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# GROWTH AND OSCILLATION OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS HAVING THE SAME ORDER

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Abstract. In this article, we investigate the growth and fixed points of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_0(z), \dots, A_{k-1}(z)$ are entire functions. Some estimates are given for the iterated order and iterated exponent of convergence of fixed points of solutions of the above equation when most of the coefficients have the same order with each other.

## 1. Introduction and statement of results

For the definition of the iterated order of an entire function, we use the same definition as in [16], [6, p. 317], [17, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all $r$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

**Definition 1.1.** Let $f$ be an entire function. Then the iterated $p$-order $\rho_p(f)$ of $f$ is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \tag{1.1}$$

where $p$ is an integer, $p \geq 1$, $T(r,f)$ is the Nevanlinna characteristic function of $f$ and $M(r,f) = \max_{|z|=r} |f(z)|$; see [15, 20]. For $p = 1$, this notation is called order and for $p = 2$ hyper-order [24].

**Definition 1.2** ([6, 17])**.** The finiteness degree of the order of an entire function $f$ is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ a polynomial,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{there exists } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \\ +\infty, & \text{when } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.2}$$

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**Definition 1.3.** Let $f$ be an entire function. Then the iterated $p$-type of an entire function $f$, with iterated $p$-order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} = \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{r^{\rho_p(f)}} \quad (1 \le p \text{ an integer}). \tag{1.3}$$

For $k \ge 2$, we consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.4}$$

where $A_0(z), \dots, A_{k-1}(z)$ are entire functions. It is well-known that all solutions of equation (1.4) are entire functions and if some of the coefficients of (1.4) are transcendental, then (1.4) has at least one solution with order $\rho(f) = +\infty$.

A natural question arises: What conditions on $A_0(z), A_1(z), \dots, A_{k-1}(z)$ will guarantee that every solution $f \not\equiv 0$ has infinite order? Also: For solutions of infinite order, how to express the growth of them explicitly, it is a very important problem. Partial results have been available since in a paper by Frei [12]. Extensive work in recent years has been concerned with the growth of solutions of complex linear differential equations. Many results have been obtained for the growth and the oscillation of solutions of the differential equation (1.4); see e.g. [2, 3, 4, 5, 7, 8, 9, 10, 11, 16, 17, 18, 19, 21, 22, 23]. Examples of such results are the following two theorems:

**Theorem 1.4** ([16]). *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions such that $i(A_0) = p$ $(0 < p < \infty)$. If either $\max\{i(A_j) : j = 1, 2, \dots, k-1\} < p$ or $\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} < \rho_p(A_0)$, then every solution $f \not\equiv 0$ of (1.4) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0)$.*

**Theorem 1.5** ([22]). *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions such that $\rho(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\tau(A_0) = \tau$ $(0 < \tau < +\infty)$, and let $\rho(A_j) \le \rho(A_0) = \rho$ $(j = 1, 2, \dots, k-1)$, $\tau(A_j) < \tau(A_0) = \tau$ $(j = 1, 2, \dots, k-1)$ if $\rho(A_j) = \rho(A_0)$ $(j = 1, 2, \dots, k-1)$. Then every solution $f \not\equiv 0$ of (1.4) satisfies $\rho(f) = +\infty$ and $\rho_2(f) = \rho(A_0) = \rho$.*

The purpose of this paper is to extend and to improve the results of Theorems 1.4 and 1.5. We will prove the following theorems:

**Theorem 1.6.** *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that $\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} \le \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$. Then every solution $f \not\equiv 0$ of (1.4) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.*

Combining Theorems 1.4 and 1.6, we obtain the following result.

