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EXISTENCE AND CONTINUITY OF GLOBAL ATTRACTORS FOR A DEGENERATE SEMILINEAR PARABOLIC EQUATION

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Abstract. In this article, we study the existence and the upper semicontinuity with respect to the nonlinearity and the shape of the domain of global attractors for a semilinear degenerate parabolic equation involving the Grushin operator.

1. Introduction

Understanding the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack this problem for dissipative dynamical systems is to consider its global attractors. A first question is to study the existence of a global attractor. Once a global attractor is obtained, a next natural question is to study the most important properties of the global attractor, such as dimension, dependence on parameters, regularity of the attractor, determining modes, etc. In the previous decades, many authors have paid attention to these problems and obtained results for a large class of PDEs; see [4, 8, 14, 15] and references therein. However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions of degenerate equations.

One of the classes of degenerate equations that has been studied widely, in recent years, is the class of equations involving an operator of Grushin type

$$G_s u = \Delta_{x_1} u + |x_1|^{2s}\Delta_{x_2} u, \quad (x_1, x_2) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},\ s \geqslant 0.$$

This operator was first introduced in [7]. Noting that $G_0 = \Delta$ and G_s, when $s > 0$, is not elliptic in domains in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ intersecting with the hyperplane $\{x_1 = 0\}$. The local properties of G_s were investigated in [3, 7]. The existence and nonexistence results for the elliptic equation

$$\begin{gathered} -G_s u + f(u) = 0, \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{gathered}$$

were proved in [16]. Furthermore, the semilinear elliptic systems with the Grushin type operator, which are in the Hamilton form or in the potential form, were also studied in [5, 6, 10].

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To study boundary value problems for equations involving Grushin operators, we have usually used the natural energy space $S_0^1(\Omega)$ defined as the completion of $C_0^1(\bar{\Omega})$ in the norm

$$\|u\|_{S_0^1(\Omega)} = \Big(\int_\Omega \big(|\nabla_{x_1} u|^2 + |x_1|^{2s} |\nabla_{x_2} u|^2\big) dx \Big)^{1/2}.$$

We have the continuous embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$, for $2 \leqslant p \leqslant 2_s^* = \frac{2N(s)}{N(s)-2}$, where $N(s) = N_1 + (s+1)N_2$. Moreover, this embedding is compact if $2 \leqslant p < 2_s^*$ (for more details, see [16]).

In a recent paper [1], we considered the initial boundary value problem

$$\begin{gathered} u_t - G_s u + f(u) + g(x) = 0, \quad x \in \Omega, t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ $(N_1, N_2 \geq 1)$ with smooth boundary $\partial\Omega$, $u_0 \in S_0^1(\Omega)$ is given, $g \in L^2(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{gathered} |f(u) - f(v)| \leqslant C_0 |u - v|(1 + |u|^\rho + |v|^\rho), \quad 0 \leqslant \rho < \frac{4}{N(s)-2}, \\ F(u) \geqslant -\frac{\mu}{2} u^2 - C_1, \\ f(u)u \geqslant -\mu u^2 - C_2, \end{gathered}$$

where $C_0, C_1, C_2 \geqslant 0$, F is the primitive $F(y) = \int_0^y f(s)ds$ of f, $\mu < \lambda_1$, λ_1 is the first eigenvalue of the operator $-G_s$ in Ω with homogeneous Dirichlet condition. Under the above assumptions of f, we proved that problem (1.1) defines a semigroup $S(t) : S_0^1(\Omega) \to S_0^1(\Omega)$, which possesses a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $S_0^1(\Omega)$. Furthermore, for each $u_0 \in S_0^1(\Omega)$, the corresponding solution $u(t)$ tends to the set E of equilibrium points in $S_0^1(\Omega)$ as $t \to +\infty$. The basic tool for the approach in this case is the following Lyapunov functional

$$\Phi(u) = \frac{1}{2}\|u\|^2_{S_0^1(\Omega)} + \int_\Omega (F(u) + gu) dx.$$

Noting that the critical exponent of the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is $2_s^* = \frac{2N(s)}{N(s)-2}$, so the condition $0 \leqslant \rho < \frac{4}{N(s)-2}$ is necessary to prove the existence of a mild solution by the fixed point method and to ensure the existence of the Lyapunov functional Φ.

In this article, we continue studying the long-time behavior of solutions to problem (1.1) by removing the restrictions on the growth of the nonlinearity f. More precisely, we assume that the initial data $u_0 \in L^2(\Omega)$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function that satisfies the following conditions:

$$C_1|u|^p - C_0 \leqslant f(u)u \leqslant C_2|u|^p + C_0, \quad p > 2, \tag{1.2}$$

$$f'(u) \geqslant -C_3, \quad \text{for all } u \in \mathbb{R}, \tag{1.3}$$

where C_0, C_1, C_2 and C_3 are positive constants. A typical example of the nonlinearity f satisfying (1.2)-(1.3) is the following

$$f(u) = \sum_{j=0}^{2p+1} b_j u^j, \quad \text{where } b_j \in \mathbb{R}, b_{2p+1} > 0.$$

It is clear that the fixed point method for proving the existence of solutions is not valid here, and the system is no longer a gradient system. However, thanks to the structure of the nonlinearity, we may use the compactness method [11] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of an absorbing set B_0 in the space $S_0^1(\Omega)$ for the semigroup $S(t)$ generated by the solutions of the problem (1.1). By the compactness of the embedding $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the semigroup $S(t)$ is asymptotically compact in $L^2(\Omega)$. This implies the existence of a compact global attractor $\mathcal{A} = \omega(B_0)$ for $S(t)$ in $L^2(\Omega)$.

