \int_\xi^\eta k_1(\xi,s) a(s) \widetilde{f_0}\Big[\mu \int_\xi^\eta k_2(s,r) b(r) \widetilde{g_0} \gamma \|u\| dr\Big] ds \\
&\ge \lambda \int_\xi^\eta k_1(\xi,s) a(s) \widetilde{f_0}\Big[\mu \int_\xi^\eta \min_{\xi \le s \le \eta} k_2(s,r) b(r) \widetilde{g_0} (\gamma_1\gamma_2) dr\Big] ds \|u\| \\
&= \Big\{\gamma_1 \Big[\lambda \int_\xi^\eta k_1(\xi,s) a(s) ds\Big] \widetilde{f_0}\Big\} \Big\{\gamma_2 \Big[\mu \int_\xi^\eta \min_{\xi \le s \le \eta} k_2(s,r) b(r) dr\Big] \widetilde{g_0}\Big\} \|u\|,
\end{aligned}$$

thus, by (3.24) and (3.25) we obtain $\mathcal{T}u(\xi) \ge \|u\|$. Consequently, we may conclude that for $u \in \mathcal{P}$ with $\|u\| = H_3$ it holds $\|\mathcal{T}u\| \ge \|u\|$, so by setting $\Omega_3 = \{x \in \mathcal{B} : \|x\| < H_3\}$, it follows that

$$\|\mathcal{T}u\| \ge \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.32}$$

Since $\overline{f_\infty}$, $\overline{g_\infty} \in [0,\infty)$ by the choice of λ and μ it follows that there exists a positive number ϵ such that

$$\begin{aligned}
\lambda &\le \Big[\int_0^1 \max_{0 \le t \le 1} k_1(t,r) a(r) (\overline{f_\infty} + \epsilon) dr\Big]^{-1}, \\
\mu &\le \Big[\int_0^1 \max_{0 \le t \le 1} k_2(\xi,r) b(r) (\overline{g_\infty} + \epsilon) dr\Big]^{-1}.
\end{aligned}$$

Set

$$\widetilde{f_\infty} = \begin{cases} (\overline{f_\infty} + \epsilon), & \text{if } \overline{f_\infty} \in (0,\infty) \\ [\lambda \int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr]^{-1}, & \text{if } \overline{f_\infty} = 0, \end{cases}$$

$$\widetilde{g_\infty} = \begin{cases} (\overline{g_\infty} + \epsilon), & \text{if } \overline{g_\infty} \in (0,\infty) \\ \big[\mu \int_0^1 \max_{0\le t\le 1} k_2(t,r)b(r)dr\big]^{-1} & \text{if } \overline{g_\infty} = 0, \end{cases}$$

and note that $\widetilde{f_\infty}$ and $\widetilde{g_\infty}$ are always positive numbers for which it holds

$$\lambda\Big[\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr\Big]\widetilde{f_\infty} \le 1, \tag{3.33}$$

$$\mu\Big[\int_0^1 \max_{0\le t\le 1} k_2(t,r)b(r)dr\Big]\widetilde{g_\infty} \le 1. \tag{3.34}$$

We consider the functions $f^*, g^* : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f^*(x) = \sup_{0\le t\le x} f(t) \quad \text{and} \quad g^*(x) = \sup_{0\le t\le x} g(t),$$

and observe that, these two functions are nondecreasing and such that

$$f(x) \le f^*(x) \text{ for } x \ge 0 \quad \text{and} \quad g(x) \le g^*(x) \;\text{ for } x \ge 0.$$

In addition, it is not difficult to verify that

$$\limsup_{x\to\infty} \frac{f^*(x)}{x} = \overline{f_\infty} \quad \text{and} \quad \limsup_{x\to\infty} \frac{g^*(x)}{x} = \overline{g_\infty},$$

and so, by the definition of $\widetilde{f_\infty}$ and $\widetilde{g_\infty}$, it follows that we can always find an $H_4 > 2H_3$ such that

$$f^*(x) \le \widetilde{f_\infty} x \quad \text{for any } x \ge H_4, \tag{3.35}$$

$$g^*(x) \le \widetilde{g_\infty} x \quad \text{for any } x \ge H_4. \tag{3.36}$$

Let $u \in \mathcal{P}$ with $\|u\| = H_4$. Taking into consideration the nondecreasing character of g^* and employing (3.36), for $s \in [0,1]$, we have

$$\begin{aligned} \mu \int_0^1 k_2(s,r)b(r)g(u(r))dr &\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)g(u(r))dr \\ &\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)g^*(u(r))dr \\ &\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)g^*(\|u\|)dr \\ &\le \mu \int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)\widetilde{g_\infty}\|u\|dr \\ &= \mu[\int_0^1 \max_{0\le s\le 1} k_2(s,r)b(r)dr]\widetilde{g_\infty}\|u\| \end{aligned}$$

which by (3.34) implies

$$\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr \le \|u\|, \quad s \in [0,1].$$

In view of the above inequality and the nondecreasing character of f^*, we may employ (3.35) to obtain for $t \in [0,1]$,

$$\begin{aligned}
\mathcal{T}u(t) &= \lambda \int_0^1 k_1(t,s)a(s)f\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds \\
&\le \lambda \int_0^1 k_1(t,s)a(s)f^*\Big(\mu \int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)f^*(\|u\|)ds \\
&\le \lambda \Big[[\int_0^1 \max_{0\le t\le 1} k_1(t,s)a(s)\widetilde{f_\infty} ds\Big]\|u\|
\end{aligned}$$

which by (3.33) implies

$$\mathcal{T}u(t) \le \|u\|, \quad \text{for all } t \in [0,1],$$

and so $\|Tu\| \le \|u\|$. Therefore, by setting $\Omega_4 = \{x \in \mathcal{P} : \|x\| < H_4\}$, we have

$$\|\mathcal{T}u\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.37}$$

In view of (3.32) and (3.37), from Theorem 2.3 it follows that the operator $\mathcal{T}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, and so (1.1) has a solution in the cone $\mathcal{P}$. The proof is now complete. □

4. Discussion

In this section we discuss the positivity of the (nonnegative) solutions yielded by Theorems 3.1 and 3.2 in Section 3. We, also, present some remarks on the intervals where the eigenvalues λ and μ may belong. Noting the similarity of the results in Theorems 3.1 and 3.2 we will focus our discussion mainly on the results of Theorem 3.1. We, also, note that a large part of the discussion below is closely related with the remarks in [28].

Though Theorems 3.1 and 3.2 yield the existence of a (nontrivial) nonnegative solution to (1.1), however it is not guaranteed that such a solution is positive on the whole interval $[0,1]$: indeed, if for some $t_0 \in [0,1]$ it holds $k_1(t_0,r) = 0$ for all $r \in [0,1]$, then $u(t_0) = 0$. Thus, if there exists a subset $J_1 \subseteq [0,1]$ such that $k_1(t,r) = 0$ for $(t,r) \in J_1 \times [0,1]$, then $u(t) = 0$ for all $t \in J_1$. Consequently, a *necessary* condition so that a solution u is positive at some point t_0 (respectively, on some set J_1) is that $\max_{r\in[0,1]} k_1(t_0,r) \ne 0$ (resp., $\max_{r\in[0,1]} k_1(t,r) \ne 0$ for all $t \in J_1$). Similarly, as one can easily verify by the definition of v by (2.2), a *necessary condition* so that the function v is positive at some point t_0 (respectively, on some interval J_2) is that $\max_{r\in[0,1]} k_2(t_0,r) \ne 0$ (resp., $\max_{r\in[0,1]} k_2(t,r) \ne 0$ for all $t \in J_2$).

As $\max_{\xi\le s\le\eta}[\min_{\xi\le t\le\eta} k_1(t,s)] > 0$ by (C), employing the continuity of k_1 we see that there exists an interval $J \subseteq [\xi,\eta]$ such that

$$\min_{\xi\le t\le\eta} k_1(t,s) > 0 \text{ for all } s \in J,$$

which, in view of (B), implies that $\min_{\xi\le t\le\eta} k_1(t,r)a(s) > 0$ on some interval $J' \subseteq J$, thus

$$\lambda \int_\xi^\eta k_1(s,r)a(r)dr > 0$$

hence $[\int_\xi^\eta k_1(s,r)a(r)dr]^{-1}$ is a well defined positive real number.

We note that if for some $s_1 \in [0,1]$ there exists some $t_1 \in [\xi,\eta]$ with $k_1(t_1,s_1)=0$, then $\min_{\xi\le t\le\eta} k_1(t,s_1)=0$, and so from (C) it follows that $k_1(t,s_1)=0$ for all $t\in[0,1]$. Clearly, if $\min_{\xi\le t\le\eta} k_1(t,s)=0$ for all $s\in[0,1]$, then $k_1\equiv 0$. Therefore, if we are looking for some suitable interval $[\xi,\eta]\subseteq[0,1]$ such that (C) is fulfilled, then $[\xi,\eta]$ should be selected so that there exists an $s\in[0,1]$ such that $k_1(t,s)>0$ for all $t\in[\xi,\eta]$.

