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CONTROLLABILITY OF 1-D COUPLED DEGENERATE PARABOLIC EQUATIONS

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Abstract. This article is devoted to the study of null controllability properties for two systems of coupled one dimensional degenerate parabolic equations. The first system consists of two forward equations, while the second one consists of one forward equation and one backward equation. Both systems are in cascade, that is, the solution of the first equation acts as a control for the second equation and the control function only acts directly in the first equation. We prove positive null controllability results when the control and coupling sets have nonempty intersection and 0 does not belong to the coupling set.

1. Statement of the problem

In this paper we are concerned with the controllability properties of systems of coupled degenerate parabolic equations. We are going to consider two different kind of systems: the first one consists of two forward equations and the second one, consists of one forward equation and one backward equation. More precisely, given two non empty open sets $\omega \subset (0,1)$ and $\mathcal{O} \subset (0,1)$ and a number $\alpha \in [0,2)$, we consider the system of equations

$$\begin{gathered} y_t - (x^\alpha y_x)_x + c(t,x)y = \xi + h\mathbb{I}_\omega \quad \text{in } Q = (0,T)\times(0,1)\,, \\ y(t,1) = 0 \quad t \in (0,T)\,, \\ y(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha y_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ y(0,\cdot) = y^0 \quad \text{in } (0,1)\,, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} u_t - (x^\alpha u_x)_x + d(t,x)u = y\mathbb{I}_{\mathcal{O}} \quad \text{in } Q\,, \\ u(t,1) = 0 \quad t \in (0,T)\,, \\ u(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha u_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ u(0,\cdot) = u^0 \quad \text{in } (0,1)\,, \end{gathered} \tag{1.2}$$

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or the system

$$\begin{gathered} y_t - (x^\alpha y_x)_x + c(t,x)y = \xi + h\mathbb{I}_\omega \quad \text{in } Q\,, \\ y(t,1) = 0 \quad t \in (0,T)\,, \\ y(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha y_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ y(0,\cdot) = y^0 \quad \text{in } (0,1)\,, \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} -q_t - (x^\alpha q_x)_x + d(t,x)q = y\mathbb{I}_{\mathcal{O}} \quad \text{in } Q\,, \\ q(t,1) = 0 \quad t \in (0,T)\,, \\ q(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha q_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ q(T,\cdot) = 0 \quad \text{in } (0,1)\,, \end{gathered} \tag{1.4}$$

where $y^0 \in L^2(0,1)$, $\xi \in L^2(Q)$, $c(t,x), d(t,x) \in L^\infty(Q)$ are given, h denotes a control function to be determined, and $\mathbb{I}_A$ denotes the characteristic function of the set A.

Models of type (1.1)-(1.2) are the linear version of more complex models that appear in mathematical biology and in a wide variety of physical situations (see e.g. [17, 20, 9]). The controllability properties of nondegenerate parabolic cascade systems have been studied in different contexts in the last fifteen years or so (see [2, 22, 3, 4, 14, 16, 18]). However, as far as we know, the degenerate case has not been analyzed in the literature.

On the other hand, coupled systems like (1.3)-(1.4) arise in a natural way when treating "insensitizing problems" (see [19] for the original formulation). To be more specific, consider the system of equations

$$\begin{gathered} \bar{y}_t - (x^\alpha \bar{y}_x)_x + c(t,x)\bar{y} = \xi + h\mathbb{I}_\omega \quad \text{in } Q\,, \\ \bar{y}(t,1) = 0 \quad t \in (0,T)\,, \\ \bar{y}(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha \bar{y}_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ \bar{y}(0,\cdot) = y_0 + \tau\bar{y}_0 \quad \text{in } (0,1)\,. \end{gathered} \tag{1.5}$$

In this system, $\xi \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ are given, $h \in L^2(\omega \times (0,T))$ is a control to be determined and $\bar{y}_0 \in L^2(\Omega)$ is unknown but τ is small and $\|\bar{y}_0\|_2 = 1$. Let $\mathcal{O} \subset \Omega$ be a nonempty set, and consider the functional

$$\Phi(h,\tau) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |\bar{y}|^2\,dx\,dt.$$

We will say that h insensitizes Φ if

$$\frac{\partial\Phi}{\partial\tau}\Big|_{\tau=0} = 0. \tag{1.6}$$

It is not difficult to see (e.g.[2]) that condition (1.6) is equivalent to obtain a control h such that system (1.3)-(1.4) satisfies $q(0,\cdot) = 0$.

In this paper we extend the Carleman estimates obtained in one dimensional domains by the first author and collaborators [6, 1] to the case of cascade systems as specified before, and recover controllability results similar to those obtained in [22] and [15].

We introduce the weight $e_M(t) = \exp(Mt^{-4})$, and define the Hilbert space

$$L^2(e_M(t)) = \big\{ f : \int_0^T \int_\Omega f^2(t,x) e_M(t)\, dx\, dt < \infty \big\}.$$

The main results in this paper are as follows.

Theorem 1.1. *Assume that $0 \notin \overline{\mathcal{O}}$ and that $\omega \cap \mathcal{O} \neq \emptyset$. There exists a positive constant $M = M(\omega, T)$ such that, if $\xi \in L^2(e_M(T-t))$ and $y^0, u^0 \in L^2(\Omega)$, then there exists $h \in L^2(Q)$ such that the corresponding solution to* (1.1)-(1.2) *satisfies* $y(T,\cdot) = u(T,\cdot) = 0$.

Theorem 1.2. *Assume that $0 \notin \overline{\mathcal{O}}$ and that $\omega \cap \mathcal{O} \neq \emptyset$. There exists a positive constant $M = M(\omega, T)$ such that, if $\xi \in L^2(e_M(t))$ and $y^0 = 0$, then there exists $h \in L^2(Q)$ such that the corresponding solution to* (1.3)-(1.4) *satisfies* $q(0,\cdot) = 0$.