Besides the problem of existence of the global attractor, the dependence of the global attractor on the parameters is also an important object of study (see [14] for an excellent review of the subject). In particular, the problem of continuity of the global attractor with respect to variations of the domain where the problem is posed has been studied recently for the reaction-diffusion equation with various boundary conditions. In [12, 13], the authors assume that Ω is a small regular perturbation of a fixed smooth domain Ω_0 and use the approach suggested by Henry [9]. This approach is simple, but quite limited since it requires that Ω_0 is C^2 and Ω is only a C^k $(k \geqslant 2)$ small perturbation of Ω, i.e. there exists a C^k-diffeomorphic $h : \Omega_0 \to \mathbb{R}^N$ such that $\Omega = h(\Omega_0)$ and $\|h - id_{\Omega_0}\|_{C^k}$ is small. In [2], the authors used a different method based on the spectral convergence which allows more irregular perturbations. However, as indicated in [12], this approach is quite technical and gives less detailed results for the regular case.

In this paper, we use another approach to study the upper semicontinuity of the global attractor with respect to the shape of the domain, which allows us to consider the more general situations and requires less smoothness of the domain than one used in [12, 13], and it is simpler than one used in [2]. We can also use this method to study the upper semicontinuity of the global attractor with respect to the nonlinear term when taking the nonlinearity as a parameter. However, the more delicate question of the lower semicontinuity of global attractor is not treated in the present paper.

The rest of the paper is organized as follows. In Section 2, we prove first the existence and uniqueness of a weak solution of the problem by using the compactness method, and then the existence of a compact global attractor $\mathcal{A}$ in $L^2(\Omega)$ for the semigroup $S(t)$ generated by (1.1). In Section 3, we study the upper semicontinuity of the global attractor with respect to the nonlinearity. In the last section, the upper-continuous dependence of the global attractors on the shape of the domain is investigated.

Notation. The $L^2(\Omega)$-norm will be denoted as $\|\cdot\|$, and the $S_0^1(\Omega)$-norm will be denoted by $\|\cdot\|_{S_0^1(\Omega)}$. By $S^{-1}(\Omega)$ we denote the dual space of $S_0^1(\Omega)$. Let (X, d) be a metric space, we usually use the semi-distance $\delta_X(.,.)$ defined on the subsets of X by

$$\delta_X(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b), \quad \forall A, B \subset X.$$

Denote by $Q_T = \Omega \times (0, T)$ the cylinder with the base Ω.

Let X_1, X_2 be two Banach spaces and Z be a topological vector space such that $X_1 \hookrightarrow Z, X_2 \hookrightarrow Z$. Then $X_1 \cap X_2$ and $X_1 + X_2$ are two Banach spaces equipped with the norms

$$\|u\|_{X_1\cap X_2} = \|u\|_{X_1} + \|u\|_{X_2},$$

$$\|u\|_{X_1+X_2} = \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2} : u = u_1 + u_2\}.$$

It is known that if $X_1 \cap X_2$ is dense both in X_1 and X_2 then $(X_1 \cap X_2)^* = X_1^* + X_2^*$.

2. Existence of the Global Attractor

In this section, we prove the global existence of a weak solution and of a global attractor of problem (1.1), under the assumptions $u_0 \in L^2(\Omega)$, $g \in L^2(\Omega)$ given, and f satisfying the conditions (1.2)-(1.3). First, we give the definition of the weak solution to the problem (1.1).

Definition 2.1. Let $T > 0$ and $u_0 \in L^2(\Omega)$ be given. A function u is called a weak solution of the problem (1.1) on $(0, T)$ if $u \in W_{0,T} = L^p(Q_T) \cap L^2(0, T; S_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(0, T; S^{-1}(\Omega)) + L^q(Q_T)$, $u(0) = u_0$, and

$$\begin{aligned} &\int_0^T \langle u_t, \varphi\rangle dt + \int_0^T \int_\Omega (\nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2}\varphi)\, dx\, dt \\ &+ \int_0^T \int_\Omega f(u)\varphi\, dx\, dt + \int_0^T \int_\Omega g(x)\varphi\, dx\, dt = 0 \end{aligned}$$

for all test functions $\varphi \in L^p(Q_T) \cap L^2(0, T; S_0^1(\Omega))$, where q is the conjugate of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$).

We remark that under condition (1.2), one can prove that $f(u) \in L^q(Q_T)$ if $u \in W_{0,T}$ (see the proof of Theorem 2.4 below). Thus, the integral $\int_0^T \int_\Omega f(u)\varphi\, dx\, dt$ is well-defined.

To prove the existence of solutions by the compactness method, we need the following Compactness Lemma (see e.g. [11, p. 58]).