Now let us suppose that $xf(x)>0$ for $x\neq 0$, and let u_0 be a (nontrivial) nonnegative solution of (1.1) belonging to $\mathcal{P}$. Then there exists a constant $H>0$ such that $\|u_0\|=H$. As $u\in\mathcal{P}$ by Lemma 2.2, we have

$$u_0(t)\ge \min_{\xi\le r\le\eta} u_0(r)\ge\gamma\|u_0\|\ge\gamma H \quad \text{for any } t\in[\xi,\eta],$$

and so

$$\gamma H\le u_0(r)\le H \quad \text{for all } r\in[\xi,\eta].$$

In view of Lemma 2.1, we see that for the function $v_0:[0,1]\to\mathbb{R}^+$ with

$$v_0(t)=\mu\int_0^1 k_2(t,r)b(r)g(u_0(r))dr, \quad t\in[0,1],$$

we have $v_0\in\mathcal{P}$ and so there exists an $H'>0$ such that

$$\gamma H'\le v_0(s)\le H' \quad \text{for all } s\in[\xi,\eta].$$

Employing the continuity of f and the assumption that f is positive on $(0,\infty)$, we may see that there exist some $m_f, M_f>0$ such that

$$m_f\le f(w)\le M_f \quad \text{for all } w\in[\gamma H',H'],$$

and so

$$m_f\le f(v_0(s))\le M_f \quad \text{for all } s\in[\xi,\eta].$$

Then for $\widetilde{t}\in[0,1]$ we have

$$\begin{aligned}
u_0(\widetilde{t}) &= \lambda\int_0^1 k_1(\widetilde{t},s)a(s)f\Big(\mu\int_0^1 k_2(s,r)b(r)g(u_0(r))dr\Big)ds\\
&= \lambda\int_\xi^\eta k_1(\widetilde{t},s)a(s)f(v_0(s))ds,
\end{aligned}$$

hence,

$$u_0(\widetilde{t})\ge\Big[\lambda\int_\xi^\eta k_1(\widetilde{t},s)a(s)ds\Big]m_f.$$

In view of assumption (B) and Lemma 2.2, from the last relation it follows that if for some given $\widetilde{t}\in[0,1]$ there exists some $\widetilde{s}\in[\xi,\eta]$ such that $k_1(\widetilde{t},\widetilde{s})>0$, then the continuity of k_1 implies that $u_0(\widetilde{t})>0$. Consequently,

$$\max_{\xi\le s\le\eta} k_1(t,s)>0, \quad \text{for } t\in J_1 \tag{4.1}$$

is a *sufficient condition* for $u_0(t)>0$, for all $t\in J_1\subseteq[0,1]$. We, thus, have the following result.

> Assume that $xf(x)>0$, $x\neq0$. Then (4.1) is a sufficient condition for a nonnegative nontrivial solution $u\in\mathcal{P}$ of the integral equation (1.1) to be positive on J_1.

In other words, if the kernel k_1 is not identically zero on each $\{t\} \times [\xi, \eta]$ for $t \in J_1 \subseteq [0,1]$, then any (nontrivial) solution $u \in \mathcal{P}$ of (1.1) is positive on J. Concerning the function v defined by (2.2), by similar arguments we may obtain the following result.

> Assume that $xg(x) > 0$ for $x \neq 0$. If $u \in \mathcal{P}$ is a nonnegative nontrivial solution of (1.1), then
> $$\max_{\xi \le s \le \eta} k_2(t,s) > 0 \quad \text{for } t \in J_2 \subseteq [0,1] \tag{4.2}$$
> is a sufficient condition so that the function v defined by (2.2) be positive on J_2.

We note that by (C) and the continuity of k_i $(i = 1,2)$ it follows that (4.1) and (4.2) are always fulfilled on $[\xi, \eta]$. In view of the above, from Theorem 3.1 (respectively, Theorem 3.2) we have the following proposition.

Proposition 4.1. *Assume conditions* (A), (B), (C), (3.3) *(resp,. Theorem 3.2) are satisfied and define* L_1^f, L_1^g *by* (3.4) *and* L_2^f, L_2^g *by* (3.5) *(resp.* L_3^f, L_3^g *by* (3.20) *and* L_4^f, L_4^g *by* (3.21)*). Furthermore, assume that* $xf(x) > 0$ *for* $x \neq 0$*. If* (4.1) *holds true on some subset* $J \subseteq [0,1]$*, then, for* λ, μ *with* $(\lambda, \mu) \in I_f \times I_g$ *there exists a nonnegative solution* u *of the integral equation* (1.1) *which is positive on* J*.*

It is not difficult to see that (C) is satisfied if we assume that

> $k_i(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,2$ are continuous functions and there are points ξ_i, η_i, $r_i \in [0,1]$, $(i = 1,2)$ with $\xi = \max\{\xi_1, \xi_2\} < r_1$, $r_2 < \min\{\eta_1, \eta_2\} = \eta$, for which $[\min_{\xi \le t \le \eta} k_i(t, r_i)] > 0$, $i = 1,2$, and positive numbers γ_i, $i = 1,2$ such that
> $$\min_{\xi_i \le r \le \eta_i} k_i(r,s) \ge \gamma_i k_i(t,s) \quad \text{for } (t,s) \in [0,1]^2, \quad i = 1,2.$$

Obviously, in order that the result of Theorem 3.1 makes sense, it is necessary that the intervals I_f and I_g are nonvoid, i.e.,

$$L_1^f < L_2^f \quad \text{and} \quad L_1^g < L_2^g.$$

In view of (3.3), L_1^f and L_1^g are nonnegative real numbers while L_2^f and L_2^g may be positive real numbers or ∞. We briefly discuss the case of L_1^f, L_2^f, L_1^g, L_2^g being positive real numbers and the case where some of L_1^f, L_2^f and some of L_1^g, L_2^g are not positive real numbers. Conclusions for the other cases may be easily deducted.

Clearly, if $\underline{f_\infty} = \infty$ then $L_1^f = 0 < L_2^f$ while if $\overline{f_0} = 0$ then $L_1^f < \infty = L_2^f$. Hence, if $\underline{f_\infty} = \infty$ or $\overline{f_0} = 0$, then $I_f \neq \emptyset$. As $\underline{f_\infty} = \infty$ implies $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ and $\overline{f_0} = 0$ implies $\lim_{x\to 0} \frac{f(x)}{x} = 0$, from Theorem 3.1 we have the following corollary.

Corollary 4.2. *Assume conditions* (A), (B), (C) *are satisfied.*

(i) If
$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty \ \textit{or} \ \lim_{x\to 0} \frac{f(x)}{x} = 0, \quad \textit{and} \quad \lim_{x\to\infty} \frac{fg(x)}{x} = \infty \ \textit{or} \ \lim_{x\to 0} \frac{g(x)}{x} = 0$$
then there exist positive numbers λ *and* μ *such that* (1.1) *has a nonnegative solution.*

(ii) If
$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty = \lim_{x\to\infty} \frac{g(x)}{x} \quad \textit{and} \quad \lim_{x\to 0} \frac{f(x)}{x} = 0 = \lim_{x\to 0} \frac{g(x)}{x}$$
then (1.1) *has a nonnegative solution for any positive numbers* λ *and* μ*.*

In the case that L_1^f, L_2^f, L_1^g, L_2^g are positive real numbers, then the inequality $L_1^f < L_2^f$ may equivalently be written

$$\Big[\gamma_1 \underline{f_\infty} \int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]^{-1} < \Big[\overline{f_0}\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr\Big]^{-1};$$

i.e.,

$$\Big[\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr\Big]\overline{f_0} < \gamma_1\Big[\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]\underline{f_\infty}, \tag{4.3}$$

and so

$$1 \le \frac{\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr}{\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr} < \gamma_1 \frac{\underline{f_\infty}}{\overline{f_0}}.$$

Hence, a *necessary condition* for I_f and I_g to be nonvoid is

$$\frac{\overline{f_0}}{\underline{f_\infty}} < \gamma_1 \quad\text{and}\quad \frac{\overline{g_0}}{\underline{g_\infty}} < \gamma_2. \tag{4.4}$$

On the other hand, from (C) we have $k_i(t,s) \le \frac{1}{\gamma_i}\min_{\xi\le r\le\eta} k_i(r,s)$ for $(t,s)\in[0,1]^2$, $i=1,2$, and so

$$\max_{t\in[0,1]} k_i(t,s) \le \frac{1}{\gamma_i}\min_{\xi\le r\le\eta} k_i(r,s) \quad\text{for } s,t\in[0,1],\quad i=1,2,$$

from which we take

$$\int_0^1 \max_{0\le t\le 1} k_1(t,r)a(r)dr \le \frac{1}{\gamma_1}\int_0^1 \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr.$$

Hence from (4.3) and the last relation it follows that a sufficient condition for the inequality $L_1^f < L_2^f$ to hold is

$$\Big[\frac{1}{\gamma_1}\int_0^1 \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]\overline{f_0} < \gamma_1\Big[\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr\Big]\underline{f_\infty},$$

or, equivalently,

$$\frac{\int_0^1 \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr}{\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr} < (\gamma_1)^2\frac{\underline{f_\infty}}{\overline{f_0}}.$$

Therefore, a *sufficient condition* for I_f and I_g to be nonvoid is

$$\frac{\int_0^1 \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr}{\int_\xi^\eta \min_{\xi\le t\le\eta} k_1(t,r)a(r)dr} < (\gamma_1)^2\frac{\underline{f_\infty}}{\overline{f_0}},\quad \frac{\int_0^1 \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr}{\int_\xi^\eta \min_{\xi\le t\le\eta} k_2(t,r)b(r)dr} < (\gamma_2)^2\frac{\underline{g_\infty}}{\overline{g_0}}. \tag{4.5}$$

In view of the above discussion, from Theorem 3.1 we have the following corollary.