**Corollary 1.7.** *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that either*

$$\max\{i(A_j) : j = 1, 2, \dots, k-1\} < p$$

*or*

$$\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} \le \rho_p(A_0) = \rho \quad (0 < \rho < +\infty)$$

*and*

$$\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau \quad (0 < \tau < +\infty).$$

*Then every solution $f \not\equiv 0$ of (1.4) satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.*

**Corollary 1.8.** *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions, and let $i(A_j)=p$ ($j=0,\dots,k-1$), ($0<p<\infty$). Assume that $\rho_p(A_j)=\rho$ ($j=0,\dots,k-1$), ($0<\rho<+\infty$) and $\max\{\tau_p(A_j): j=1,2,\dots,k-1\}<\tau_p(A_0)=\tau$ ($0<\tau<+\infty$). Then every solution $f\not\equiv 0$ of (1.4) satisfies $i(f)=p+1$ and $\rho_{p+1}(f)=\rho$.*

**Corollary 1.9.** *Let $h_j(z)$ ($j=0,1,\dots,k-1$) ($k\geq 2$) be entire functions with $h_0\not\equiv 0$, $\rho_p(h_j)<n$ ($0<p<\infty$), and let $A_j(z)=h_j(z)\exp_p(P_j(z))$, where $P_j(z)=\sum_{i=0}^{n}a_{ji}z^i$ ($j=0,\dots,k-1$) are polynomials with degree $n\geq 1$, $a_{jn}$ ($j=0,1,\dots,k-1$) are complex numbers. If $|a_{0n}|>\max\{|a_{jn}|: j=1,\dots,k-1\}$, then every solution $f\not\equiv 0$ of (1.4) satisfies $i(f)=p+1$ and $\rho_{p+1}(f)=n$.*

Replacing the dominant coefficient $A_0$ by an arbitrary coefficient $A_s$, where $s\in\{1,\dots,k-1\}$, we obtain the following result which is an extension of Theorems 1.4, 1.5 and 1.6.

**Theorem 1.10.** *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions. Suppose that there exists an $A_s$ ($1\leq s\leq k-1$) with $i(A_s)=p$ ($0<p<\infty$). Assume that $\max\{\rho_p(A_j): j\neq s\}\leq\rho_p(A_s)=\rho$ ($0<\rho<+\infty$) and $\max\{\tau_p(A_j):\rho_p(A_j)=\rho_p(A_s)\}<\tau_p(A_s)=\tau$ ($0<\tau<+\infty$). Then every transcendental solution $f$ of (1.4) satisfies $p\leq i(f)\leq p+1$ and $\rho_{p+1}(f)\leq\rho_p(A_s)\leq\rho_p(f)$, and every non-transcendental solution $f$ of (1.4) is a polynomial of degree $\deg(f)\leq s-1$.*

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [25]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [10]).

In the following, we investigate the relationship between solutions of the differential equation (1.4) and entire functions with finite iterated $p$-order. We obtain some precise estimates of their fixed points.

To give the precise estimate of fixed points, we define the following concept.

**Definition 1.11** ([16, 18])**.** Let $f$ be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f)=\limsup_{r\to+\infty}\frac{\log_p\overline{N}(r,\frac{1}{f})}{\log r}\quad(1\leq p\text{ an integer}),\tag{1.5}$$

where $\overline{N}(r,1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{|z|<r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyper-exponent of convergence of the sequence of distinct zeros.

**Definition 1.12** ([18])**.** Let $f$ be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\lambda}_p(f-z)=\limsup_{r\to+\infty}\frac{\log_p\overline{N}(r,\frac{1}{f-z})}{\log r}\quad(1\leq p\text{ an integer}).\tag{1.6}$$

For $p=1$, this notation is called exponent of convergence of the sequence of distinct fixed points and for $p=2$ hyper-exponent of convergence of the sequence of distinct fixed points (see [19, 23]). Thus $\overline{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

We obtain the following results.

**Theorem 1.13.** *Suppose that $A_0(z), \dots, A_{k-1}(z)$ satisfy the hypotheses of Theorem 1.6, and let $\varphi \not\equiv 0$ be a finite iterated $p$-order entire function. Then every solution $f \not\equiv 0$ of (1.4) satisfies*

$$\overline{\lambda}_p(f-\varphi) = \lambda_p(f-\varphi) = \rho_p(f) = +\infty$$

*and $\overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f) = \rho$.*

Setting $\varphi(z) = z$ in Theorem 1.13, we obtain the following corollary.