Lemma 2.2. *Let $X_0, X,$ and X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, the injection of X into X_1 is continuous, the injection of X_0 into X is compact, and X_0, X_1 are reflexive. Let $1 < \alpha_0, \alpha_1 < \infty$, we set*

$$E = \Big\{u \in L^{\alpha_0}(0, T; X_0), \frac{du}{dt} \in L^{\alpha_1}(0, T; X_1)\Big\}$$

equipped with the norm

$$\|u\|_E = \|u\|_{L^{\alpha_0}(0,T;X_0)} + \|\frac{du}{dt}\|_{L^{\alpha_1}(0,T;X_1)}.$$

Then the inclusion $E \hookrightarrow L^{\alpha_0}(0, T; X)$ is compact.

The following lemma shows the continuity of solutions.

Lemma 2.3. *If $u \in L^2(0, T; S_0^1(\Omega)) \cap L^p(Q_T)$ and $\frac{\partial u}{\partial t} \in L^2(0, T; S^{-1}(\Omega)) + L^q(Q_T)$ then $u \in C([0, T]; L^2(\Omega))$.*

Proof. We select a sequence $u_n \in C^1([0, T]; S_0^1(\Omega))$ such that

$$\begin{aligned} u_n \to u \quad &\text{in } L^2(0, T; S_0^1(\Omega)) \cap L^p(Q_T) \\ \frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad &\text{in } L^2(0, T; S^{-1}(\Omega)) + L^q(Q_T). \end{aligned}$$

Then, for all $t, t_0 \in [0, T]$, we have

$$\|u_n(t) - u_m(t)\|^2 = \|u_n(t_0) - u_m(t_0)\|^2 + 2\int_{t_0}^t \langle u_n'(s) - u_m'(s), u_n(s) - u_m(s)\rangle ds.$$

We choose t_0 so that

$$\|u_n(t_0) - u_m(t_0)\|^2 = \frac{1}{T}\int_0^T \|u_n(t) - u_m(t)\|^2 dt.$$

Setting $X(t_1, t_2) = L^2(t_1, t_2; S_0^1(\Omega)) \cap L^p(Q_T)$ and $X^*(t_1, t_2) = L^2(t_1, t_2; S^{-1}(\Omega)) + L^q(Q_T)$, we have

$$\begin{aligned}
&\int_\Omega |u_n(t) - u_m(t)|^2 dx \\
&= \frac{1}{T}\int_\Omega \int_0^T |u_n(t) - u_m(t)|^2 dt dx + 2\int_\Omega \int_{t_0}^t (u_n'(s) - u_m'(s))(u_n(s) - u_m(s)) ds dx \\
&\leqslant \frac{1}{T}\int_\Omega \int_0^T |u_n(t) - u_m(t)|^2 dt dx + 2\|u_n' - u_m'\|_{X^*(t_0,t)}\|u_n - u_m\|_{X(t_0,t)} \\
&\leqslant \frac{1}{T}\int_\Omega \int_0^T |u_n(t) - u_m(t)|^2 dt dx + 2\|u_n' - u_m'\|_{X^*(0,T)}\|u_n - u_m\|_{X(0,T)}.
\end{aligned}$$

Hence, $\{u_n\}$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$ thanks to the choosing of the sequence u_n. Thus the sequence $\{u_n\}$ converges in $C([0,T]; L^2(\Omega))$ to a function $v \in C([0,T]; L^2(\Omega))$. Since $u_n(t) \longrightarrow u(t) \in L^2(\Omega)$ for a.e. $t \in [0,T]$, we deduce that $u = v$ a.e. It implies that $u \in C([0,T]; L^2(\Omega))$ (after possibly being redefined on a set of measure zero). □

Theorem 2.4. *Under the conditions* (1.2)-(1.3), *problem* (1.1) *has a unique weak solution* $u(t)$ *satisfying*

$$\begin{gathered}
u \in C([0,\infty); L^2(\Omega)) \cap L^2_{loc}(0,\infty; S_0^1(\Omega)) \cap L^p_{loc}(0,\infty; L^p(\Omega)), \\
\frac{\partial u}{\partial t} \in L^2_{loc}(0,\infty; S^{-1}(\Omega)) + L^q_{loc}(0,\infty; L^q(\Omega)),
\end{gathered}$$

where q is the conjugate of p. Moreover, the mapping $u_0 \mapsto u(t)$ is continuous on $L^2(\Omega)$.

Proof. **(i) Existence.** We will use the compactness method for showing the existence of a weak solution to the problem (1.1).

We look for an approximate solution $u_n(t)$ that belongs to the finite-dimensional space spanned by the first n eigenfunctions of $-G_s$ such that

$$u_n(t) = \sum_{j=1}^n u_{nj}(t) e_j,$$

and solves the problem

$$\begin{gathered}
\langle \frac{\partial u_n}{\partial t}, e_j\rangle - \langle G_s u_n, e_j\rangle + \langle\ f(u_n), e_j\rangle + (g, e_j) = 0, \quad 1 \leqslant j \leqslant n, \\
(u_n(0), e_j) = (u_0, e_j).
\end{gathered} \tag{2.1}$$

Hence we have a system of first-order ordinary differential equations for the functions $u_{n1}, u_{n2}, \dots, u_{nn}$,

$$\begin{gathered}
u_{nj}' + \lambda_j u_{nj} + \langle f(u_n), e_j\rangle + (g, e_j) = 0, \quad j = \overline{1,n} \\
u_{nj}(0) = (u_0, e_j).
\end{gathered}$$