Corollary 4.3. *Assume that conditions* (A), (B), (C) *hold and that* $\overline{f_0}$, $\underline{f_\infty}$, $\overline{g_0}$, $\underline{g_\infty}$ *are real numbers. Moreover, assume that* (4.5) *is satisfied. Then there exist positive numbers* λ *and* μ *such that* (1.1) *has a nonnegative solution.*

Having in mind that $\gamma_1,\gamma_2\in[0,1]$ we may see that in case that at least one of the functions f and g is linear then the condition (4.4) is violated, hence Theorem 3.1 cannot be applied. Obviously, if $f\equiv c_0\ne 0$ then (1.1) has always a (positive) solution obtained by a simple integration $u(t)=\lambda c_0\int_0^1 k_1(t,s)a(s)ds$, $0\le t\le 1$. For such a case, we have $\overline{f_0}=\infty$ and $\underline{f_\infty}=0$, and so Theorem 3.1 does not apply. However, as $\underline{f_0}=\infty$ and $\overline{f_\infty}=0$ we may consider that g is any appropriate function that satisfies (3.19) thus existence is yielded by Theorem 3.2. Next, let us suppose that f is a polynomial of first degree, i.e.,

$$f(x) = c_1 x + b_1, \quad x \in [0, \infty)$$

where $c_1 > 0$ and $b_1 \geq 0$ are real numbers. Having in mind that Theorem 3.1 may be applied provided that $\overline{f_0}, \overline{g_0} \in [0, \infty)$ and $\underline{f_\infty}$, $\underline{g_\infty} \in (0, \infty]$, we find

$$f_0 := \overline{f_0} = \lim_{u \to 0+} \frac{f(u)}{u} = \lim_{u \to 0+} \frac{c_1 u + b_1}{u} = \begin{cases} +\infty, & \text{if } b_1 > 0 \\ c_1, & \text{if } b_1 = 0, \end{cases} \tag{4.6}$$

and

$$f_\infty := \underline{f_\infty} = \lim_{u \to \infty} \frac{f(u)}{u} = \lim_{u \to \infty} \frac{c_1 u + b_1}{u} = c_1. \tag{4.7}$$

Hence, in order that Theorem 3.1 may be applied we must have $b_1 = 0$ and in this case $\underline{f_\infty} = c_1 = \overline{f_0}$, and so

$$L_1^f = \Big[\gamma_1 c_1 \int_\xi^\eta \min_{\xi \leq t \leq \eta} k_1(t, r) a(r) dr\Big]^{-1}, \quad L_2^f = \Big[c_1 \int_0^1 \max_{0 \leq t \leq 1} k_1(t, r) a(r) dr\Big]^{-1}.$$

It follows that $L_1^f < L_2^f$ is equivalent to

$$\Big[\gamma_1 c_1 \int_\xi^\eta \min_{\xi \leq t \leq \eta} k_1(t, r) a(r) dr\Big]^{-1} < \Big[c_1 \int_0^1 \max_{0 \leq t \leq 1} k_1(t, r) a(r) dr\Big]^{-1},$$

or

$$\int_0^1 \max_{0 \leq t \leq 1} k_1(t, r) a(r) dr < \gamma_1 \int_\xi^\eta \min_{\xi \leq t \leq \eta} k_1(t, r) a(r) dr.$$

This inequality cannot hold even if $\xi = 0$, $\eta = 1$, $\gamma_1 = 1$ and $\max_{0 \leq t \leq 1} k_1(t, r) = \min_{\xi \leq t \leq \eta} k_1(t, r) = k(r)$, $r \in [0, 1]$. Hence, in case that f is polynomial of first degree, then Theorem 3.1 cannot be applied, and by similar arguments, neither does it in the case of g being a polynomial of first degree. Therefore, we may conclude that Theorem 3.1 cannot be applied in the case that some of the functions f, g is a first degree polynomial

Now let us see if Theorem 3.2 can be applied. In order that (3.19) hold, we must have $\underline{f_0}, \underline{g_0} \in (0, \infty]$ and $\overline{f_\infty}$, $\overline{g_\infty} \in [0, \infty)$. As $c_1 > 0$, in view of (4.6) and (4.7) we find that

$$L_3^f = \begin{cases} \big[c_1 \gamma_1 \int_\xi^\eta \min_{\xi \leq t \leq \eta} k_1(t, r) a(r) dr\big]^{-1}, & \text{if } b_1 = 0 \\ 0, & \text{if } b_1 > 0, \end{cases}$$

$$L_4^f = \big[c_1 \int_0^1 \max_{0 \leq t \leq 1} k_1(t, r) a(r) dr\big]^{-1} > 0.$$

Consequently, if $c_1 b_1 > 0$ then $L_3^f = 0 < L_4^f$ and so (L_3^f, L_4^f) is not void, while if $b_1 = 0$ then in order that $L_3^f < L_4^f$ it is necessary that

$$c_1 \int_0^1 \max_{0 \leq t \leq 1} k_1(t, r) a(r) dr < c_1 \gamma_1 \int_\xi^\eta \min_{\xi \leq t \leq \eta} k_1(t, r) a(r) dr,$$

which contradicts the fact that $\gamma_1 \in [0, 1]$. It follows that for $c_1 > 0$ Theorem 3.2 applies if and only if $b_1 > 0$, and in this case $(L_3^f, L_4^f) \neq \emptyset$. Observe that if $g(x) = c_2 x + b_2$, $x \geq 0$ then employing similar arguments we see that $\underline{g_0} \in (0, \infty]$ and $\overline{g_\infty} \in [0, \infty)$ only if $c_2 > 0$ and $b_2 > 0$, which contradicts the assumption $g(0) = 0$ posed in Theorem 3.2. Therefore, in case that $c_1 > 0$, Theorem 3.2 cannot be applied when both f and g are first degree polynomials. In conclusion, Theorem 3.2 can be applied if $f(x) = c_1x + b_1$, $x \geq 0$ with either $c_1 = 0$ or with $c_1b_1 > 0$, and g is a nonlinear function for which it holds $L_3^g < L_4^g$.

The next few lines are devoted to giving some weaker, but easier to verify alternatives of Theorems 3.1 and 3.2 by introducing the following assumption:

(K) There exist functions $k_i^m : [\xi, \eta] \to [0, \infty)$ $i = 1, 2$ and $k_i^M : [0, 1] \to [0, \infty)$, $i = 1, 2$ with

$$k_i^m(r) \leq \min_{\xi \leq t \leq \eta} k_i(t, r) \quad \text{for all } r \in [\xi, \eta], \quad i = 1, 2, \tag{4.8}$$

$$\max_{0 \leq t \leq 1} k_i(t, r) \leq k_i^M(r) \quad \text{for all } \ r \in [0, 1], \quad i = 1, 2. \tag{4.9}$$

Assuming that (K) holds true, we may consider the positive numbers K_3^f, K_3^g and K_4^f, K_4^g defined by

$$K_1^f := \begin{cases} [\gamma_1 \int_\xi^\eta k_1^m(r)a(r)\underline{f_\infty}dr]^{-1}, & \text{if } \underline{f_\infty} \in (0, \infty) \\ 0, & \text{if } \underline{f_\infty} = \infty, \end{cases}$$

$$K_1^g := \begin{cases} \big[\gamma_2 \int_\xi^\eta k_2^m(r)b(r)\underline{g_\infty}dr\big]^{-1}, & \text{if } \underline{g_\infty} \in (0, \infty) \\ 0, & \text{if } \underline{g_\infty} = \infty, \end{cases}$$

$$K_2^f := \begin{cases} \big[\int_0^1 k_1^M(r)a(r)\overline{f_0}dr\big]^{-1}, & \text{if } \overline{f_0} \in (0, \infty) \\ +\infty, & \text{if } \overline{f_0} = 0\,, \end{cases}$$

$$K_2^g := \begin{cases} \big[\int_0^1 k_2^M(r)b(r)\overline{g_0}dr\big]^{-1}, & \text{if } \overline{g_0} \in (0, \infty) \\ +\infty, & \text{if } \overline{g_0} = 0\,. \end{cases}$$

Then it it is not difficult to see that $L_i^f \leq K_i^f$ and $L_i^g \leq K_i^g$ $(i = 1, 2)$ and so from Theorems 3.1 and 3.2 we have the following two results.