Remark 1.3. Observe that in Theorem 1.2, we require y_0 to be equal to zero. In [22], for the non degenerate case, it is proved that there exists initial data $y^0 \in L^2(\Omega)$ such that the solution q to (1.4) does not vanish at $t = 0$ for any $h \in L^2(\omega \times (0,T))$. In other words, system (1.3)-(1.4) is not null controllable for general initial data in L^2. This situation is due to the fact that equation (1.3) is forward in time and equation (1.4) is backward. A more complete analysis of this phenomenon (in the non degenerate case) can be found in [22] and in [23].

It is by now well understood that the null controllability of systems is equivalent to the validity of an observability inequality for the adjoint system. To be more specific, instead of proving Theorems 1.1 and 1.2 directly, we will prove equivalent results. That is, we consider the adjoint system to (1.1)-(1.2),

$$\begin{gathered} z_t + (x^\alpha z_x)_x - c(t,x)z = v\mathbb{I}_{\mathcal{O}} \quad \text{in } Q\,, \\ z(t,1) = 0 \quad t \in (0,T)\,, \\ z(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha z_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\ z(T,\cdot) = z^0 \quad \text{in } (0,1) \end{gathered} \tag{1.7}$$

and

$$\begin{gathered} v_t + (x^\alpha v_x)_x - d(t,x)v = 0 \quad \text{in } Q\,, \\ v(t,1) = 0 \quad t \in (0,T)\,, \\ v(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha v_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T), \\ v(T,\cdot) = v^0 \quad \text{in } (0,1)\,, \end{gathered} \tag{1.8}$$

and the adjoint system to (1.3)-(1.4):

$$\begin{gathered} z_t + (x^\alpha z_x)_x - c(t,x)z = p\mathbb{I}_{\mathcal{O}} \quad \text{in } Q\,, \\ z(t,1) = 0 \quad t \in (0,T)\,, \\ z(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\ (x^\alpha z_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T), \\ z(T,\cdot) = 0 \quad \text{in } \ (0,1). \end{gathered} \tag{1.9}$$

and

$$
\begin{gathered}
p_t-(x^\alpha p_x)_x+d(t,x)p=0 \quad \text{in } Q\,,\\
p(t,1)=0 \quad t\in(0,T),\ t\in(0,T)\,,\\
p(t,0)=0 \quad \text{if } 0\le\alpha<1,\ t\in(0,T),\\
(x^\alpha p_x)(t,0)=0 \quad \text{if } 1\le\alpha<2,\ t\in(0,T),\\
p(0,\cdot)=p^0 \quad \text{in } (0,1)\,.
\end{gathered}
\tag{1.10}
$$

Then we have the following observability inequalities.

Proposition 1.4. *Suppose $\mathcal{O}\cap\omega\neq\emptyset$ and suppose that $0\notin\overline{\mathcal{O}}$. Then, there exist constants $M>0$ large enough and $C>0$ such that for every solution to* (1.7)-(1.8) *the following holds*

$$
\int_\Omega (v^2(0)+z^2(0))dx+\iint_Q e^{-M/(T-t)^4}z^2\,dx\,dt\le C\int_0^T\!\!\int_\omega z^2\,dx\,dt\,. \tag{1.11}
$$

Moreover, there exist positive constants M and C such that for every solution to (1.9)-(1.10) *the following holds*

$$
\iint_Q e^{-M/t^4}z^2\,dx\,dt\le C\int_0^T\!\!\int_\omega z^2\,dx\,dt. \tag{1.12}
$$

The rest of the paper is structured in the following way. In the next section we prove a Carleman inequality for a single parabolic degenerate heat equation. This inequality will be used in Section 3 to prove Carleman inequalities for the cascade systems (1.7)-(1.8) and (1.9)-(1.10). In the last section we prove (1.11) and (1.12), and sketch a proof of Theorem 1.1, the proof of Theorem 1.2 being similar.

2. Degenerate parabolic equations

In this section we are concerned with the solutions of a degenerate parabolic equation of the form

$$
\begin{gathered}
v_t+(x^\alpha v_x)_x+c(t,x)v=F \quad \text{in } Q\,,\\
v(t,1)=0 \quad t\in(0,T)\,,\\
v(t,0)=0 \quad \text{if } 0\le\alpha<1, t\in(0,T)\,,\\
(x^\alpha v_x)(t,0)=0 \quad \text{if } 1\le\alpha<2,\ t\in(0,T)\,,\\
v(0,\cdot)=v^0 \quad \text{in } (0,1)\,.
\end{gathered}
\tag{2.1}
$$

In the first part of this chapter we prove existence and uniqueness and, in the second part, we prove the Carleman inequality for (2.1) that we will use in Chapter 3.

2.1. **Well-posedness.** First, we briefly describe the weighted spaces where the above problem is well-posed. Let us set $a(x)=x^\alpha$. For $0\le\alpha<1$, define the Hilbert space

$$
\begin{aligned}
H^1_a(0,1):=\big\{u\in L^2(0,1):\ & u \text{ is absolutely continuous in } [0,1],\\
& \sqrt{a}u_x\in L^2(0,1) \text{ and } u(0)=u(1)=0\big\},
\end{aligned}
$$

and the unbounded operator $A:D(A)\subset L^2(0,1)\to L^2(0,1)$ by

$$
\begin{gathered}
\forall u\in D(A),\quad Au:=(au_x)_x,\\
D(A):=\{u\in H^1_a(0,1): au_x\in H^1(0,1)\}.
\end{gathered}
$$

Notice that, if $u \in D(A)$ (or even $u \in H^1_a(0,1)$), then u satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$.

For $1 \le \alpha < 2$, let us change the definition of $H^1_a(0,1)$ to

$$\begin{aligned} H^1_a(0,1) := \{u \in L^2(0,1) &: u \text{ is locally absolutely continuous in } (0,1], \\ &\sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0\}\,. \end{aligned}$$

Then, the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ will be defined by

$$\forall u \in D(A), \quad Au := (au_x)_x,$$

$$\begin{aligned} D(A) := \big\{u \in L^2(0,1) &: u \text{ is locally absolutely continuous in } (0,1], \\ &au \in H^1_0(0,1),\ au_x \in H^1(0,1) \text{ and } (au_x)(0) = 0\big\}. \end{aligned}$$

In fact, it can be proved (see, e.g., [7]) that

$$D(A) = \{u \in H^1_a(0,1) : au_x \in H^1(0,1)\}\,.$$

Notice that when $u \in D(A)$, then u satisfies the Neumann boundary condition $(au_x)(0) = 0$ and the Dirichlet boundary condition $u(1) = 0$.