According to theory of ODEs, we obtain the existence of approximate solutions $u_n(t)$. We now establish some *a priori* estimates for u_n. Since

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \|u_n\|^2_{S_0^1(\Omega)} + \int_\Omega f(u_n)u_n dx + \int_\Omega g u_n dx = 0,$$

it follows from (1.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2 + \|u_n(t)\|^2_{S_0^1(\Omega)} + C_1\int_\Omega |u_n(t)|^p dx \\
&- C_0|\Omega| - \frac{1}{2\lambda_1}\|g\|^2 - \frac{\lambda_1}{2}\|u_n(t)\|^2 \leqslant 0,
\end{aligned} \tag{2.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-G_s$ in Ω with the homogeneous Dirichlet condition (noting that $\|u\|^2_{S_0^1(\Omega)} \geq \lambda_1\|u\|^2$ for all $u \in S_0^1(\Omega)$). Hence

$$\frac{d}{dt}\|u_n(t)\|^2 \leqslant -\lambda_1\|u_n(t)\|^2 + C_4,$$

where $C_4 = \frac{1}{\lambda_1}\|g\|^2 + 2C_0|\Omega|$. Using the Gronwall inequality, we obtain

$$\|u_n(t)\|^2 \leqslant e^{-\lambda_1 t}\|u_n(0)\|^2 + \frac{C_4}{\lambda_1}(1 - e^{-\lambda_1 t}). \tag{2.3}$$

This estimate implies that the solution $u_n(t)$ of (2.1) can be extended to $+\infty$.

From (2.2), we have

$$\frac{d}{dt}\|u_n(t)\|^2 + \|u_n(t)\|^2_{S_0^1(\Omega)} + 2C_1\int_\Omega |u_n(t)|^p dx \leqslant C_4.$$

Let T be an arbitrary positive number, integrating both sides of the above inequality from 0 to T, we obtain

$$\|u_n(T)\|^2 + \int_0^T \|u_n(t)\|^2_{S_0^1(\Omega)} dt + 2C_1\int_0^T\int_\Omega |u_n|^p \,dx\,dt \leqslant \|u_n(0)\|^2 + C_4T.$$

This inequality yields $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, in $L^2(0,T;S_0^1(\Omega))$, and in $L^p(Q_T)$.

We first use the boundedness of $\{u_n\}$ in $L^p(Q_T)$ to prove the boundedness of $\{f(u_n)\}$ in $L^q(Q_T)$, where q is conjugate of p. Indeed, the condition (1.2) implies

$$|f(u)| \leqslant C_5\Big(1 + |u|^{p-1}\Big).$$

Therefore,

$$\begin{aligned}
\|f(u_n)\|^q_{L^q(Q_T)} &= \int_0^T\int_\Omega |f(u_n)|^q \,dx\,dt \\
&\leqslant C\int_0^T\int_\Omega \Big(1 + |u_n|^{q(p-1)}\Big)\,dx\,dt \\
&\leqslant C\int_0^T\int_\Omega \Big(1 + |u_n|^p\Big)\,dx\,dt.
\end{aligned}$$

Hence $\{f(u_n)\}$ is bounded in $L^q(Q_T)$.

Next, we show that $\{\frac{\partial u_n}{\partial t}\}$ is bounded in the space $L^q(0,T;S^{-1}(\Omega))$. Indeed, since

$$\frac{\partial u_n}{\partial t} = G_s u_n - f(u_n) - g$$

we have $\frac{\partial u_n}{\partial t} \in L^2(0,T;S^{-1}(\Omega)) + L^q(Q_T)$. Combining this with the fact that $L^2(0,T;S^{-1}(\Omega))$ and $L^q(Q_T)$ are continuously embedded into $L^q(0,T;S^{-1}(\Omega))$, we obtain the boundedness of $\{\frac{\partial u_n}{\partial t}\}$ in $L^q(0,T;S^{-1}(\Omega))$. Hence, by choosing a subsequence, we can assume that $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^q(0,T;S^{-1}(\Omega))$.

From the above results, we can assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } L^2(0,T;S_0^1(\Omega)), \\ u_n \rightharpoonup u \quad \text{in } L^p(Q_T), \\ f(u_n) \rightharpoonup \eta \quad \text{in } L^q(Q_T). \end{gathered}$$

From the fact that $u \in L^2(0,T;S_0^1(\Omega)) \cap L^p(Q_T)$ and $u_t \in L^2(0,T;S^{-1}(\Omega)) + L^q(Q_T)$, by Lemma 2.3, we infer that $u \in C([0,T];L^2(\Omega))$ and thus $u \in W_{0,T}$.

It remains to be shown that $\eta = f(u)$ and $u(0) = u_0$. Since $\{u_n\}$ is bounded in $L^2(0,T;S_0^1(\Omega))$ and $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $L^q(0,T;S^{-1}(\Omega))$, it follows from the Compactness Lemma that

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence we can choose a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k}(t,x) \to u(t,x) \quad \text{for a.e. } (t,x) \in Q_T.$$

It follows from the continuity of the function f that

$$f(u_{n_k}(t,x)) \to f(u(t,x)) \quad \text{for a.e. } (t,x) \in Q_T.$$

In view of the boundedness of $\{f(u_{n_k})\}$ in $L^q(Q_T)$, by [11, Lemma 1.3], we conclude that

$$f(u_{n_k}) \rightharpoonup f(u) \quad \text{in } L^q(Q_T).$$

Taking into account the uniqueness of a weak limit, we get $\eta = f(u)$.