Theorem 4.4. *Assume conditions* (A), (B), (C), (3.3) *are satisfied. Furthermore, assume that* (K) *holds and that*

$$K_1^f < K_2^f \quad \textit{and} \quad K_1^g < K_2^g, \tag{4.10}$$

where K_1^f, K_1^g *and* K_2^f, K_2^g *are defined as above. Then, for* λ, μ *with* $(\lambda, \mu) \in (K_1^f, K_2^f) \times (K_1^g, K_2^g)$ *equation* (1.1) *has a nonnegative solution.*

Theorem 4.5. *Assume conditions* (A), (B), (C), (3.19) *are satisfied. Moreover, assume that* $g(0) = 0$ *and* (K) *holds . If*

$$K_3^f < K_4^f \quad \textit{and} \quad K_3^g < K_4^g, \tag{4.11}$$

where K_3^f, K_3^g *and* K_4^f, K_4^g *are defined as above, then, for* λ, μ *with* $(\lambda, \mu) \in (K_3^f, K_4^f) \times (K_3^g, K_4^g)$ *equation* (1.1) *has a nonnegative solution.*

In some cases it seems easier to use Theorems 4.4 and 4.5 than Theorems 3.1 and 3.2, respectively, as it is rather simpler to spot some functions k_i^m, k_i^M, $i = 1, 2$ for which (4.8) and (4.9) hold than to calculate $\min_{\xi \leq t \leq \eta} k_i(t, r)$, $r \in [\xi, \eta]$, $i = 1, 2$ and $\max_{0 \leq t \leq 1} k_i(t, r)$, $r \in [0, 1]$, $i = 1, 2$. However, one may easily see that the corresponding intervals (K_i^f, K_{i+1}^f), (K_i^g, K_{i+1}^g), $i = 1, 3$ get shorter or, in case that some of (4.10) or (4.11) fail to hold, one or both of these intervals may not even make sense, hence Theorems 4.4 or 4.5 do not apply.

Finally, let us deal with the relation between the constants γ_i, $i = 1, 2$, the kernels k_i, $i = 1, 2$, and the intervals I_f and I_g in the case that $\overline{f_0}$, $\underline{f_\infty}$, $\overline{g_0}$, $\underline{g_\infty} \in (0, \infty)$. Assume, first, that there exist positive numbers m_i and M_i ($i = 1, 2$) such that

$$m_i \le k_i(t, s) \le M_i \quad \text{for } (t, s) \in [0, 1], \quad i = 1, 2, \tag{4.12}$$

and set

$$\gamma_i = \frac{m_i}{M_i}, \quad i = 1, 2 \quad \text{and} \quad \xi = 0,\ \eta = 1.$$

Then

$$\gamma_i k_i(t, s) \le \gamma_i M_i = m_i \le k_i(t, s) \quad \text{for } (t, s) \in [0, 1]\ i = 1, 2,$$

and so,

$$\gamma_i \max_{0 \le t \le 1} k_i(t, s) \le \gamma_i M_i = m_i \le \min_{\xi \le t \le \eta} k_i(t, s) \quad \text{for all } s \in [0, 1],\ i = 1, 2,$$

from which it follows that for $i = 1, 2$ it holds

$$\max_{0 \le t \le 1} k_i(t, s) \le \frac{M_i}{m_i} \min_{0 \le t \le 1} k_i(t, s) \le \frac{M_i}{m_i} \min_{\xi \le t \le \eta} k_i(t, s) \quad s \in [0, 1],$$

and so

$$\int_0^1 \max_{0 \le t \le 1} k_i(t, r) a(r) dr \le \frac{M_i}{m_i} \int_0^1 \min_{0 \le t \le 1} k_i(t, r) a(r) dr \le \frac{M_i}{m_i} \int_0^1 \min_{\xi \le t \le \eta} k_i(t, r) a(r) dr.$$

Having in mind that in case that $\overline{f_0}$, $\underline{f_\infty} \in (0, \infty)$ then $L_1^f < L_2^f$ may equivalently be written as

$$\frac{\overline{f_0}}{\underline{f_\infty}} \int_0^1 \max_{0 \le t \le 1} k_1(t, r) a(r) dr < \frac{m_1}{M_1} \int_0^1 \min_{\xi \le t \le \eta} k_1(t, r) a(r) dr,$$

it follows that a sufficient condition for $L_1^f < L_2^f$ is

$$\frac{\overline{f_0}}{\underline{f_\infty}} \Big(\frac{M_1}{m_1} \Big)^2 < \frac{\int_0^1 \min_{\xi \le t \le \eta} k_1(t, r) a(r) dr}{\int_0^1 \min_{0 \le t \le 1} k_1(t, r) a(r) dr}. \tag{4.13}$$

Similarly, if $\overline{g_0}$, $\underline{g_\infty} \in (0, \infty)$ then a sufficient condition for $L_1^g < L_2^g$ is

$$\frac{\overline{g_0}}{\underline{g_\infty}} \Big(\frac{M_2}{m_2} \Big)^2 < \frac{\int_0^1 \min_{\xi \le t \le \eta} k_2(t, r) b(r) dr}{\int_0^1 \min_{0 \le t \le 1} k_2(t, r) b(r) dr}. \tag{4.14}$$

We have the following corollary.

Corollary 4.6. *Assume conditions* (A), (B), (C) *are satisfied and that* $\overline{f_0}$, $\underline{f_\infty}$, $\overline{g_0}$, $\underline{g_\infty} \in (0, \infty)$. *In addition, suppose that there exist positive numbers* m_i *and* M_i *(*$i = 1, 2$*) such that* (4.12) *holds true. If* (4.13) *and* (4.14) *are fulfilled then there exist positive numbers* λ *and* μ *such that* (1.1) *has a nonnegative solution.*

As $\min_{0 \le t \le 1} k_i(t, r) \le \min_{\xi \le t \le \eta} k_i(t, r)$, $r \in [0, 1]$ ($i = 1, 2$), from Corollary 4.6 we have the following result which gives weaker but easier to verify sufficient conditions for I_f and I_g to be nonvoid.

Corollary 4.7. *Assume conditions* (A), (B), (C) *are satisfied and that* $\overline{f_0}$, $\underline{f_\infty}$, $\overline{g_0}$, $\underline{g_\infty} \in (0, \infty)$. *In addition, suppose that there exist positive numbers* m_i *and* M_i *(*$i = 1, 2$*) such that* (4.12) *holds. If*

$$\frac{\overline{f_0}}{\underline{f_\infty}} < \Big(\frac{m_1}{M_1} \Big)^2 \quad \text{and} \quad \frac{\overline{g_0}}{\underline{g_\infty}} < \Big(\frac{m_2}{M_2} \Big)^2,$$

then there exist positive numbers λ *and* μ *such that* (1.1) *has a nonnegative solution.*

Now let us suppose that $m_1 = 0$ and that

$$\text{there exists } \widehat{t} \in [0,1] \text{ such that } k_1(\widehat{t}, s) \text{ is nonzero for any } s \in [0,1]. \tag{4.15}$$

In view of the continuity of the kernel k_1 it follows that there exists some $\widehat{\xi}_1$, $\widehat{\eta}_1 \in [0,1]$ with $\widehat{\xi}_1 < \widehat{\eta}_1$ such that $k_1(t,s)$ is positive on the block $[\widehat{\xi}_1, \widehat{\eta}_1] \times [0,1]$ and so there exist some real numbers $\widehat{m}_1, \widehat{M}_1$ with $\max\limits_{0 \le t,s \le 1} k_i(t,s) = \widehat{M}_1 > 0$ and

$$\widehat{m}_1 = \min_{r \in [\widehat{\xi}_1, \widehat{\eta}_1]} k_1(r,s), \quad \text{for all } s \in [0,1].$$

Thus, setting $\gamma_1 = \widehat{m}_1 / \widehat{M}_1$, we have

$$\gamma_1 k_1(t,s) \le \gamma_1 \widehat{M}_1 = \widetilde{m}_1 = \inf_{r \in [\widehat{\xi}_1, \widehat{\eta}_1]} k_1(r,s), \quad \text{for all } t, s \in [0,1]$$

i.e.,

$$\min_{r \in [\widehat{\xi}_1, \widehat{\eta}_1]} k_1(r,s) \ge \gamma_1 k_1(t,s) \quad \text{for } (t,s) \in [0,1]^2.$$

Consequently, if (4.15) holds then $\max\limits_{\xi \le r \le \eta} [\min_{\xi \le t \le \eta} k_1(t,r)] > 0$. We conclude that (4.15) is a *sufficient condition* so that (C) is fulfilled. However, (4.15) *is not a necessary condition* for (C) to hold as it may happen that for any $t \in [\xi, \eta]$ we have $k_1(t,s) > 0$ for all $s \in [\xi, \eta]$ while $k_1(t, s_t) = 0$ for some $s_t \in [0,1] \setminus [\xi, \eta]$.

From the above discussion it follows that there may be more than one valid choice of ξ_i, η_i, γ_i for each kernel k_i $(i = 1,2)$ for which assumption (C) is fulfilled. This is an advantage of the results of the present investigation as we are allowed to look for the best choice of these parameters that optimize the eigenvalue intervals. However, this may not be an easy task since the longer we take the interval $[\xi_i, \eta_i]$ the smaller the positive constant γ_i becomes.