In both cases $0 \le \alpha < 1$ and $1 \le \alpha < 2$, the following results hold, (see, e.g., [5] and [6]).

Proposition 2.1. *The operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is closed self-adjoint negative, with dense domain.*

Hence, A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$. Consequently, we have the following well-posedness result.

Theorem 2.2. *Let F be given in $L^2(Q_T)$. For all $v_0 \in L^2(0,1)$, problem (2.1) has a unique solution*

$$v \in \mathcal{U} := \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1)). \tag{2.2}$$

Moreover, if $v_0 \in D(A)$, then

$$v \in \mathcal{C}^0([0,T]; H^1_a(0,1)) \cap L^2(0,T; D(A)) \cap H^1(0,T; L^2(0,1)). \tag{2.3}$$

Remark 2.3. Most of the results of this paper hold (and will be stated) for solutions in the above class (2.2). However, in the proofs, we will assume–often without further notice–that solutions belong to the stronger class (2.3). This can yields no loss of generality, since the general result can always be recovered by a standard density argument.

2.2. **Carleman inequalities.** For $\omega = (a,b)$ let us call $\kappa = \frac{2a+b}{3}$, $\lambda = \frac{a+2b}{3}$, and let $\xi \in C^2(\mathbb{R})$ be such that $0 \le \xi \le 1$ and

$$\xi(x) = \begin{cases} 1 & \text{if } x \in (0,\kappa) \\ 0 & \text{if } x \in (\lambda,1). \end{cases}$$

Let us define

$$\theta(t) = \frac{1}{(t(T-t))^4} \quad \forall t \in (0,T),$$

$$\psi(x) = \begin{cases} (x^{2-\alpha} - c_1), & 0 \le \alpha < 2,\ \alpha \ne 1,\ \forall x \in [0,1] \\ (e^x - c_1), & \alpha = 1,\ \forall x \in [0,1] \end{cases}$$

where c_1 is such that $\psi(x) < 0$ for every $x \in [0,1]$. Now, let us set

$$\begin{gathered}\zeta(x) = \frac{1 - x^{\alpha/2}}{1 - \alpha/2},\\ \Psi(x) = e^{2r\zeta(0)} - e^{r\zeta(x)}\\ \Phi(t,x) = \theta(t)[\xi(x)\psi(x) - (1 - \xi(x))\Psi(x)].\end{gathered}$$

The main result of this section is as follows.

Theorem 2.4. *Let $0 \le \alpha < 2$ and $T > 0$ be given. Then there exists two positive constants C, s_0 such that for all $s \ge s_0$ and for every solution $v \in \mathcal{U}$ to (2.1),*

$$\begin{aligned}&\iint_Q (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2) e^{2s\Phi}\,dx\,dt\\ &\le C\Big(\iint_Q e^{2s\Phi} F^2\,dx\,dt + \int_0^T\int_\omega e^{2s\Phi} v^2\,dx\,dt\Big)\end{aligned}\tag{2.4}$$

Remark 2.5. This inequality was basically proved in [6, 1, 8]. The reason why we provide the proof is that, here, we need the locally distributed term in the right-hand side of (2.4) to appear with the same exponential weight as in the left-hand side of the inequality. In [6, 1, 8] such a term was replaced by a boundary term involving the normal derivative of the solution.

The proof of Theorem 2.4 will be given at the end of this section as a consequence of the following result. Let us consider any solution v to the system

$$\begin{gathered}v_t + (x^\alpha v_x)_x = F \quad \text{in } Q\,,\\ v(t,1) = 0 \quad t \in (0,T)\,,\\ v(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,,\\ (x^\alpha v_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,,\\ v(0,\cdot) = v^0 \quad \text{in } (0,1)\,.\end{gathered}\tag{2.5}$$

Theorem 2.6. *Let $0 \le \alpha < 2$ and $T > 0$ be given. Then there exists two positive constants C, s_0 such that for all $s \ge s_0$ and for every solution $v \in \mathcal{U}$ to (2.5),*

$$\begin{aligned}&\iint_Q (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2) e^{2s\Phi}\,dx\,dt\\ &\le C\Big(\iint_Q e^{2s\Phi} F^2\,dx\,dt + \int_0^T\int_\omega e^{2s\Phi} v^2\Big)\end{aligned}\tag{2.6}$$

The proof of Theorem 2.6 follows the ideas of [1]. That is, we prove first a Carleman inequality for the degenerate part and combine it with a classical Carleman inequality for the non degenerate part. We will see that the appropriate combination of both inequalities drives to (2.6).

Let $\varphi(t,x) = \psi(x)\theta(t)$. Then we will prove the following result.

Theorem 2.7. *Let $0 \le \alpha < 2$ and $T > 0$ be given. Then there exists two positive constants C, s_0 such that for all $s \ge s_0$ and for every solution $v \in \mathcal{U}$ to (2.5),*

$$\begin{aligned}&\iint_Q \Big(\frac{|(x^\alpha v_x)_x|^2}{s\theta} + \frac{|v_t|^2}{s\theta} + s^3\theta^3 x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\Big) e^{2s\varphi}\,dx\,dt\\ &\le C\Big(\iint_Q e^{2s\varphi} F^2\,dx\,dt + \int_0^T s\theta e^{2s\varphi} v_x^2|_{x=1}\Big).\end{aligned}\tag{2.7}$$

For the proof of Theorem 2.7 we follow the ideas in [6, 1], that is we use an appropriate change of variables and the following Hardy type inequality.