**Corollary 1.14.** *Under the hypotheses of Theorem 1.13, every solution $f \not\equiv 0$ of the equation (1.4) satisfies $\overline{\lambda}_p(f-z) = \lambda_p(f-z) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f-z) = \lambda_{p+1}(f-z) = \rho_{p+1}(f) = \rho$.*

**Corollary 1.15.** *Suppose that $A_j(z) = h_j(z)\exp_p(P_j(z))$ $(j = 0, 1, \dots, k-1)$ and the complex numbers $a_{jn}$ $(j = 0, 1, \dots, k-1)$ satisfy the hypotheses of Corollary 1.9. If $\varphi \not\equiv 0$ is a finite iterated $p$-order entire function, then every solution $f \not\equiv 0$ of (1.4) satisfies $\overline{\lambda}_p(f-\varphi) = \lambda_p(f-\varphi) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f) = n$. In particular, every solution $f(z) \not\equiv 0$ of (1.4) satisfies $\overline{\lambda}_p(f-z) = \lambda_p(f-z) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f-z) = \lambda_{p+1}(f-z) = \rho_{p+1}(f) = n$.*

## 2. Preliminary Lemmas

Our proofs depend mainly upon the following lemmas. Before starting these lemmas, we recall the concepts of linear and logarithmic measure. For $E \subset [0, +\infty)$, we define the linear measure of a set $E$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ by $\operatorname{lm}(F) = \int_1^{+\infty} (\chi_F(t)/t)dt$, where $\chi_H$ is the characteristic function of a set $H$.

**Lemma 2.1** ([13, p. 90])**.** *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on $\alpha$ and $(m, n)$ $(m, n$ positive integers with $m < n)$ such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\Big|\frac{f^{(n)}(z)}{f^{(m)}(z)}\Big| \le B\Big[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\Big]^{n-m}. \tag{2.1}$$

**Lemma 2.2.** *Let $f(z)$ be an entire function of iterated $p$-order $0 < \rho_p(f) < +\infty$ and iterated $p$-type $0 < \tau_p(f) < \infty$. Then for any given $\beta < \tau_p(f)$, there exists a set $E_2 \subset [1, +\infty)$ that has infinite logarithmic measure, such that for all $r \in E_2$, we have*

$$\log_p M(r, f) > \beta r^{\rho_p(f)}. \tag{2.2}$$

*Proof.* When $p = 1$, the Lemma is due to Tu and Yi [22]. Thus we assume that $p \ge 2$. By definitions of iterated order and iterated type, there exists an increasing sequence $\{r_n\}$, $r_n \to \infty$ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{r_n \to +\infty} \frac{\log_p M(r_n, f)}{r_n^{\rho_p(f)}} = \tau_p(f) > \beta. \tag{2.3}$$

Then there exists a positive integer $n_0$ such that for all $n > n_0$ and for any given $\varepsilon$ $(0 < \varepsilon < \tau_p(f) - \beta)$, we have

$$\log_p M(r_n, f) > (\tau_p(f) - \varepsilon)r_n^{\rho_p(f)}. \tag{2.4}$$

For any given $\beta < \tau_p(f)$, there exists a positive integer $n_1$ such that for all $n > n_1$, we have

$$\Big(\frac{n}{n+1}\Big)^{\rho_p(f)} > \frac{\beta}{\tau_p(f) - \varepsilon}\,. \tag{2.5}$$

Taking $n \geq n_2 = \max\{n_0, n_1\}$. By (2.4) and (2.5) for any $r \in [r_n, (1+\frac{1}{n})r_n]$, we obtain

$$\begin{aligned}\log_p M(r,f) &\geq \log_p M(r_n, f)\\ &> (\tau_p(f)-\varepsilon) r_n^{\rho_p(f)}\\ &\geq (\tau_p(f)-\varepsilon)\Big(\frac{n}{n+1}r\Big)^{\rho_p(f)}\\ &> \beta r^{\rho_p(f)}.\end{aligned} \tag{2.6}$$