Recalling the notation in (2.3), i.e., setting

$$v(t) = \mu \int_0^1 k_2(t,s) b(s) g(u(s)) ds, \quad 0 \le t \le 1,$$

one may see that (1.1) can equivalently be written as the system of integral equations

$$\begin{gathered} u(t) = \lambda \int_0^1 k_1(t,s) a(s) f(v(t)) ds, \quad 0 \le t \le 1, \\ v(t) = \mu \int_0^1 k_2(t,r) b(r) g(u(r)) dr, \quad 0 \le t \le 1. \end{gathered} \tag{4.16}$$

We say that a pair (u,v) of functions $u, v \in C([0,1],[0,\infty))$ is a (nonnegative) *solution* of (4.16) if (u,v) satisfies (4.16) for all $t \in [0,1]$. As it concerns the notion of positivity for solutions to the system of integral equations (4.16), we will say that a solution (u,v) of (4.16) is *positive on the (nonvoid) set* $I \times J \subseteq [0,1]^2$ if $u(t) > 0$ for $t \in I$ and $v(t) > 0$ for $t \in J$. As it seems more convenient to work with (1.1) than with the integral system (4.16), we have chosen to establish our results for (1.1) and then show how these results may be applied on an integral system such as (4.16). In particular, the next section contains applications of our results to systems of BVP which may be formulated as systems of integral equations of the type of (4.16).

Finally, we note that our results may easily be applied to the special case of (1.1) taken for $\lambda = \mu = 1$, i.e., the integral equation

$$u(t) = \int_0^1 k_1(t,s)a(s)f\Big(\int_0^1 k_2(s,r)b(r)g(u(r))dr\Big)ds, \quad 0 \le t \le 1, \tag{4.17}$$

or to the system of integral equations

$$\begin{aligned} u(t) &= \int_0^1 k_1(t,s)a(s)f(v(t))ds, \quad 0 \le t \le 1,\\ v(t) &= \int_0^1 k_2(t,r)b(r)g(u(r))dr, \quad 0 \le t \le 1. \end{aligned} \tag{4.18}$$

As an example, we state the following result which is an immediate consequence of Theorem 3.1.

Theorem 4.8. *Assume conditions* (A), (B), (C), (3.3) *are satisfied and define* L_1^f, L_1^g *by* (3.4) *and* L_2^f, L_2^g *by* (3.5)*. If*

$$L_1^f < 1 < L_2^f \quad \text{and} \quad L_1^g < 1 < L_2^g$$

then (4.17) *has a nonnegative solution* u *(or, equivalently,* (4.18) *has a nonnegative solution* (u,v)*).*

5. Applications to systems of boundary value problems

This section is devoted to applying our results to systems of BVP concerning differential equations.

The applications below bend on the observation that a large class of BVP concerning differential equations may be converted to integral equations by the use of Green's functions and so a system of BVP can equivalently be written as system of integral equations. In case that the integral system can be formulated as a single integral equation such as (1.1), then results valid for (1.1) may yield analogous results for the initial system of BVP. It is not difficult to see that the above argument may still hold even in the case that the starting system consists of differential BVP together with integral equations. In this section we show how existence results for systems of BVP may be deducted from corresponding results obtained for (1.1). It comes out that (1.1) is general enough to include a variety of systems of BVP and so the results of this paper include (and in certain cases extend or generalize) several known existence results concerning nonnegative/positive solutions (e.g., see [5] - [9]). We note that in a large number of such problems the assumptions (A), (B) are always fulfilled while condition (C) comes as a property of the Green's function(s) for suitable values of the constants γ, ξ and η. Hence, our results may easily be applied to a large class of integral equations or systems of BVP for which the corresponding Green's functions satisfy conditions (A), (B) and (C).

The systems of BVP considered in the applications below have been selected mainly for two reasons: to illustrate the routine by which existence of nonnegative/positive solutions may be obtained and to underline the variety of BVP for which the results of the paper are applicable.

The first application concerns a system of two multi-point second-order BVP where the sets of points at which the boundary conditions are considered may be different and the cardinality of these sets may not be the same. The second application deals with a system of two-point BVP of third order with different types of boundary conditions. We note that in this BVP the choice of the constants γ, ξ and η is not unique as ξ_i and η_i $(i = 1, 2)$ may arbitrarily be chosen. Finally, in the third application we consider a system of two BVP that differ not only in the boundary conditions but, also, in order. To the best of our knowledge, such type of system has not been considered so far. The results obtained in the first two applications improve known results.

5.1. **A system of multi-point second-order bvp.** Consider the system of BVP consisting of the second order ordinary differential equations

$$\begin{aligned} u''(t) + \lambda a(t) f(v(t)) = 0, \quad 0 < t < 1 \\ v''(t) + \mu b(t) g(u(t)) = 0, \quad 0 < t < 1 \end{aligned} \tag{5.1}$$

along with the multi-point boundary value conditions ($m, n \geq 3$ are positive integers)

$$\begin{aligned} u(0) = 0, \quad u'(1) = \sum_{i=1}^{m-2} \widehat{a}_i u'(\widehat{\zeta}_i), \\ v(0) = 0, \quad v'(1) = \sum_{j=1}^{n-2} \widetilde{a}_i v'(\widetilde{\zeta}_j), \end{aligned} \tag{5.2}$$

where $0 < \widehat{a}_i$, $(i = 1, \dots, m-2)$, $0 < \widehat{\zeta}_1 < \dots < \widehat{\zeta}_{m-2} < \widehat{\zeta}_{m-1} = 1$, $\widetilde{a}_j$ $(j = 1, \dots, n-2)$, $0 = \widetilde{\zeta}_0 < \widetilde{\zeta}_1 < \dots < \widetilde{\zeta}_{n-2} < \widetilde{\zeta}_{n-1} = 1$. We assume that the functions f, g and a, b satisfy (A) and (B).

We will make use of the following lemma taken from [19].

Lemma 5.1. *Let $0 < a_i, (i = 1, \dots, k-2)$, $0 = \zeta_0 < \zeta_1 < \dots < \zeta_{k-2} < 1$, $0 < \sum_{i=1}^{k-2} a_i \zeta_i < 1$ ($k \geq 3$ is a positive integer). The Green's function G_2 for the BVP*

$$\begin{gathered} -u''(t) = 0, \quad 0 < t < 1 \\ u(0) = 0, \quad u'(1) = \sum_{i=1}^{k-2} a_i u'(\zeta_i) \end{gathered}$$

is given by

$$\begin{aligned} &G_2(t, s) \\ &= \begin{cases} s + \frac{\sum_{i=1}^{w-1} a_i}{1 - \sum_{i=1}^{k-2} a_i} t, & 0 \leq t \leq 1,\ \zeta_{w-1} \leq s \leq \min\{\zeta_w, t\},\ w = 1, 2, \dots, k-1 \\ \frac{1 - \sum_{i=w}^{k-2} a_i}{1 - \sum_{i=1}^{k-2} a_i} t, & 0 \leq t \leq 1,\ \max\{\zeta_{w-1}, t\} \leq s \leq \zeta_w,\ w = 1, 2, \dots, k-1. \end{cases} \end{aligned}$$

It follows that a pair (u, v) is a solution of the system of BVP (5.1)-(5.2) if and only if (u, v) is a solution of the system

$$\begin{aligned} u(t) = \lambda \int_0^1 G_2^1(t, s) a(s) f(v(s)) ds, \quad 0 \leq t \leq 1, \\ v(t) = \mu \int_0^1 G_2^2(t, r) b(r) g(u(r)) dr, \quad 0 \leq t \leq 1. \end{aligned}$$

i.e., if u satisfies (1.1) with $k_1 = G_2^1$ and $k_2 = G_2^2$. By Lemma 5.1, for $t \in [\zeta_{k-2}, 1]$ and $0 \leq s \leq \zeta_{k-2}$ we have $\zeta_{k-2} = \min\{\zeta_{k-2}, t\}$ and $\max\{\zeta_{k-2}, t\} = t$, and so, in view of the convention $\sum_{j=k-1}^{k-2} a_j = 0$, we have for $t \in [\zeta_{k-2}, 1]$

$$G_2(t,s) = \begin{cases} s + \frac{\sum_{i=1}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} t, & 0 \le s \le t \le 1, t \in [\zeta_{k-2}, 1] \\ \frac{1}{1-\sum_{i=1}^{k-2} a_i} t, & \zeta_{k-2} \le t \le s \le 1, \end{cases}$$

from which we find

$$\min_{\zeta_{k-2} \le t \le 1} G_2(t,s) = \begin{cases} s + \frac{\sum_{i=1}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} \zeta_{k-2}, & 0 \le s \le t, \\ \frac{1}{1-\sum_{i=1}^{k-2} a_i} \zeta_{k-2}, & t \le s \le 1, \end{cases}$$

hence

$$\min_{\zeta_{k-2} \le t \le 1} G_2(t,s) \ge \frac{\sum_{i=1}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} \zeta_{k-2}. \tag{5.3}$$

On the other hand for $t \in [0,1]$, we have

$$\begin{aligned} G_2(t,s) &= \begin{cases} s + \frac{\sum_{i=1}^{w-1} a_i}{1-\sum_{i=1}^{k-2} a_i} t, & \zeta_{w-1} \le s \le \min\{\zeta_w, t\}, w = 1, 2, \dots, k-1 \\ \frac{1-\sum_{i=w}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} t, & \max\{\zeta_{w-1}, t\} \le s \le \zeta_w,\ w = 1, 2, \dots, k-1 \end{cases} \\ &\le \begin{cases} t + \frac{\sum_{i=1}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} t, & \zeta_{w-1} \le s \le \min\{\zeta_w, t\},\ w = 1, 2, \dots, k-1 \\ \frac{1}{1-\sum_{i=1}^{k-2} a_i} t, & \max\{\zeta_{w-1}, t\} \le s \le \zeta_w,\ w = 1, 2, \dots, k-1 \end{cases} \\ &\le \frac{1}{1-\sum_{i=1}^{k-2} a_i} t, \end{aligned}$$

i.e.,

$$G_2(t,s) \le \frac{1}{1-\sum_{i=1}^{k-2} a_i} t, \quad \text{for } s, t \in [0,1]. \tag{5.4}$$