Lemma 2.8. (1) *Let* $0 \le \alpha^* < 1$. *Then, for all locally absolutely continuous function* $u \in (0,1)$ *satisfying*

$$u(x) \to 0 \text{ as } x \to 0^+ \quad \text{and} \quad \int_0^1 x^{\alpha^*} u_x^2 dx < \infty,$$

the following inequality holds

$$\int_0^1 x^{\alpha^*-2} u^2 dx \le \frac{4}{(1-\alpha^*)^2} \int_0^1 x^{\alpha^*} u_x^2 dx. \tag{2.8}$$

(2) *Let* $1 < \alpha^* < 2$, *then the above inequality* (2.8) *still holds for all locally absolutely continuous function* u *in* $(0,1)$ *satisfying*

$$u(x) \to 0 \text{ as } x \to 1^- \quad \text{and} \quad \int_0^1 x^{\alpha^*} u_x^2 dx < \infty.$$

Remark 2.9. Observe that (2.8) is false for $\alpha^* = 1$.

Sketch of the proof of Theorem 2.7. Let us define $w(t,x) = e^{s\varphi(t,x)} v(t,x)$ where v satisfies (2.5). Then w solves

$$\begin{gathered}
(e^{-s\varphi} w)_t + (x^\alpha (e^{-s\varphi} w)_x)_x = F \quad \text{in } Q\,, \\
w(t,1) = 0 \quad t \in (0,T)\,, \\
w(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\
(x^\alpha w_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\
w(0,\cdot) = w(T,\cdot) = 0 \quad \text{in } (0,1)\,,
\end{gathered} \tag{2.9}$$

We can rewrite the above system as

$$P_s w = P_s^+ w + P_s^- w = F e^{s\varphi}$$

where

$$\begin{gathered}
P_s^+ w = -s\varphi_t w + s^2 x^\alpha \varphi_x^2 w + (x^\alpha w_x)_x, \\
P_s^- = w_t - s(x^\alpha \varphi_x)_x w - 2s x^\alpha \varphi_x w_x.
\end{gathered}$$

We observe that, for $\alpha \ne 1$,

$$(x^\alpha w_x)_x = P_s^+ w + s\theta_t (x^{2-\alpha} - c_1) w - s^2 c_2 x^{2-\alpha} \theta^2 w \tag{2.10}$$

with c_2 a generic constant, whereas, for $\alpha = 1$,

$$(x^\alpha w_x)_x = P_s^+ w + s\theta_t (e^x - c_1) w - s^2 x e^{2x} \theta^2 w. \tag{2.11}$$

Observe that

$$\|F e^{s\varphi}\|^2 \ge \|P_s^+ w\|^2 + \|P_s^- w\|^2 + 2\langle P_s^+ w, P_s^- w\rangle.$$

Following [6], we conclude that, for every $0 \le \alpha < 2$,

$$\begin{aligned}
\|F e^{s\varphi}\|^2 &\ge \|P_s^+ w\|^2 + \|P_s^- w\|^2 + 2\langle P_s^+ w, P_s^- w\rangle \\
&\ge \|P_s^+ w\|^2 + \|P_s^- w\|^2 + C s^3 \iint_Q \theta^3 x^{2-\alpha} w^2 + C s \iint \theta x^\alpha w_x^2 \\
&\quad - C' \int_0^T \{s\theta w_x^2\}\Big|_{x=1}.
\end{aligned} \tag{2.12}$$

Now, we consider the case $\alpha \neq 1$. From (2.10) and the fact that $|\theta_t| \leq C\theta^{5/4} \leq C\theta^2$ we obtain

$$\begin{aligned} &\iint_Q \frac{|(x^\alpha w_x)_x|^2}{\theta s}\,dx\,dt \\ &\leq C\Big(\iint_Q \frac{|P_s^+|^2}{\theta s} + s\frac{\theta_t^2}{\theta}w^2 + s\frac{\theta_t^2}{\theta}x^{2(2-\alpha)}w^2 + s^3\theta^3 x^{2(2-\alpha)}w^2\,dx\,dt\Big). \end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned} \iint_Q s\frac{\theta_t^2}{\theta}w^2 &\leq C\iint_Q s\theta^{3/2}w^2\,dx\,dt \\ &= C\iint s^{1/2}\theta^{1/2}wx^{\frac{\alpha-2}{2}}\theta wx^{-(\frac{\alpha-2}{2})}s^{1/2}\,dx\,dt \\ &\leq C\Big[\iint_Q s\theta w^2x^{\alpha-2} + \iint s\theta^2w^2x^{2-\alpha}\,dx\,dt\Big] \end{aligned}$$

and, since $x \leq 1$ and $\theta^{3/2} \leq C(T)\theta^2$,

$$\begin{aligned} \iint_Q s\frac{\theta_t^2}{\theta}x^{2(2-\alpha)}w^2 &\leq C\iint_Q s\theta^{3/2}x^{2(2-\alpha)}w^2\,dx\,dt \\ &= C\iint s\theta^2w^2x^{2-\alpha}\,dx\,dt. \end{aligned}$$

In conclusion,

$$\iint_Q \frac{|(x^\alpha w_x)_x|^2}{\theta s}\,dx\,dt \leq C\Big(\iint_Q \frac{|P_s^+|^2}{\theta s} + s^3\theta^3w^2x^{2-\alpha}\,dx\,dt + \iint_Q s\theta w^2x^{\alpha-2}\,dx\,dt\Big).$$

Applying Hardy's inequality, we obtain

$$\begin{aligned} &\iint_Q \frac{|(x^\alpha w_x)_x|^2}{\theta s}\,dx\,dt \\ &\leq C\Big(\iint_Q \frac{|P_s^+|^2}{\theta s}\,dx\,dt + \iint_Q s^3\theta^3w^2x^{2-\alpha}\,dx\,dt + \iint_Q s\theta w_x^2x^\alpha\,dx\,dt\Big). \end{aligned} \tag{2.14}$$

Proceeding as before, it is not difficult to prove that

$$\begin{aligned} &\iint_Q \frac{|w_t|^2}{\theta s}\,dx\,dt \\ &\leq C\Big(\iint_Q \frac{|P_s^-|^2}{\theta s}\,dx\,dt + \iint_Q s^3\theta^3w^2x^{2-\alpha}\,dx\,dt + \iint_Q s\theta w_x^2x^\alpha\,dx\,dt\Big). \end{aligned} \tag{2.15}$$