We are in a position to show that $u(0) = u_0$. Choosing a test function $\varphi \in C^1([0,T];S_0^1(\Omega) \cap L^p(\Omega))$ with $\varphi(T) = 0$, we see that $\varphi \in L^p(Q_T) \cap L^2(0,T;S_0^1(\Omega))$. Taking integration by parts in the t variable, we have

$$\begin{aligned} &\int_0^T -(u,\varphi') + \int_0^T \int_\Omega (\nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2}\varphi) + \int_0^T \int_\Omega (f(u)+g)\varphi \\ &= (u(0),\varphi(0)). \end{aligned}$$

Doing the same in the Galerkin approximations yields

$$\begin{aligned} &\int_0^T -(u_n,\varphi') + \int_0^T \int_\Omega (\nabla_{x_1} u_n \nabla_{x_1}\varphi + |x_1|^{2s}\nabla_{x_2} u_n \nabla_{x_2}\varphi) + \int_0^T \int_\Omega (f(u_n)+g)\varphi \\ &= (u_n(0),\varphi(0)). \end{aligned}$$

Taking limits as $n \to \infty$ we conclude that

$$\begin{aligned} &\int_0^T -(u,\varphi)' + \int_0^T \int_\Omega (\nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2}\varphi) + \int_0^T \int_\Omega (f(u)+g)\varphi \\ &= (u_0,\varphi(0)) \end{aligned}$$

since $u_n(0) \to u_0$. Thus, $u(0) = u_0$.

We now prove existence of a global solution u. Analogously to (2.3) we have

$$\|u(t)\|^2 \leqslant e^{-\lambda_1 t}\|u(0)\|^2 + \frac{C_4}{\lambda_1}(1 - e^{-\lambda_1 t}). \tag{2.4}$$

This implies that the solution u exists globally in time.

(ii) Uniqueness and continuous dependence. Let $u_0, v_0 \in L^2(\Omega)$. Denote by u, v two corresponding solutions of the problem (1.1) with initial data u_0, v_0. Then $w = u - v$ satisfies

$$\begin{gathered} w_t - G_s w + f(u) - f(v) = 0, \\ w_{|\partial\Omega} = 0 \\ w(0) = u_0 - v_0. \end{gathered}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|w\|^2_{S_0^1(\Omega)} + \int_\Omega (u-v)(f(u)-f(v))dx = 0, \quad \text{for a.e. } t \in [0,T].$$

Using (1.3), we have

$$\frac{d}{dt}\|w\|^2 + 2\|w\|^2_{S_0^1(\Omega)} \leqslant 2C_3\|w\|^2, \quad \text{for a.e. } t \in [0,T].$$

Applying the Gronwall inequality, we obtain $\|w(t)\| \leqslant \|w(0)\|e^{2C_3 t}$. This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of solutions. □

Note that Theorem 2.4 allows us to define a continuous semigroup

$$S(t) : u_0 \in L^2(\Omega) \mapsto u(t) \in L^2(\Omega)$$

associated with problem (1.1). We now prove that the semigroup $S(t)$ possesses a compact connected global attractor $\mathcal{A}$ in $L^2(\Omega)$.

First, from (2.4) we deduce the existence of an absorbing set in $L^2(\Omega)$: There is a constant R and a time $t_0(\|u_0\|)$ such that, for the solution $u(t) = S(t)u_0$,

$$\|u(t)\| \leqslant R \quad \text{for all } t \geqslant t_0(\|u_0\|).$$

Multiplying (1.1) by u and using (1.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|^2_{S_0^1(\Omega)} + C_1\int_\Omega |u(t)|^p dx - C_0|\Omega| + \int_\Omega gu\,dx \leqslant 0.$$

Integrating between t and $t+1$, we obtain

$$\int_t^{t+1}\Big[\frac{1}{2}\|u(s)\|^2_{S_0^1} + C_1\int_\Omega |u(s)|^p dx + \int_\Omega gu\,dx\Big]ds \leqslant C_0|\Omega| + \frac{1}{2}\|u(t)\|^2.$$

This shows that

$$\int_t^{t+1}\Big[\frac{1}{2}\|u(s)\|^2_{S_0^1} + C_1\int_\Omega |u(s)|^p dx + \int_\Omega gudx\Big]ds \leqslant C_0|\Omega| + \frac{1}{2}R^2, \quad \forall t \geqslant t_0(\|u_0\|).$$

Noting that

$$C_5(|u|^p - 1) \leqslant F(u) \leqslant C_6(|u|^p + 1), \tag{2.5}$$

where $F(u) = \int_0^u f(\sigma)d\sigma$, we obtain

$$\int_t^{t+1}\Big[\frac{1}{2}\|u(s)\|^2_{S_0^1} + \int_\Omega (F(u) + gu)dx\Big]ds \leqslant C_7, \quad \text{for all } t \geqslant t_0(\|u_0\|). \tag{2.6}$$

In what follows, we shall formally derive an *a priori* estimate in $S_0^1(\Omega) \cap L^p(\Omega)$ on the solutions which holds for smooth functions and will become rigorous by using a Galerkin truncation and a limiting process. Taking the inner product of (1.1) with u_t, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2_{S_0^1} + \int_\Omega (F(u) + gu)dx\Big] = -2\|u_t\|^2 \leqslant 0. \tag{2.7}$$

To deduce the existence of an absorbing set in $S_0^1(\Omega)$, we need the uniform Gronwall inequality that we recall (see e.g. [15, p. 91]).