From (5.3) and (5.4), for $s, t \in [0,1]$, we find

$$\begin{aligned} G_2(t,s) &\le \frac{1}{1-\sum_{i=1}^{k-2} a_i} t \\ &\le \frac{t}{\zeta_{k-2} \sum_{i=1}^{k-2} a_i} \frac{\sum_{i=1}^{k-2} a_i}{1-\sum_{i=1}^{k-2} a_i} \zeta_{k-2} \\ &= \frac{1}{\zeta_{k-2} \sum_{i=1}^{k-2} a_i} \min_{\zeta_{k-2} \le t \le 1} G_2(t,s) \end{aligned}$$

which implies

$$G_2(t,s) \le \frac{1}{\zeta_{k-2} \sum_{i=1}^{k-2} a_i} \min_{\zeta_{k-2} \le t \le 1} G_2(t,s) \quad \text{for all } (t,s) \in [0,1]. \tag{5.5}$$

From (5.5) it follows that

$$\begin{aligned} G_2^1(t,s) &\le \gamma_1 \min_{\widehat{\zeta}_{m-2} \le t \le 1} G_2^1(t,s) \quad \text{for all } (t,s) \in [0,1], \\ G_2^2(t,s) &\le \gamma_2 \min_{\widetilde{\zeta}_{n-2} \le t \le 1} G_2^2(t,s) \quad \text{for all } (t,s) \in [0,1], \end{aligned}$$

where

$$\gamma_1 = \frac{1}{\widehat{\zeta}_{m-2} \sum_{i=1}^{m-2} \widehat{a}_i}, \quad \gamma_2 = \frac{1}{\widetilde{\zeta}_{n-2} \sum_{j=1}^{n-2} \widetilde{a}_j},$$

and so condition (C) is fulfilled with $\xi = \max\{\widehat{\zeta}_{m-2}, \widehat{\zeta}_{n-2}\}$, $\eta = 1$ and $\gamma = \min\{\gamma_1, \gamma_2\}$.

From the definition of G_2 in Lemma 5.1 it follows that condition (4.1) is satisfied on the interval $I = (0,1]$. In connection to the discussion at the beginning of Section 3, we note that $u(0) = 0$ is yielded by the fact that $G_2(0,s) = 0$, $s \in [0,1]$ while $u(1) > 0$ follows from the fact that $G_2(1,s) > 0$ for $s \in [\zeta_{m-2}, 1]$. Applying Corollary 4.2 we have the following Proposition.

Proposition 5.2. *Assume conditions* (A), (B) *are satisfied. Moreover, assume that* $\overline{f_0}, \overline{g_0} \in [0,\infty)$, $\underline{f_\infty}$, $\underline{g_\infty} \in (0,\infty]$ *where* $\overline{f_0}$, $\overline{g_0}$, $\underline{f_\infty}$ *and* $\underline{g_\infty}$ *are defined by* (3.1) *and define* $\ell^A_{f,1}$, $\ell^A_{f,2}$ *and* $\ell^A_{g,1}$ *and* $\ell^A_{g,2}$ *by*

$$\ell^A_{f,1} := \begin{cases} \Big[\gamma_1 \underline{f_\infty} \frac{\min\{1-\sum_{i=j}^{m-2}\widehat{a}_j, \sum_{i=1}^{j-1}\widehat{a}_j\}}{1-\sum_{i=1}^{m-2}a_i} \int_\xi^1 a(r)dr\Big]^{-1}, & \text{if } \underline{f_\infty} \in (0,\infty) \\ 0, & \text{if } \underline{f_\infty} = \infty, \end{cases}$$

$$\ell^A_{f,2} := \begin{cases} \Big[\overline{f_0} \frac{\max\{1-\sum_{i=j}^{m-2}a_j, \sum_{i=1}^{j-1}a_j\}}{1-\sum_{i=1}^{m-2}a_i} \int_0^1 a(r)dr\Big]^{-1}, & \text{if } \overline{f_0} \in (0,\infty) \\ +\infty, & \text{if } \overline{f_0} = 0, \end{cases}$$

$$\ell^A_{g,1} := \begin{cases} \Big[\gamma_2 \underline{g_\infty} \frac{\min\{1-\sum_{i=j}^{n-2}a_j, \sum_{i=1}^{j-1}a_j\}}{1-\sum_{i=1}^{m-2}a_i} \int_\xi^1 b(r)dr\Big]^{-1}, & \text{if } \underline{g_\infty} \in (0,\infty) \\ 0, & \text{if } \underline{g_\infty} = \infty, \end{cases}$$

$$\ell^A_{g,2} := \begin{cases} \Big[\overline{g_0} \frac{\max\{1-\sum_{i=j}^{m-2}a_j, \sum_{i=1}^{j-1}a_j\}}{1-\sum_{i=1}^{m-2}a_i} \int_0^1 b(r)dr\Big]^{-1}, & \text{if } \overline{g_0} \in (0,\infty) \\ +\infty, & \text{if } \overline{g_0} = 0. \end{cases}$$

Then, for any $\lambda \in (\ell^A_{f,1}, \ell^A_{f,2})$ *and* $\mu \in (\ell^A_{g,1}, \ell^A_{g,2})$ *there exists a nonnegative solution* (u,v) *of* (5.1)-(5.2). *Furthermore, if in addition it holds* $xf(x) > 0$ *for* $x \neq 0$ *and* $xg(x) > 0$ *for* $x \neq 0$ *then* $u(x) > 0$ *and* $v(x) > 0$ *for* $x \in (0,1]$.

The existence of positive eigenvalues for the special case of the system (5.1)-(5.2) taken for $m = n$ and $\widehat{\zeta}_i = \widetilde{\zeta}_i$ $i = 1, \dots, m-1$, has also been discussed in [9]. However, Proposition 5.2 (as well as the analogous proposition corresponding to Theorem 3.2) improves and generalizes these existence results discussed in [9] not only by allowing the points in the boundary conditions to be arbitrarily chosen (and not necessarily of the same number) but also by replacing lim by lim sup or lim inf.

5.2. **A system of third order bvp.** In this subsection we show how our results may be applied to a system of BVP consisting of two differential equations of third order but different boundary conditions concerning the same points (endpoints) of the interval $[0,1]$. It is interesting that the points ξ, η may arbitrarily be chosen in the interval $(0,1)$ (provided that $\xi < \eta$). More precisely, we consider the system consisting of the third order differential equations

$$\begin{aligned} u'''(t) + \lambda a(t) f(v(t)) = 0, \quad 0 < t < 1 \\ v'''(t) + \mu b(t) g(u(t)) = 0, \quad 0 < t < 1 \end{aligned} \tag{5.6}$$

along with the two-point boundary conditions

$$u'(0) = u''(0) = u(1) = 0, \tag{5.7}$$

$$v(0) = v'(0) = v''(1) = 0. \tag{5.8}$$

Concerning the BVP (5.6)-(5.7) we have the following lemma taken from [34].

Lemma 5.3. *For any $y \in C([0,1],\mathbb{R})$, the boundary value problem consisting of the third order differential equation*

$$u'''(t) + y(t) = 0, \quad t \in (0,1) \tag{5.9}$$

along with the initial condition (5.7) *has the unique solution*

$$u(t) = \int_0^1 G_3^1(t,s)y(s)ds, \quad t \in [0,1],$$

where

$$G_3^1(t,s) = \frac{1}{2}\begin{cases}(1-s)^2 - (t-s)^2, & 0 \le s \le t \le 1,\\ (1-s)^2, & 0 \le t \le s \le 1.\end{cases}$$

It is not difficult to verify (see, [28]) that

$$G_3^1(t,s) \le \frac{1}{2}(1-s)^2 \le \frac{1}{2}, \quad \text{for all } (t,s) \in [0,1]^2.$$

For estimating $\min_{t\in[\xi_1,\eta_1]} G_3^1(t,s)$ where $0 \le \xi_1 < \eta_1 < 1$ we consider the two cases below.