Combining (2.12), (2.14) and (2.15) we conclude that, for s large enough,

$$\begin{aligned} C\|Fe^{s\varphi}\|^2 \geq &\iint_Q \frac{|w_t|^2}{\theta s}\,dx\,dt + \iint_Q \frac{|(x^\alpha w_x)_x|^2}{\theta s}\,dx\,dt \\ &+ s^3\iint_Q \theta^3x^{2-\alpha}w^2 + s\iint \theta x^\alpha w_x^2 - C'\int_0^T \{s\theta w_x^2\}\Big|_{x=1}. \end{aligned} \tag{2.16}$$

For $\alpha \neq 1$ recall that $\varphi = \theta(t)\psi(x)$ with

$$\psi_x = c_1(2-\alpha)x^{1-\alpha} \quad \text{and} \quad \psi_{xx} = c_1(2-\alpha)(1-\alpha)x^{-\alpha}.$$

Then $x^{2\alpha}\psi_x^4 = Cx^{2(2-\alpha)}$ and $x^\alpha\psi_x^2 = Cx^{2-\alpha}$. Moreover, $v(t,x) = e^{-s\varphi}w(t,x)$, $v_t = -s\theta_t\psi e^{-s\varphi}w + e^{-s\varphi}w_t$ and $v_x(t,x) = -s\theta\psi_x e^{-s\varphi}w + e^{-s\varphi}w$. Therefore,

$$
\begin{aligned}
&\iint_Q \Big(s^3\theta^3x^{2-\alpha}v^2 + s\theta x^\alpha v_x^2 + \frac{v_t^2}{\theta s} + \frac{(x^\alpha v_x)_x^2}{\theta s}\Big)\,dx\,dt \\
&\le \iint_Q \Big(s^3\theta^3x^{2-\alpha}e^{-2s\varphi}w^2 + s\theta x^\alpha(2s^2\theta^2\psi_x^2e^{-2s\varphi}w^2) + 2e^{-2s\varphi}w_x^2\Big)\,dx\,dt \\
&\quad + \iint_Q \Big(2\frac{e^{-2s\varphi}w_t^2}{\theta s} + 2\frac{s^2\theta_t^2\psi^2e^{-2s\varphi}w^2}{\theta s} + \frac{2}{\theta s}(x^\alpha w_x)_x^2e^{-2s\varphi}\Big)\,dx\,dt \\
&\quad + \iint_Q \Big(2\frac{s^2\theta^2}{\theta s}\alpha x^{2(\alpha-1)}\psi_x^2e^{-2s\varphi}w^2 + 2\frac{s^2\theta^2}{s\theta}x^{2\alpha}\psi_{xx}^2e^{-2s\varphi}w^2\Big)\,dx\,dt \\
&\quad + \iint_Q \Big(2\frac{s^4\theta^4}{s\theta}x^{2\alpha}\psi_x^2e^{-2s\varphi}w^2 + 4\frac{s^2\theta^2}{s\theta}x^{2\alpha}\psi_x^2e^{-2s\varphi}w_x^2\Big)\,dx\,dt.
\end{aligned}
$$

Using several times the Hardy type estimate and the bounds on φ and on its derivatives, it is not difficult to conclude that

$$
\begin{aligned}
&\iint_Q e^{2s\varphi}\Big(s^3\theta^3x^{2-\alpha}v^2 + s\theta x^\alpha v_x^2 + \frac{v_t^2}{\theta s} + \frac{(x^\alpha v_x)_x^2}{\theta s}\Big) \\
&\le C\iint_Q \Big(s^3\theta^3x^{2-\alpha}w^2 + s\theta x^\alpha w_x^2 + \frac{w_t^2}{\theta s} + \frac{(x^\alpha w_x)_x^2}{\theta s}\Big).
\end{aligned}
\tag{2.17}
$$

Observe that $v|_{x=1} = 0$ and then $v_x|_{x=1} = e^{s\varphi}w_x|_{x=1}$. The latter combined with (2.16) and (2.17) leads to (2.7).

We now consider the case $\alpha = 1$. From (2.11) we have

$$
\begin{aligned}
&\iint_Q \frac{|(x^\alpha w_x)_x|^2}{\theta s}\,dx\,dt \\
&\le C\Big(\iint_Q \frac{|P_s^+|^2}{\theta s} + s\frac{\theta_t^2}{\theta}w^2 + s\frac{\theta_t^2}{\theta}xe^{2x}w^2 + s^3\theta^3x^2e^{4x}w^2\,dx\,dt\Big).
\end{aligned}
\tag{2.18}
$$

Observe that

$$
\begin{aligned}
|\int_0^1 s\frac{\theta_t^2}{\theta}w^2dx| &\le C\int_0^1 s\theta^{3/2}\left(x^{-1/4}w^{3/2})(x^{1/4}w^{1/2}\right)dx \\
&\le C\int_0^1 s\left(\theta x^{-1/3}w^2\right)^{3/4}\left(\theta^3xw^2\right)^{1/4}dx \\
&\le C\left(\int_0^1 s\theta x^{-1/3}w^2dx\right)^{3/4}\left(\int_0^1 \theta^3xw^2dx\right)^{1/4}.
\end{aligned}
$$

We now use Hardy's inequality with $\alpha = 5/3$ to obtain

$$
|\int_0^1 s\frac{\theta_t^2}{\theta}w^2dx| \le C\Big(\int_0^1 s\theta x^{5/3}w_x^2dx\Big)^{3/4}\Big(\int_0^1 \theta^3xw^2dx\Big)^{1/4}. \tag{2.19}
$$

Since $5/3 > 1$, using Young's inequality we get, by integrating in time,

$$
|\iint_Q s\frac{\theta_t^2}{\theta}xe^{2x}w^2dxdt| \le C\Big(\iint_Q s\theta xw_x^2\,dx\,dt + \iint_Q s^3\theta^3xw^2\,dx\,dt\Big).
$$

Proceeding as before it is not difficult to see that

$$\iint_Q \frac{|(xw_x)_x|^2}{\theta s}\,dx\,dt \le C\Big(\iint_Q \frac{|P_s^+|^2}{\theta s} + \iint_Q s\theta x w_x^2\,dx\,dt + \iint_Q s^3\theta^3 x w^2\,dx\,dt\Big).$$

In a similar way the following inequality can be proved

$$\iint_Q \frac{|w_t|^2}{\theta s}\,dx\,dt \le C\Big(\iint_Q \frac{|P_s^-|^2}{\theta s} + \iint_Q s\theta x w_x^2\,dx\,dt + \iint_Q s^3\theta^3 x w^2\,dx\,dt\Big).$$

The last part of the proof is similar to the case $\alpha \neq 1$, the only difference being the use of Hardy's inequality (false if $\alpha = 1$) with the same exponent as in (2.19). $\square$

We will also need the following Carleman estimates, valid in the nondegenerate case.