Lemma 2.5. *Let g, h, y be three locally integrable functions on $(t_0, +\infty)$ which satisfy*

$$\frac{dy}{dt} \in L^1_{loc}(t_0, +\infty) \quad \text{and} \quad \frac{dy}{dt} \leqslant gy + h, \quad \text{for } t \geqslant t_0,$$

$$\int_t^{t+r} g(s)ds \leqslant a_1, \quad \int_t^{t+r} h(s)ds \leqslant a_2, \quad \int_t^{t+r} y(s)ds \leqslant a_3, \quad \text{for } t \geqslant t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t) \leqslant (\frac{a_3}{r} + a_2)e^{a_1}, \quad \text{for all } t \geqslant t_0 + r.$$

Combining (2.6), (2.7) and using the above lemma, we obtain

$$\frac{1}{2}\|u\|^2_{S_0^1} + \int_\Omega (F(u) + gu)dx \leqslant C_7, \quad \text{for all } t \geqslant t_0(\|u_0\|) + 1.$$

Using (2.5), the Cauchy inequality and the fact that $\|u\|^2_{S_0^1(\Omega)} \geqslant \lambda_1 \|u\|^2_{L^2(\Omega)}$, we deduce from the last inequality that

$$\|u(t)\|^2_{S_0^1} + \int_\Omega |u|^p dx \leqslant C_8$$

provided that $t \geqslant t_0(\|u_0\|) + 1$. It follows from here that the ball B_0 centered at 0 with radius C_8 is an absorbing set for $S(t)$ in $S_0^1(\Omega) \cap L^p(\Omega)$.

Using the absorbing set B_0 in $S_0^1(\Omega)$, and noting that the embedding $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and that $L^2(\Omega)$ is connected, we obtain the following theorem.

Theorem 2.6. *Under conditions* (1.2)-(1.3)*, the semigroup $S(t)$ generated by the problem* (1.1) *possesses a compact connected global attractor $\mathcal{A} = \omega(B_0)$ in $L^2(\Omega)$.*

Remark 2.7. In fact, if we are only concerned with the existence of the global attractor for the semigroup $S(t)$ in $L^2(\Omega)$, then the assumption (1.3) can be replaced by the weaker assumption

$$\big(f(u) - f(v)\big)(u - v) \geqslant -C|u - v|^2 \quad \text{for any } u, v \in \mathbb{R}.$$

However, we need to use the stronger assumptions, namely $f \in C^1(\mathbb{R})$ and (1.3), in the next section (for proving (3.2)).

3. Continuous dependence of attractors on the nonlinearity

In this section we consider a family of C^1 functions $f_\lambda, \lambda \in \Lambda$, such that for each $\lambda \in \Lambda$, f_λ satisfies conditions (1.2)-(1.3) with the constants independent of λ. The family Λ is considered with a topology $\mathcal{T}$ such that the convergence $\lambda_j \to \lambda$ with respect to $\mathcal{T}$ implies that

$$f_{\lambda_j}(u) \to f_\lambda(u) \quad \text{for any } u.$$

Let $S_t(\lambda, u_0)$ be the semigroup generated by the problem

$$u_t - G_s u + f_\lambda(u) + g(x) = 0, \quad x \in \Omega, t > 0$$
$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0$$
$$u(x,0) = u_0(x), \quad x \in \Omega.$$

From the results in Section 2, this semigroup has a compact absorbing set

$$B_\lambda = \{u \in L^2(\Omega) : \|u\|_{S_0^1(\Omega)} \leqslant R_\lambda\}$$

and a compact global attractor $\mathcal{A}_\lambda = \omega(B_\lambda)$ in $X = L^2(\Omega)$.

Lemma 3.1. *$S_t(.,.)$ is continuous in $\Lambda \times X$ for any fixed $t > 0$.*

Proof. Let $(\lambda_0, u_0) \in \Lambda \times X$ and $(\lambda_j, u_{j0}) \in \Lambda \times X$ such that $\lambda_j \to \lambda_0$ and $u_{j0} \to u_0$. Let $u_j(t) = S_t(\lambda_j, u_{j0})$ be the solution of (1.1) with the nonlinearity f_{λ_j} and the initial data u_{j0}. Since f_{λ_j} satisfies (1.2)-(1.3) with the same constants and $\{u_{j0}\}$ is bounded, by using arguments as in the proof of Theorem 2.4, we have

$$\begin{gathered} \{u_j\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \\ \{u_j\} \text{ is bounded in } L^2(0,T;S_0^1(\Omega)) \\ \{f_{\lambda_j}(u_j)\} \text{ is bounded in } L^q(0,T;L^q(\Omega)) \\ \{\partial_t u_j\} \quad \text{is bounded in } L^2(0,T,S^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \end{gathered}$$