Set $E_2 = \cup_{n=n_2}^{\infty}[r_n, (1+\frac{1}{n})r_n]$, then there holds

$$\operatorname{lm}(E_2) = \sum_{n=n_2}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_2}^{\infty} \log(1+\frac{1}{n}) = +\infty. \tag{2.7}$$

□

Using similar arguments as in the proof of Lemma 2.2, we obtain the following result.

**Lemma 2.3.** *Let $f(z)$ be an entire function of iterated p-order $0 < \rho_p(f) < +\infty$ and iterated p-type $0 < \tau_p(f) < \infty$. Then for any given $\beta < \tau_p(f)$, there exists a set $E_3 \subset [1, +\infty)$ that has infinite logarithmic measure, such that for all $r \in E_3$, we have*

$$\log_{p-1} m(r,f) = \log_{p-1} T(r,f) > \beta r^{\rho_p(f)}. \tag{2.8}$$

To avoid some problems caused by the exceptional set we recall the following Lemmas.

**Lemma 2.4** ([1, p. 68])**.** *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E_4$ of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

**Lemma 2.5** ([14])**.** *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin E_5 \cup [0, 1]$, where $E_5 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r > r_1$.*

**Lemma 2.6** ([5, 7])**.** *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite iterated p-order meromorphic functions. If $f$ is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_1 f' + A_0 f = F, \tag{2.9}$$

*then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

**Lemma 2.7** ([6, 17])**.** *Suppose that $k \geq 2$ and $A_0(z), \dots, A_{k-1}(z)$ are entire functions of finite iterated p-order. If $f(z)$ is a solution of (1.4), then $i(f) \leq p+1$ and $\rho_{p+1}(f) \leq \max\{\rho_p(A_j) : j = 0, \dots, k-1\} = \rho$.*

## 3. Proof of Theorem 1.6

Suppose that $f \not\equiv 0$ is a solution of equation (1.4). From (1.4), we can write

$$|A_0(z)| \le \Big|\frac{f^{(k)}}{f}\Big| + |A_{k-1}(z)|\Big|\frac{f^{(k-1)}}{f}\Big| + \dots + |A_1(z)|\Big|\frac{f'}{f}\Big|. \tag{3.1}$$

By Lemma 2.1, there exist a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_1 \cup [0, 1]$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \le B[T(2r, f)]^{k+1} \quad (j = 1, 2, \dots, k). \tag{3.2}$$

If $\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} < \rho_p(A_0) = \rho$, then by Theorem 1.4, we obtain $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.

If $\max\{\rho_p(A_j) : j = 1, 2, \dots, k-1\} = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$. Then, there exists a set $I \subseteq \{1, 2, \dots, k-1\}$ such that $\rho_p(A_j) = \rho_p(A_0) = \rho (j \in I)$ and $\tau_p(A_j) < \tau_p(A_0)$ $(j \in I)$. Thus, we choose $\alpha_1$, $\alpha_2$ satisfying $\max\{\tau_p(A_j) : (j \in I)\} < \alpha_1 < \alpha_2 < \tau_p(A_0) = \tau$ such that for sufficiently large $r$, we have

$$|A_j(z)| \le \exp_p(\alpha_1 r^\rho) \quad (j \in I), \tag{3.3}$$

$$|A_j(z)| \le \exp_p(r^{\alpha_0}) \quad (j \in \{1, 2, \dots, k-1\}\backslash I), \tag{3.4}$$

where $0 < \alpha_0 < \rho$. By Lemma 2.2, there exists a set $E_2 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2$, we have

$$M(r, A_0) > \exp_p(\alpha_2 r^\rho). \tag{3.5}$$

Hence from (3.1)-(3.5), for all $z$ satisfying $|A_0(z)| = M(r, A_0)$ and for sufficiently large $|z| = r \in E_2 \backslash E_1 \cup [0, 1]$, we have

$$\exp_p(\alpha_2 r^\rho) \le kB \exp_p(\alpha_1 r^\rho)[T(2r, f)]^{k+1}. \tag{3.6}$$