Case I. $0 \le s \le t \le 1$. We have

$$\begin{aligned}\min_{t\in[\xi_1,\eta_1]} G_3^1(t,s) &= \min_{t\in[\xi_1,\eta_1]} \frac{1}{2}[(1-s)^2 - (t-s)^2]\\ &= \frac{1}{2}[(1-s)^2 - (\eta_1 - s)^2]\\ &= \frac{1}{2}(1-\eta_1)(1+\eta_1 - 2s)\\ &\ge \frac{1}{2}(1-\eta_1)(1+\eta_1-2\eta_1)\\ &\ge \frac{1}{2}(1-\eta_1)^2\end{aligned}$$

and so

$$\min_{t\in[\xi_1,\eta_1]} G_3^1(t,s) \ge \frac{1}{2}(1-\eta_1)^2 G_3^1(t,s), \quad \text{for } 0 \le s \le t \le 1. \tag{5.10}$$

Case II. $0 \le t \le s \le 1$. We have

$$\min_{t\in[\xi_1,\eta_1]} G_3^1(t,s) = \frac{1}{2}(1-s)^2 \ge G_3^1(t,s), \quad \text{for } 0 \le t \le s \le 1. \tag{5.11}$$

By (5.10) and (5.11) it follows that

$$\frac{1}{2}(1-\eta_1)^2 G_3^1(t,s) \le \inf_{t\in[\xi_1,\eta_1]} G_3^1(t,s), \quad (t,s) \in [0,1]^2. \tag{5.12}$$

Concerning the BVP (5.9)-(5.8) we have the next lemma taken from [21].

Lemma 5.4. *For any $y \in C([0,1],\mathbb{R})$, the boundary value problem* (5.9)-(5.8) *has the unique solution*

$$u(t) = \int_0^1 G_3^2(t,s)y(s)ds, \quad t \in [0,1]$$

where

$$G_3^2(t,s) = \frac{1}{2}\begin{cases}t^2, & 0 \le t \le s \le 1\\ t^2 - (t-s)^2, & 0 \le s \le t \le 1.\end{cases}$$

It follows that

$$G_3^2(t,s) \le ts, \quad \text{for all } (t,s) \in [0,1]^2. \tag{5.13}$$

For estimating $\inf_{t\in[\xi_2,\eta_2]} G_3^2(t,s)$ where $0 \le \xi_2 < \eta_2 \le 1$, we first note that

$$\min_{t\in[\xi_2,\eta_2]} G_3^2(t,s) = \frac{1}{2}\xi_2^2, \quad 0 \le t \le s \le 1 \tag{5.14}$$

while for $0 \le s \le t \le 1$, we have

$$\min_{t\in[\xi_2,\eta_2]} G_3^2(t,s) = \min_{t\in[\xi_2,\eta_2]} \frac{1}{2}[t^2 - (t-s)^2] = \min_{t\in[\xi_2,\eta_2]} \frac{1}{2}(2t-s)s \ge \min_{t\in[\xi_2,\eta_2]} \frac{1}{2}ts$$

from which it follows that

$$\min_{t\in[\xi_2,\eta_2]} G_3^2(t,s) \ge \frac{1}{2}\xi_2 s, \quad \text{for } 0 \le s \le t \le 1. \tag{5.15}$$

By (5.14) and (5.15) we have

$$\min_{t\in[\xi_2,\eta_2]} G_3^2(t,s) \ge \frac{1}{2}\xi_2 \min\{\xi_2, s\} \quad \text{for } s \in [0,1],$$

from which in view of $\min\{\xi_2, s\} \ge \xi_2 s$ we have

$$\min_{t\in[\xi_2,\eta_2]} G_3^2(t,s) \ge \frac{1}{2}\xi_2^2 s, \quad \text{for } s \in [0,1].$$

Combining the last inequality with (5.13), we take for $(t,s) \in [0,1]^2$,

$$\frac{1}{2\eta_2}\xi_2^2 G_3^2(t,s) \le \frac{1}{2\eta_2}\xi_2^2 ts \le \min_{t\in[\xi_2,\eta_2]} G_3^2(t,s), \tag{5.16}$$

thus

$$\gamma_2 G_3^2(t,s) \le \min_{t\in[\xi_2,\eta_2]} G_3^2(t,s), \quad (t,s) \in [0,1]^2 \tag{5.17}$$

where $\gamma_2 = \frac{1}{2\eta_2}\xi_2^2$. Setting $\xi = \max\{\xi_1, \xi_2\}$, $\eta = \min\{\eta_1, \eta_2\}$ and

$$\gamma = \frac{1}{2}\min\big\{(1-\eta_1)^2, \frac{\xi_2^2}{\eta_2}\big\},$$

from (5.12) and (5.17)) we conclude that (C) is fulfilled on any nonvoid arbitrarily chosen interval $[\xi, \eta] \subseteq (0,1)$, provided that $\xi < \eta$. Observing that (4.1) is satisfied on $I = (0,1]$, from Corollary 4.2 we have the next proposition, which, to the best of our knowledge, is a new result.

Proposition 5.5. *Assume conditions* (A), (B) *are satisfied. Moreover, assume that* $\overline{f_0}, \overline{g_0} \in [0,\infty)$ *and* $\underline{f_\infty}, \underline{g_\infty} \in (0,\infty]$ *where* $\overline{f_0}, \overline{g_0}$, *and* $\underline{f_\infty}, \underline{g_\infty}$ *are defined by*

(3.1) and define $\ell_1^{B_1}$, $\ell_2^{B_1}$, $\ell_1^{B_2}$ and $\ell_2^{B_2}$ by

$$\ell_1^{B_1} := \begin{cases} \left[(1-\eta_1)^2 \int_\xi^\eta \min_{\xi\le t\le\eta} G_3^1(t,r)a(r)\underline{f_\infty}dr\right]^{-1}, & \text{if } \underline{f_\infty} \in (0,\infty) \\ 0, & \text{if } \underline{f_\infty} = \infty, \end{cases}$$

$$\ell_2^{B_1} := \begin{cases} \left[\int_0^1 \max_{0\le t\le 1} G_3^1(t,r)a(r)\overline{f_0}dr\right]^{-1}, & \text{if } \overline{f_0} \in (0,\infty) \\ +\infty, & \text{if } \overline{f_0} = 0, \end{cases}$$

$$\ell_1^{B_2} := \begin{cases} \left[\frac{1}{2\eta_2}\xi_2^2 \int_\xi^\eta \min_{\xi\le t\le\eta} G_3^2(t,r)a(r)\underline{f_\infty}dr\right]^{-1}, & \text{if } \underline{f_\infty} \in (0,\infty) \\ 0, & \text{if } \underline{f_\infty} = \infty, \end{cases}$$

$$\ell_2^{B_2} := \begin{cases} \left[\int_0^1 \max_{0\le t\le 1} G_3^2(t,r)a(r)\overline{f_0}dr\right]^{-1}, & \text{if } \overline{f_0} \in (0,\infty) \\ +\infty, & \text{if } \overline{f_0} = 0. \end{cases}$$

Then, for any $(\lambda,\mu) \in (\ell_1^{B_1}, \ell_2^{B_1}) \times (\ell_1^{B_2}, \ell_2^{B_2})$ there exists a nonnegative solution (u,v) of (5.6)-(5.7)-(5.8). If, in addition, $xf(x) > 0$ for $x \neq 0$ and $xg(x) > 0$ for $x \neq 0$ then there exists a nonnegative solution (u,v) of (5.6)-(5.7)-(5.8) such that $u(x) > 0$ and $v(x) > 0$ for $x \in (0,1]$.

5.3. **A system of mixed type.** Here, we show that the results of this paper can easily be applied to obtain eigenvalue intervals for systems of BVP where the differential equations are not of the same order. For simplicity, we consider a system of BVP consisting of types of BVP already mentioned, namely the differential equations

$$\begin{gathered} u''(t) + \lambda a(t) f(v(t)) = 0, \quad 0 < t < 1 \\ v'''(t) + \mu b(t) g(u(t)) = 0, \quad 0 < t < 1 \end{gathered} \tag{5.18}$$

along the boundary value conditions

$$\begin{gathered} u(0) = 0, \quad u'(1) = \sum_{i=1}^{k-2} a_i u'(\zeta_i) \\ v(0) = v'(0) = v''(1) = 0, \end{gathered} \tag{5.19}$$

where $0 < a_i$, $(i = 1,\dots,k-2)$, $0 = \zeta_0 < \zeta_1 < \dots < \zeta_{k-2} < 1$, $0 < \sum_{i=1}^{k-2} a_i\zeta_i < 1$, i.e., the boundary condition (5.2) and the boundary condition (5.8).

Taking into consideration Lemma 5.1 and Lemma 5.3, it is not difficult to see that (u,v) is a solution of the system (5.18)-(5.19) if and only if u is a solution of the integral equation

$$u(t) = \lambda \int_0^1 G_2(t,s)a(s) f\Big(\mu \int_0^1 G_3^2(s,r)b(r)g(u(r))dr\Big)ds, \quad 0 \le t \le 1, \tag{5.20}$$

and v is given by

$$v(t) = \mu \int_0^1 G_3^2(t,r)b(r)g(u(r))dr, \quad 0 \le t \le 1,$$

with G_2 and G_3^2 given in Lemma 5.1 and Lemma 5.4, respectively.

Then for $\xi,\eta \in [\zeta_{k-2}, 1]$ with $\xi < \eta$, in view of (5.5) and (5.16) we may see that (C) is satisfied with

$$\gamma = \min\Big\{\frac{1}{\zeta_{k-2}\sum_{i=1}^{k-2} a_i}, \frac{1}{2\eta}\xi^2\Big\}.$$

From Theorem 3.1, we have the following result concerning the system (5.18)-(5.19).