We may apply the Compactness Lemma to conclude that $\{u_j\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. Hence, there exists a subsequence (still denoted by) u_j such that

$$\begin{gathered} u_j \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \\ u_j \rightharpoonup u \quad \text{in } L^2(0,T;S_0^1(\Omega)) \\ u_j \to u \quad \text{almost everywhere in } \Omega \times (0,T) \\ f_{\lambda_j}(u_j) \rightharpoonup \omega \quad \text{in } L^q(0,T;L^q(\Omega)) \\ \partial_t u_j \rightharpoonup \partial_t u \quad \text{in } L^2(0,T;S^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \end{gathered} \tag{3.1}$$

Combining (3.1) with the hypotheses imposed on f_λ and the fact that f_{λ_j} converges almost everywhere to f_{λ_0} we have

$$f_{\lambda_j}(u_j) \to f_{\lambda_0}(u) \quad \text{almost everywhere in } \Omega \times (0,T). \tag{3.2}$$

From [11, Lemma 1.3], we have $\omega = f_{\lambda_0}(u)$. By passing to limit in the weak form, we obtain that u is the solution of the problem (1.1).

Now, let $t \in (0,T)$. Since $u_j(t)$ is bounded in $S_0^1(\Omega)$, there is a subsequence, still denoted by u_j, such that $u_j(t) \to v(t)$ strongly in $L^2(\Omega)$. Therefore,

$$S_t(\lambda_j, u_{j0}) \to S_t(\lambda_0, u_0).$$

We have proved that for any $(\lambda_j, u_{j0}) \to (\lambda_0, u_0)$, there exists a subsequence of $S_t(\lambda_j, u_{j0})$ which converges to $S_t(\lambda_0, u_0)$ and the limit is independent of the subsequence, so the whole sequence $S_t(\lambda_j, u_{j0})$ converges to $S_t(\lambda_0, u_0)$. This completes the proof. $\square$

Theorem 3.2. *The family $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ depends upper semi-continuously on the parameter λ, i.e.*

$$\limsup_{\lambda \to \lambda_0} \delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.$$

Proof. For any $\lambda_j \in \Lambda$ the semigroup $S_t(\lambda_j, u)$ has a compact absorbing set

$$B_{\lambda_j} = \{u \in L^2(\Omega) : \|u\|_{S_0^1(\Omega)} \leqslant R\},$$

where R is sufficiently large constant depending only on the constants in (1.2)-(1.3). Hence, we can choose R independent of λ_j. Hence, there exists

$$B_0 = \{u \in L^2(\Omega) : \|u\|_{S_0^1(\Omega)} \leqslant R\}$$

such that for any bounded set $B \subset L^2(\Omega)$ and for any λ, there is $\tau = \tau(\lambda, B)$ with the property

$$S_t(\lambda, B) \subset B_0 \text{ for } t \geq \tau.$$

Let $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that

$$\delta_X(S_T(\lambda_0, B_0), \mathcal{A}_{\lambda_0}) < \varepsilon.$$

By Lemma 3.1, for any $x \in B_0$, there are open neighborhoods $V(x)$ and $W(\lambda_0)$ in X and Λ such that

$$\delta_X(S_T(\lambda, V(x)), \mathcal{A}_{\lambda_0}) < \varepsilon \quad \text{for any } \lambda \in W(\lambda_0).$$

Since B_0 is compact in X, there exists a neighborhood W of λ_0 such that

$$\delta_X(S_T(\lambda, B_0), \mathcal{A}(\lambda_0)) < \varepsilon \quad \text{for any } \lambda \in W.$$

Therefore,

$$\delta_X(\mathcal{A}(\lambda), \mathcal{A}(\lambda_0)) < \varepsilon \quad \text{for any } \lambda \in W.$$

The proof is complete. □

4. Continuous dependence of attractors on the shape of domain

Let Ω_0 be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega_0$. We consider a family $\mathcal{G}$ of diffeomorphism G such that:

- Any $G \in \mathcal{G}$ is a diffeomorphism of class C^1 of a neighborhood of $\overline{\Omega}_0$. We denote $\Omega_G = G(\Omega_0)$ and let
$$\|G\|_{C^0(\overline{\Omega}_0)} = \max_{x\in\overline{\Omega}_0} |G(x)|$$
$$\|G\|_{C^1(\overline{\Omega}_0)} = \|G\|_{C^0(\overline{\Omega}_0)} + \max_{x\in\overline{\Omega}_0} |\frac{\partial G}{\partial x}(x)|.$$
- Assume that
$$\sup_{G\in\mathcal{G}} \|G\| < +\infty, \quad \sup_{G\in\mathcal{G}} \|G^{-1}\| < +\infty. \tag{4.1}$$
- The family $\mathcal{G}$ is equipped with the topology $\mathcal{T}$ such that $G_j \to G$ with respect to $\mathcal{T}$ if and only if
$$\|G_j - G\|_{C^0(\overline{\Omega}_0)} \to 0.$$

Let $X = L^2(\Omega_0)$ and $X_G = L^2(\Omega_G)$, we define $G^* : X_G \to X$ as follows:

$$G^*u(x) = u(G(x)) \quad \text{for } u \in X_G.$$

We consider (1.1) on $\Omega_G \times [0, +\infty)$ and assume that (1.2)-(1.3) are satisfied. Denote by $\Sigma_t(G, u_0)$ the semigroup in X_G generated by this problem. From the results in Section 2, this semigroup has a compact absorbing set

$$B_G = \{u \in X_G : \|u\|_{S_0^1(\Omega_G)} \leqslant R_G\}$$

and has a global attractor $\mathcal{A}_G = \omega(B_G)$.