Since $\alpha_2 > \alpha_1 > 0$, we get from (3.6) that

$$\exp((1-\gamma)\exp_{p-1}(\alpha_2 r^\rho)) \le kB[T(2r, f)]^{k+1}, \tag{3.7}$$

where $\gamma$ $(0 < \gamma < 1)$ is a real number. By (3.7), Lemma 2.5 and the definition of iterated order, we have $i(f) \ge p+1$ and $\rho_{p+1}(f) \ge \rho_p(A_0) = \rho$. On the other hand by Lemma 2.7, we have $i(f) \le p+1$ and $\rho_{p+1}(f) \le \rho_p(A_0) = \rho$, hence $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.

## 4. Proof of Theorem 1.10

Assume that $f$ is a transcendental solution of (1.4). It follows from (1.4) that

$$\begin{aligned} A_s(z) &= -\Big(\frac{f^{(k)}}{f^{(s)}} + A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} + \dots + A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}} \\ &\quad + A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} + \dots + A_1(z)\frac{f'}{f^{(s)}} + A_0(z)\frac{f}{f^{(s)}}\Big) \\ &= -\frac{f}{f^{(s)}}\Big(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \dots + A_{s+1}(z)\frac{f^{(s+1)}}{f} \\ &\quad + A_{s-1}(z)\frac{f^{(s-1)}}{f} + \dots + A_1(z)\frac{f'}{f} + A_0(z)\Big). \end{aligned} \tag{4.1}$$

By Lemma of logarithmic derivative [15] and (4.1), we obtain

$$\begin{aligned} T(r,A_s) &= m(r,A_s) \\ &\le m\big(r,\frac{f}{f^{(s)}}\big) + \sum_{j\neq s} m(r,A_j) + O(\log(rT(r,f))) \\ &= m\big(r,\frac{f}{f^{(s)}}\big) + \sum_{j\neq s} T(r,A_j) + O(\log(rT(r,f))) \end{aligned} \tag{4.2}$$

holds for all $r$ outside a set $E\subset(0,+\infty)$ with a finite linear measure $m(E)<+\infty$. Noting that

$$\begin{aligned} m(r,\frac{f}{f^{(s)}}) &\le m(r,f) + m\big(r,\frac{1}{f^{(s)}}\big) \\ &\le T(r,f) + T(r,\frac{1}{f^{(s)}}) \\ &= T(r,f) + T(r,f^{(s)}) + O(1) \\ &\le (s+2)T(r,f) + o(T(r,f)) + O(1). \end{aligned} \tag{4.3}$$

For sufficiently large $r$, we have

$$O(\log r + \log T(r,f)) \le \frac{1}{2}T(r,f). \tag{4.4}$$

Thus, by (4.2)-(4.4), for sufficiently large $r\notin E$, we have

$$T(r,A_s) \le cT(r,f) + \sum_{j\neq s} T(r,A_j), \tag{4.5}$$

where $c$ is a positive constant.

If $\max\{\rho_p(A_j) : j=0,1,\dots,s-1,s+1,\dots,k-1\} < \rho_p(A_s)=\rho$, then for sufficiently large $r$, we have

$$T(r,A_j) \le \exp_{p-1}(r^{\beta_0}) \quad (j=0,1,\dots,s-1,s+1,\dots,k-1), \tag{4.6}$$

where $0<\beta_0<\rho$. Since $\rho_p(A_s)=\rho$, there exists $\{r_n'\}$ $(r_n'\to+\infty)$ such that

$$\lim_{r_n'\to+\infty} \frac{\log_p T(r_n',A_s)}{\log r_n'} = \rho. \tag{4.7}$$

Set the linear measure of $E$, $m(E)=\delta<+\infty$, then there exists a point $r_n\in[r_n',r_n'+\delta+1]-E$. From