Proposition 2.10 (Classical Carleman Estimates)**.** *Let z be solution of*

$$\begin{gathered} z_t + (a(x)z_x)_x - c(t,x)z = h \quad in\ Q\,, \\ z(t,1) = 0, \quad z(t,0) = 0 \quad t \in (0,T)\,, \end{gathered} \tag{2.20}$$

where $a \in C^1([0,1])$ is a strictly positive function. Let us define $\varrho(t,x) = \theta(t)\Psi(x)$. Then there exist two positive constants r and s_0 such that for any $s > s_0$, the solution of (2.20) *satisfies*

$$\begin{aligned} &\iint_Q \Big(\frac{|(a(x)z_x)_x|^2}{s\theta} + \frac{|z_t|^2}{s\theta} + s e^{r\zeta(x)}\theta z_x^2 + s^3\theta^3 e^{3r\zeta(x)} z^2\Big) e^{-2s\varrho}\,dx\,dt \\ &\le C\Big(\iint_Q e^{-2s\varrho} h^2\,dx\,dt + \int_0^T\int_\omega e^{-2s\varrho} z^2\,dx\,dt\Big) \end{aligned} \tag{2.21}$$

for some positive constant C.

The proof of the above result is by now classical and can be found, e.g., in [12]. We are now almost ready to prove Theorem 2.6. First, we recall Caccioppoli's inequality. For completeness, we give a sketch of its proof in the appendix at the end of the paper. A complete proof can be found in [1].

Lemma 2.11 (Caccioppoli's inequality)**.** *Suppose $\omega' \subset\subset \omega$, then there exists a constant $C > 0$ such that, for every solution of* (2.5)*, the following inequality holds*

$$\int_0^T\int_{\omega'} v_x^2 e^{2s\Phi}\,dx\,dt \le C\Big(\int_0^T\int_\omega v^2 e^{2s\Phi}\,dx\,dt + \iint_Q F^2\,dx\,dt\Big).$$

Proof of Theorem 2.6. Observe that $v = \xi v + (1-\xi)v$. Define $w = \xi v$, clearly w is solution of equation (2.5) with second member $G = \xi F + (x^\alpha \xi_x v)_x + \xi_x x^\alpha v_x$. We can then apply inequality (2.6) to w. Observe that, by construction, $w_x|_{x=1} = 0$. Then

$$\begin{aligned} &\iint_Q \Big(\frac{|(x^\alpha w_x)_x|^2}{s\theta} + \frac{|w_t|^2}{s\theta} + s^3\theta^3 x^\alpha w_x^2 + s^3\theta^3 x^{2-\alpha} w^2\Big) e^{2s\varphi}\,dx\,dt \\ &\le C\Big(\iint_Q e^{2s\varphi} F^2\,dx\,dt + \int_0^T\int_{\omega'} e^{2s\varphi}(v_x^2 + v^2)\,dx\,dt\Big). \end{aligned}$$

Since, for $x \in (0,\kappa)$, $\varphi(x) = \Phi(x)$ and $w = v$, we have

$$
\begin{aligned}
&\int_0^T\int_0^\kappa \Big(\frac{|(x^\alpha v_x)_x|^2}{s\theta} + \frac{|v_t|^2}{s\theta} + s^3\theta^3 x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\Big) e^{2s\Phi}\,dx\,dt \\
&\le C\Big(\iint_Q e^{2s\varphi} F^2\,dx\,dt + \int_0^T\int_\kappa^\lambda e^{2s\varphi}(v_x^2 + v^2)\,dx\,dt\Big).
\end{aligned}
\tag{2.22}
$$

Define $z = (1-\xi)v$, then z is solution to (2.20) (in fact in an smaller set $Q_\delta = (\delta,1)\times(0,T)$) with $h = (1-\xi)F - (x^\alpha \xi_x v)_x - \xi_x x^\alpha v_x$ and inequality

$$
\begin{aligned}
&\iint_{Q_\delta} \Big(\frac{|(a(x)z_x)_x|^2}{s\theta} + \frac{|z_t|^2}{s\theta} + s e^{r\zeta(x)}\theta z_x^2 + s^3\theta^3 e^{3r\zeta(x)} z^2\Big) e^{-2s\varrho}\,dx\,dt \\
&\le C\iint_Q e^{-2s\varrho} F^2\,dx\,dt + C\int_0^T\int_\kappa^\lambda e^{-2s\varrho}(v^2 + v_x^2)\,dx\,dt \\
&\quad + C\int_0^T\int_\omega e^{-2s\varrho} z^2\,dx\,dt.
\end{aligned}
\tag{2.23}
$$

Again, since $-\varrho(t,x) = \Psi(t,x)$ and $z = v$ for $x \in (\lambda,1)$, we obtain

$$
\begin{aligned}
&\int_0^T\int_\lambda^1 \Big(\frac{|(x^\alpha v_x)_x|^2}{s\theta} + \frac{|v_t|^2}{s\theta} + s e^{r\zeta(x)}\theta v_x^2 + s^3\theta^3 e^{3r\zeta(x)} v^2\Big) e^{2s\Phi}\,dx\,dt \\
&\le C\Big(\iint_Q e^{-2s\varrho} F^2\,dx\,dt + \int_0^T\int_\kappa^\lambda e^{-2s\varrho}(v^2 + v_x^2)\,dx\,dt\Big).
\end{aligned}
\tag{2.24}
$$