Proposition 5.6. *Assume conditions* (A), (B), *are satisfied. Moreover, assume that* $\overline{f_0}, \overline{g_0} \in [0,\infty)$ *and* $\underline{f_\infty}, \underline{g_\infty} \in (0,\infty]$ *where* $\overline{f_0}, \overline{g_0}$, *and* $\underline{f_\infty}, \underline{g_\infty}$ *are defined by* (3.1). *Let* $\ell^A_{f,1}$, $\ell^A_{f,2}$ *and* $\ell_1^{B_2}$, $\ell_2^{B_2}$ *be defined as in Proposition 5.2 and Proposition 5.5, respectively. Then, for any* $(\lambda,\mu) \in (\ell^A_{f,1}, \ell^A_{f,2}) \times (\ell_1^{B_2}, \ell_2^{B_2})$ *there exists a nonnegative solution* (u,v) *of the system* (5.18)-(5.19) *(equivalently, a nonnegative solution* u *of* (5.20)*). If, in addition,* $xf(x) > 0$ *for* $x \neq 0$ *and* $xg(x) > 0$ *for* $x \neq 0$ *then there exists a nonnegative solution* (u,v) *of* (5.18)-(5.19) *with* $u(x) > 0$ *and* $v(x) > 0$ *for* $x \in (0,1]$ *(equivalently, a nonnegative solution* u *of* (5.20) *which is positive on* $(0,1]$*).*

The above result is a new one and maybe the first of its kind as systems of BVP concerning differential equations of different order seem not to have been considered before.

6. A generalization

For the sake of simplicity, we have chosen to focus, in some detail, to nonnegative solutions of (1.1) than to deal with the existence of positive eigenvalues λ_i ($i = 1,\dots,n$) yielding nonnegative solutions to the more general equation

$$\begin{aligned} u(t) = \lambda_1 \int_0^1 & k_1(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 k_2(s_1,s_2)a_2(s_2)f_2\Big(\dots \\ & f_{n-1}\Big(\int_0^1 k_n(s_{n-1},s_n)a_n(s_n)f_n(u(s_n))ds_n\Big)ds_{n-1}\Big)\dots ds_2\Big)ds_1, \end{aligned} \tag{6.1}$$

where $0 \le t \le 1$ and $n \ge 2$ is a positive integer. We study this equation under the following assumptions:

- (An) $f_i \in C([0,\infty),[0,\infty))$, $i = 1,\dots,n$;
- (Bn) $a_i \in C([0,1],[0,\infty))$, $i = 1,\dots,n$, and each does not vanish identically on any subinterval of $[0,1]$;
- (Cn) $k_i(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,\dots,n$ are continuous functions and there are points ξ_i, $\eta_i \in [0,1]$, $i = 1,\dots,n$ and positive numbers γ_i, $i = 1,\dots,n$ such that the kernels k_i are nonzero on $[\xi_i,\eta_i]$, $i = 1,\dots,n$ and satisfy
$$\min_{\xi_i \le t,s \le \eta_i} k_i(t,s) \ge \gamma_i k_i(t,s) \quad \text{for } (t,s) \in [0,1]^2, \quad i = 1,\dots,n.$$

Clearly, the equation (6.1) may equivalently be written as the system of integral equations

$$\begin{gathered} u_1(t) = \lambda_1 \int_0^1 k_1(t,s)a_1(s)f_1(u_2(s))ds, \quad 0 \le t \le 1, \\ u_2(t) = \lambda_2 \int_0^1 k_2(t,s)a_2(s)f_2(u_3(s))ds, \quad 0 \le t \le 1, \\ \dots \\ u_n(t) = \lambda_n \int_0^1 k_n(t,s)a_n(s)f_n(u_1(s))ds, \quad 0 \le t \le 1. \end{gathered} \tag{6.2}$$

It is not difficult to see that following the arguments used to prove Theorems 3.1 and 3.2, one can obtain results on the nonnegative solutions to (6.1) that are similar to the ones obtained for (1.1); these results also hold for the integral system (6.2). Below we state only the generalization of Theorem 3.1 and leave the corresponding one of Theorem 3.2 to the interested reader.

Theorem 6.1. *Assume conditions* (An) (Bn), (Cn). *Furthermore, we assume that*

$$\overline{f_0^i} \in [0, \infty), \quad \underline{f_\infty^i} \in (0, \infty], \quad i = 1, \dots, n, \tag{6.3}$$

where

$$\overline{f_0^i} = \limsup_{u \to 0+} \frac{f_i(u)}{u}, \quad \underline{f_\infty^i} = \liminf_{u \to \infty} \frac{f(u)}{u}, \quad i = 1, \dots, n, \tag{6.4}$$

and define $L_1^{f_i}$ and $L_2^{f_i}$ $(i = 1, \dots, n)$ by

$$L_1^{f_i} := \begin{cases} \big[\gamma_i \int_\xi^\eta \min_{\xi_i \le t \le \eta_i} k_i(t,r) a_i(r) \underline{f_\infty^i} dr\big]^{-1}, & \text{if } \underline{f_\infty^i} \in (0, \infty), \\ 0, & \text{if } \underline{f_\infty^i} = \infty, \end{cases} \tag{6.5}$$

$$L_2^{f_i} := \begin{cases} \big[\int_0^1 \max_{0 \le t \le 1} k_i(t,r) a_i(r) \overline{f_0^i} dr\big]^{-1}, & \text{if } \overline{f_0^i} \in (0, \infty), \\ +\infty, & \text{if } \overline{f_0^i} = 0. \end{cases} \tag{6.6}$$

Then, for λ_i with $\lambda_i \in (L_1^{f_i}, L_2^{f_i})$, $i = 1, \dots, n$, there exists a nonnegative solution u of (6.1) *(or, equivalently, a nonnegative solution $(u_1, \dots, u_n)$, of* (6.2)*).*

We note that comments similar to the ones made in Section 3 for (1.1) (also valid for the integral system (4.16)) may easily be extended to (6.1) (also valid for (6.2)).

Working in a similar way as in the applications in Section 3, one can apply Theorem 4.4 to obtain existence results for the systems of BVP consisting of n differential equations of arbitrary order. In particular, we may consider the system of BVP consisting of n differential equations of second order

$$\begin{gathered} u_i''(t) + \lambda_i a_i(t) f_i(u_{i+1}(t)) = 0, \quad t \in (0,1)\ i = 1, \dots, n, \\ u_{n+1}(t) = u_1(t), \quad t \in [0,1], \end{gathered} \tag{6.7}$$

along with the boundary value conditions

$$u_i(0) = 0 = u_i(1), \quad i = 1, \dots, n. \tag{6.8}$$

The Green's function for the associated problem

$$\begin{gathered} -u''(t) = 0, \quad t \in (0,1) \\ u(0) = 0 = u(1) \end{gathered}$$

is given by

$$\widehat{G}_2(t,s) = \begin{cases} t(1-s), & \text{if } 0 \le t \le s \le 1, \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

(see, [26]). It is easy to verify that

$$\widehat{G}_2(t,s) \le \widehat{G}_2(s,s) \le \frac{1}{4}, \quad (t,s) \in [0,1]^2. \tag{6.9}$$

and that for $s \in [0,1]$, it holds

$$\begin{aligned}
\min_{r\in[\xi,\eta]} \widehat{G}_2(r,s) &= \min_{r\in[\xi,\eta]} \begin{cases} r(1-s), & \text{if } 0 \le r \le s \le 1 \\ s(1-r), & \text{if } 0 \le s \le r \le 1 \end{cases} \\
&= \begin{cases} \xi(1-s), & \text{if } r \in [\xi,\eta] \text{ and } \xi \le r \le s \le 1 \\ s(1-\eta), & \text{if } r \in [\xi,\eta] \text{ and } 0 \le s \le r \le \eta \end{cases} \\
&\ge \xi(1-\eta)(1-s)s \\
&\ge \xi(1-\eta)\widehat{G}_2(s,s)
\end{aligned}$$

and so by (6.9) we obtain

$$\min_{t\in[\xi,\eta]} \widehat{G}_2(t,s) \ge \xi(1-\eta)\widehat{G}_2(t,s), \quad (t,s) \in [0,1]^2.$$

In view of the above inequality, applying Theorem 6.1 we have the following result.

Proposition 6.2. *Assume conditions* (An), $(\underline{\text{B}}\text{n})$, (Cn) *are satisfied. Moreover, suppose that* (6.3) $(i = 1,\dots,n)$ *hold, where* $\overline{f_0^i}$ *and* $\underline{f_\infty^i}$ $(i = 1,\dots,n)$ *are given by* (6.4) *and define* $L_1^{f_i}, L_2^{f_i}$ *by* (6.5) *and* (6.6) *with* $\gamma_i = \xi(1-\eta)$ *and* $k_i = \widehat{G}_2$ $(i = 1,\dots,n)$*. Then, for* λ_i*,* $i = 1,\dots,n$*, with* $\lambda_i \in (L_1^{f_i}, L_2^{f_i})$*,* $i = 1,\dots,n$*, there exists a nonnegative solution* $(u_1,\dots,u_n)$ *of* (6.7)-(6.8).

The existence of positive eigenvalues yielding nonnegative solutions to a BVP concerning an iterative system of the type of (6.2) on a time scale $\mathbb{T}$ has been investigated by the authors in [2]. The results of this paper extend some particular results in [2] taken for the special case $\mathbb{T} = \mathbb{R}$ by replacing lim by lim sup or lim inf.

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