$$\frac{\log_p T(r_n,A_s)}{\log r_n} \ge \frac{\log_p T(r_n',A_s)}{\log(r_n'+\delta+1)} = \frac{\log_p T(r_n',A_s)}{\log r_n' + \log(1+(\delta+1)/r_n')} \tag{4.8}$$

we get

$$\lim_{r_n\to+\infty} \frac{\log_p T(r_n,A_s)}{\log r_n} = \rho. \tag{4.9}$$

So for any given $\varepsilon$ $(0<\varepsilon<\rho-\beta_0)$, and for $j\neq s$,

$$T(r_n,A_j) \le \exp_{p-1}(r_n^{\beta_0}) \tag{4.10}$$

$$T(r_n,A_s) \ge \exp_{p-1}(r_n^{\rho-\varepsilon}) \tag{4.11}$$

hold for sufficiently large $r_n$. By (4.5), (4.10), (4.11) and Lemma 2.4, we obtain for sufficiently large $r_n$

$$\exp_{p-1}(r_n^{\rho-\varepsilon}) \le cT(r_n,f) + (k-1)\exp_{p-1}(r_n^{\beta_0}). \tag{4.12}$$

Therefore, by (4.12) we obtain $i(f) \geq p$ and

$$\limsup_{r_n \to +\infty} \frac{\log_p T(r_n, f)}{\log r_n} \geq \rho - \varepsilon \tag{4.13}$$

and since $\varepsilon > 0$ is arbitrary, we get $\rho_p(f) \geq \rho_p(A_s) = \rho$. On the other hand by Lemma 2.7, we have $i(f) \leq p + 1$ and $\rho_{p+1}(f) \leq \rho_p(A_s)$. Hence, we obtain $p \leq i(f) \leq p + 1$ and $\rho_{p+1}(f) \leq \rho_p(A_s) \leq \rho_p(f)$.

If $\max\{\rho_p(A_j) : j = 0, 1, \dots, s-1, s+1, \dots, k-1\} = \rho_p(A_s) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_s)\} < \tau_p(A_s) = \tau$ $(0 < \tau < +\infty)$. Then, there exists a set $J \subseteq \{0, 1, \dots, s-1, s+1, \dots, k-1\}$ such that $\rho_p(A_j) = \rho_p(A_s) = \rho$ $(j \in J)$ and $\tau_p(A_j) < \tau_p(A_s)$ $(j \in J)$. Thus, we choose $\beta_2$, $\beta_3$ satisfying $\max\{\tau_p(A_j) : (j \in J)\} < \beta_2 < \beta_3 < \tau_p(A_s) = \tau$ such that for sufficiently large $r$, we have

$$T(r, A_j) \leq \exp_{p-1}(\beta_2 r^{\rho}) \quad (j \in J), \tag{4.14}$$

$$T(r, A_j) \leq \exp_{p-1}(r^{\beta_1}) \quad (j \in \{0, 1, \dots, s-1, s+1, \dots, k-1\} \backslash J), \tag{4.15}$$

where $0 < \beta_1 < \rho$. By Lemma 2.3, there exists a set $E_3 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have

$$T(r, A_s) > \exp_{p-1}(\beta_3 r^{\rho}). \tag{4.16}$$

Hence from (4.5), (4.14), (4.15), (4.16) and Lemma 2.4, for sufficiently large $|z| = r \in E_3$, we have

$$\exp_{p-1}(\beta_3 r^{\rho}) \leq cT(r, f) + (k-1) \exp_{p-1}(\beta_2 r^{\rho}). \tag{4.17}$$

By this inequality and the definition of iterated order, we have $i(f) \geq p$ and $\rho_p(f) \geq \rho_p(A_s) = \rho$. On the other hand by Lemma 2.7, we have $i(f) \leq p+1$ and $\rho_{p+1}(f) \leq \rho_p(A_s)$. Hence, we obtain $p \leq i(f) \leq p + 1$ and $\rho_{p+1}(f) \leq \rho_p(A_s) \leq \rho_p(f)$.