Observe that, for $x \in (\kappa,1)$, $x^\alpha \le Ce^{r\zeta(x)}$ and $x^{2-\alpha} \le Ce^{3r\zeta(x)}$. So, combining inequalities (2.24) and (2.23), and adding to both sides of the inequality the term

$$
\int_0^T\int_\kappa^\lambda e^{2s\Phi}\left(s^3\theta^3 x^{2-\alpha} v^2 + s\theta x^\alpha v_x^2\right)\,dx\,dt
$$

we obtain

$$
\begin{aligned}
&\iint_Q \Big(\frac{|(x^\alpha v_x)_x|^2}{s\theta} + \frac{|v_t|^2}{s\theta} + s x^\alpha\theta v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\Big) e^{2s\Phi}\,dx\,dt \\
&\le C\Big(\iint_Q (e^{-2s\varrho} + e^{2s\varphi}) F^2\,dx\,dt + \int_0^T\int_\kappa^\lambda (e^{-2s\varrho} + e^{2s\varphi} + e^{2s\Phi})(v^2 + v_x^2)\,dx\,dt\Big)
\end{aligned}
$$

Observe that $-\varrho$, φ and Φ are equivalent for $x \in (\kappa,\lambda)$, which means that, for some $C > 0$,

$$
\begin{aligned}
&\iint_Q \Big(\frac{|(x^\alpha v_x)_x|^2}{s\theta} + \frac{|v_t|^2}{s\theta} + s x^\alpha\theta v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\Big) e^{2s\Phi}\,dx\,dt \\
&\le C\Big(\iint_Q e^{2s\Phi} F^2\,dx\,dt + \int_0^T\int_\kappa^\lambda e^{2s\Phi}(v^2 + v_x^2)\,dx\,dt\Big).
\end{aligned}
$$

We conclude the proof of Theorem 2.6 combining this last inequality with Caccioppoli's inequality. $\square$

Proof of Theorem 2.4. Apply Theorem 2.7 to (2.5) for $\overline{F} = F - c(t,x)v$. Then, clearly v the solution to (2.1) satisfies

$$\begin{aligned} &\iint_Q (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2) e^{2s\Phi}\,dx\,dt \\ &\le C\Big(\iint_Q e^{2s\Phi}(F^2 + c^2(t,x)v^2)\,dx\,dt + \int_0^T\int_\omega e^{2s\Phi}v^2\Big). \end{aligned} \tag{2.25}$$

Observe that $x^{\alpha-2}$ is a decreasing function in $(0,1)$ and $\lim_{x\to 0^+} x^{\alpha-2} = \infty$. That means that

$$c^2(t,x) \le \|c\|_\infty^2 x^{\alpha-2} \quad \forall (t,x)\in Q,$$

so

$$\iint_Q e^{2s\Phi}c^2(t,x)v^2\,dx\,dt \le C\|c\|_\infty^2 \iint_Q e^{2s\Phi}x^{\alpha-2}v^2\,dx\,dt\,. \tag{2.26}$$

For $\alpha \neq 1$ we apply Hardy inequality to $w = e^{s\Phi}v$. Then,

$$\iint_Q e^{2s\Phi}x^{\alpha-2}v^2\,dx\,dt \le C\Big(\iint_Q x^\alpha s^2\Phi_x^2 v^2 e^{2s\Phi} + x^\alpha v_x^2 e^{2s\Phi}\,dx\,dt\Big).$$

Observe that for $x \in (0,\kappa)$, $\Phi_x = (2-\alpha)x^{1-\alpha}\theta(t)$ and for $1 \ge x \ge \kappa$ there exists C such that $\Phi_x \le C(2-\alpha)x^{1-\alpha}\theta(t)$. Then, the last inequality with (2.26) implies that there exists $C > 0$ such that

$$\begin{aligned} &\iint_Q (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha}v^2)e^{2s\Phi}\,dx\,dt \\ &\le C\Big(\iint_Q e^{2s\Phi}F^2\,dx\,dt + \iint_Q (x^{2-\alpha}s^2\theta^2v^2 + x^\alpha v_x^2)e^{2s\Phi}\,dx\,dt + \int_0^T\int_\omega e^{2s\Phi}v^2\Big). \end{aligned}$$

Observe that in the right hand side we have smaller exponents of s so for s large enough we obtain (2.4).

The proof for $\alpha = 1$ is similar but, instead of (2.26), observe that

$$\iint_Q e^{2s\Phi}c^2(t,x)v^2\,dx\,dt \le C\|c\|_\infty^2 \iint_Q e^{2s\Phi}x^{-1/3}v^2\,dx\,dt \tag{2.27}$$

to obtain

$$\begin{aligned} &\iint_Q (s\theta x v_x^2 + s^3\theta^3 x v^2)e^{2s\Phi}\,dx\,dt \\ &\le C\Big(\iint_Q e^{2s\Phi}F^2\,dx\,dt + \iint_Q (x^{5/3}s^2\theta^2v^2 + x^{5/3}v_x^2)e^{2s\Phi}\,dx\,dt + \int_0^T\int_\omega e^{2s\Phi}v^2\Big). \end{aligned}$$

The conclusion is then straightforward. □

3. Carleman inequality for cascade systems

In this section we will prove a Carleman inequality that is valid for both: the adjoint system to (1.1)-(1.2), i.e.,

$$\begin{gathered} z_t + (x^\alpha z_x)_x - c(t,x)z = v\mathbb{I}_{\mathcal{O}} \quad \text{in } Q\,, \\ z(t,1) = 0 \quad t\in(0,T)\,, \\ z(t,0) = 0 \quad \text{if } 0\le\alpha<1,\ t\in(0,T)\,, \quad \textit{and} \\ (x^\alpha z_x)(t,0) = 0 \quad \text{if } 1\le\alpha<2,\ t\in(0,T)\,, \\ z(T,\cdot) = z^0 \quad \text{in } (0,1)\,, \end{gathered} \tag{3.1}$$