Denote $\mathcal{A}(G) = G^*(\mathcal{A}_G)$ and we define the semigroup of operators $S_t(G, .) : X \longrightarrow X$, $G \in \mathcal{G}$, by

$$S_t(G, u_0) = G^*\Sigma_t(G, (G^*)^{-1}u_0).$$

Lemma 4.1. *S_t is continuous in $\mathcal{G} \times X$ for any fixed $t > 0$.*

Proof. Let $(G_0, u_0) \in \mathcal{G} \times X$ and assume that $G_j \to G_0$ and $u_{j0} \to u_0$. Putting $v_{j0} = (G_j^*)^{-1}(u_{j0})$ and $u_j(t) = \Sigma_t(G_j, v_{j0})$, then $u_j(t)$ is the solution of (1.1) in $\Omega_{G_j} \times (0,T)$, $u_j(0) = v_{j0} = (G_j^*)^{-1}(u_{j0})$. By (4.1), there exists $R > 0$ such that

$$\|u_j\|_{L^2(0,T;S_0^1(\Omega_{G_j}))} + \|u_j\|_{L^\infty(0,T;L^2(\Omega_{G_j}))} \leqslant R \quad \text{for all } j. \tag{4.2}$$

Putting $v_j(t) = G_j^*(u_j(t))$ then $v(0) = u_{j0}$. It follows from (4.1) and (4.2) that v_j is uniformly bounded in $L^2(0,T;S_0^1(\Omega_{G_0})) \cap L^\infty(0,T;L^2(\Omega_{G_0}))$. There exists a subsequence, still denoted by v_j, such that

$$\begin{aligned} v_j &\overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;L^2(\Omega)) \\ v_j &\rightharpoonup v \quad \text{in } L^2(0,T;S_0^1(\Omega)). \end{aligned}$$

Putting $u(t) = (G_0^*)^{-1}(v(t))$. Since u_j is the solution of (1.1) in $\Omega_{G_j} \times (0,T)$, u is the solution in the sense of distributions of (1.1) in $\Omega_{G_0} \times (0,T)$. Moreover,

$$u(0) = (G_0^*)^{-1}(v(0)) = \lim_j (G_0^*)^{-1}(u_{j0}) = (G_0^*)^{-1}(u_0).$$

On the other hand, putting $\tilde{u}(t) = \Sigma_t(G_0, v_0)$ where $v_0 = (G_0^*)^{-1}(u_0)$ then by the uniqueness of solution we have $\tilde{u} = u$.

Now, let $t \in (0,T)$. Since $v_j(t)$ is bounded in $S_0^1(\Omega_0)$, there is a subsequence (still denoted by) v_j such that $v_j(t) \to v(t)$ strongly in $L^2(\Omega_0)$. Therefore,

$$S_t(G_j, u_{j0}) \to S_t(G_0, u_0).$$

For any $(G_j, u_{j0}) \to (G_0, u_0)$, there exists a subsequence of $S_t(G_j, u_{j0})$ which converges to $S_t(G_0, u_0)$, the limit is independent on the subsequence, so the whole sequence $S_t(G_j, u_{j0})$ converges to $S_t(G_0, u_0)$. The proof is complete. □

Theorem 4.2. *The family $\{\mathcal{A}(G) : G \in \mathcal{G}\}$ depends upper semi-continuously on the parameter G, i.e.*

$$\limsup_{G \to G_0} \delta_X(\mathcal{A}(G), \mathcal{A}(G_0)) = 0.$$

Proof. For any $G \in \mathcal{G}$, the semigroup $\Sigma_t(G,u)$ has a compact absorbing set

$$B_G = \{u \in X_G : \|u\|_{S_0^1(\Omega_G)} \leqslant R\},$$

where R is a sufficiently large constant depending on the constants in (1.2)-(1.3) and on the volume of Ω_G. Hence, we can choose R independent on G. It follows from (4.1) that there exists

$$B_0 = \{u \in X : \|u\|_{S_0^1(\Omega_0)} \leqslant R\}$$

such that for any bounded set $B \subset X$ and for any $G \in \mathcal{G}$, there is $\tau = \tau(G,B)$ with the property

$$S_t(G,B) \subset B_0 \text{ for } t \geq \tau.$$

Let $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that

$$\delta_X(S_T(G_0,B_0), \mathcal{A}(G_0)) < \varepsilon.$$

By Lemma 4.1, for any $x \in B_0$, there are open neighborhoods $V(x)$ and $W(G_0)$ in X and $\mathcal{G}$ such that

$$\delta_X(S_T(G,V(x)), \mathcal{A}(G_0)) < \varepsilon \quad \text{for any } G \in W(G_0).$$

Since B_0 is compact in X, there exists a neighborhood W of G_0 such that
$$\delta_X(S_T(G, B_0), \mathcal{A}(G_0)) < \varepsilon \quad \text{for any } G \in W.$$
Therefore,
$$\delta_X(\mathcal{A}(G), \mathcal{A}(G_0)) < \varepsilon \quad \text{for any } G \in W.$$
The proof is complete. $\square$

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