Suppose that $f$ is a polynomial of $\deg f = m \geq s$. If $\max\{\rho_p(A_j) : j = 0, 1, \dots, s-1, s+1, \dots, k-1\} < \rho_p(A_s) = \rho$, then

$$i(0) = i(f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f) = i(A_s) \tag{4.18}$$

and

$$\rho_p(0) = \rho_p(f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f) = \rho_p(A_s) > 0 \tag{4.19}$$

this is a contradiction by (1.4).

If $\max\{\rho_p(A_j) : j = 0, 1, \dots, s-1, s+1, \dots, k-1\} = \rho_p(A_s) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_s)\} < \tau_p(A_s) = \tau$ $(0 < \tau < +\infty)$. Then, there exists a set $K \subseteq \{0, 1, \dots, s-1, s+1, \dots, k-1\}$ such that $\rho_p(A_j) = \rho_p(A_s) = \rho$ $(j \in K)$ and $\tau_p(A_j) < \tau_p(A_s)$ $(j \in K)$. Thus, we choose $\beta_4$, $\beta_5$ satisfying $\max\{\tau_p(A_j) : (j \in K)\} < \beta_4 < \beta_5 < \tau_p(A_s) = \tau$ such that for sufficiently large $r$, we have

$$|A_j(z)| \leq \exp_p(\beta_4 r^{\rho}) \quad (j \in K), \tag{4.20}$$

$$|A_j(z)| \leq \exp_p(r^{\beta_6}) \quad (j \in \{0, 1, \dots, s-1, s+1, \dots, k-1\} \backslash K), \tag{4.21}$$

where $0 < \beta_6 < \rho$. By Lemma 2.2, there exists a set $E_2 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2$, we have

$$M(r, A_s) > \exp_p(\beta_5 r^{\rho}). \tag{4.22}$$

Hence from (1.4), (4.20)-(4.22), for all $z$ satisfying $|A_s(z)| = M(r, A_s)$ and for sufficiently large $|z| = r \in E_2$, we have

$$\begin{aligned} d_1 r^{m-s} \exp_p(\beta_5 r^\rho) &\leq |A_s(re^{i\theta}) f^{(s)}(re^{i\theta})| \\ &\leq \sum_{j \neq s} |A_j(re^{i\theta}) f^{(j)}(re^{i\theta})| \\ &\leq d_2 r^m \exp_p(\beta_4 r^\rho), \end{aligned} \tag{4.23}$$

where $d_1, d_2$ are positive constants. By (4.23), we get

$$\exp\{\exp_{p-1}(\beta_5 r^\rho) - \exp_{p-1}(\beta_4 r^\rho)\} \leq \frac{d_2}{d_1} r^s. \tag{4.24}$$

Hence by $\beta_5 > \beta_4 > 0$, from (4.24) we obtain a contradiction. Therefore, if $f$ is a non-transcendental solution, then it must be a polynomial of $\deg f \leq s - 1$. This proves Theorem 1.10.

## 5. Proof of Theorem 1.13

Suppose that $f \not\equiv 0$ is a solution of equation (1.4). Then by Theorem 1.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Set $w = f - \varphi$. Since $\rho_p(\varphi) < \infty$, then we have $\rho_p(w) = \rho_p(f - \varphi) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. Substituting $f = w + \varphi$ into equation (1.4), we obtain

$$\begin{aligned} w^{(k)} + A_{k-1}(z) w^{(k-1)} + \cdots + A_0(z) w &= -(\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_0(z)\varphi) \\ &= W. \end{aligned} \tag{5.1}$$

Since $\varphi \not\equiv 0$ and $\rho_p(\varphi) < \infty$, we have $W \not\equiv 0$. Then by Lemma 2.6, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w) = \rho_p(A_0) = \rho$; i.e., $\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$.

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