$$
\begin{gathered}
v_t + (x^\alpha v_x)_x - d(t,x)v = 0 \quad \text{in } Q\,, \\
v(t,1) = 0 \quad t \in (0,T)\,, \\
v(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\
(x^\alpha v_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\
v(T,\cdot) = v^0 \quad \text{in } (0,1)\,,
\end{gathered} \tag{3.2}
$$

and the adjoint system to (1.3)-(1.4), i.e.,

$$
\begin{gathered}
z_t + (x^\alpha z_x)_x - c(t,x)z = p\mathbb{I}_{\mathcal{O}} \quad \text{in } \ Q\,, \\
z(t,1) = 0 \quad t \in (0,T)\,, \\
z(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\
(x^\alpha z_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\
z(T,\cdot) = z^0 \quad \text{in } \ (0,1)\,.
\end{gathered} \tag{3.3}
$$

and

$$
\begin{gathered}
p_t - (x^\alpha p_x)_x + d(t,x)p = 0 \quad \text{in } Q\,, \\
p(t,1) = 0 \quad t \in (0,T)\,, \\
p(t,0) = 0 \quad \text{if } 0 \le \alpha < 1,\ t \in (0,T)\,, \\
(x^\alpha p_x)(t,0) = 0 \quad \text{if } 1 \le \alpha < 2,\ t \in (0,T)\,, \\
p(0,\cdot) = p^0 \quad \text{in } (0,1)\,.
\end{gathered} \tag{3.4}
$$

Remark 3.1. Observe that in (3.3) we have allowed for $z(T)$ any value z^0 in $L^2(0,1)$. This can be so since the Carleman inequality is valid for general data. However, in the next section, where the observability inequality is proved, it is necessary to consider $z(T) = 0$.

We have the following result.

Theorem 3.2. *Assume $\mathcal{O} \cap \omega \neq \emptyset$ and suppose that $0 \notin \overline{\mathcal{O}}$. Then there exist two positive constants C, s_0 such that, for all $s \ge s_0$ and every solution to* (3.1)-(3.2)*, the following holds*

$$
\begin{aligned}
&\iint_Q \left(s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2 + s\theta x^\alpha z_x^2 + s^3\theta^3 x^{2-\alpha} z^2\right) e^{2s\Phi}\, dx\, dt \\
&\le C \int_0^T\!\!\int_\omega e^{2s\Phi} z^2\, dx\, dt.
\end{aligned} \tag{3.5}
$$

Moreover, there exist two positive constants C, s_0 such that, for all $s \ge s_0$ and every solution to (3.3)-(3.4)*, the following holds*

$$
\begin{aligned}
&\iint_Q \left(s\theta x^\alpha p_x^2 + s^3\theta^3 x^{2-\alpha} p^2 + s\theta x^\alpha z_x^2 + s^3\theta^3 x^{2-\alpha} z^2\right) e^{2s\Phi}\, dx\, dt \\
&\le C \int_0^T\!\!\int_\omega e^{2s\Phi} z^2\, dx\, dt.
\end{aligned} \tag{3.6}
$$

Proof. We will prove only (3.6). Indeed, the proof of (3.5) is similar because the boundary conditions at $t = 0, T$ are made irrelevant by the fact that the weight $\theta^j e^{2s\Phi}$, with $j = 1, 3$, vanishes as $t \to 0$ and $t \to T$. Let us define $p(t) = v(T-t)$, with v solution to (3.2), and observe that p solves (3.4) (with an appropriate choice of $\tilde{d}$).

The proof is to be completed in several steps.

Step 1. Take $\mathcal{O}' \subset\subset \omega \cap \mathcal{O}$. Observe that $w(t,x) := p(T-t,x)$ solves (2.1) and apply Theorem 2.4 to p, which is a solution of (3.4). Then, for $s > s_1$, we get

$$\iint_Q \left(s\theta x^\alpha p_x^2 + s^3\theta^3 x^{2-\alpha} p^2\right) e^{2s\Phi}\, dx\, dt \le C \int_0^T\int_{\mathcal{O}'} e^{2s\Phi} p^2\, dx\, dt. \tag{3.7}$$

Theorem 2.4 can also be applied to z yielding

$$\begin{aligned}
&\iint_Q \left(s\theta x^\alpha p_x^2 + s^3\theta^3 x^{2-\alpha} p^2\right) e^{2s\Phi}\, dx\, dt + \iint_Q \left(s\theta x^\alpha z_x^2 + s^3\theta^3 x^{2-\alpha} z^2\right) e^{2s\Phi}\, dx\, dt\\
&\le C\Big[\int_0^T\int_{\mathcal{O}} e^{2s\Phi} p^2\, dx\, dt + \int_0^T\int_{\mathcal{O}'} e^{2s\Phi}(p^2+z^2)\, dx\, dt\Big].
\end{aligned}$$

Now, observe that, since $0 \notin \overline{\mathcal{O}}$,

$$\int_0^T\int_{\mathcal{O}} e^{2s\Phi} p^2\, dx\, dt \le C \iint_Q s^3\theta^3 x^{2-\alpha} p^2 e^{2s\Phi}\, dx\, dt \le C \int_0^T\int_{\mathcal{O}'} e^{2s\Phi} p^2\, dx\, dt\ . \tag{3.8}$$

All together, we obtain

$$\begin{aligned}
&\iint_Q (s\theta x^\alpha p_x^2 + s^3\theta^3 x^{2-\alpha} p^2) e^{2s\Phi}\, dx\, dt + \iint_Q (s\theta x^\alpha z_x^2 + s^3\theta^3 x^{2-\alpha} z^2) e^{2s\Phi}\, dx\, dt\\
&\le C\Big[\int_0^T\int_{\mathcal{O}'} e^{2s\Phi}(p^2+z^2)\, dx\, dt\Big]\,.
\end{aligned} \tag{3.9}$$

Step 2. Take $\mathcal{O}' \subset\subset \omega' \subset\subset \omega \cap \mathcal{O}$. Let $\xi_1 \in C_0^\infty(\Omega)$ be such that

$$1 \ge \xi_1 \ge 0, \quad \xi_1(x) = 1 \text{ if } x \in \mathcal{O}', \quad \xi_1(x) = 0 \text{ if } x \in \Omega\backslash\omega'. \tag{3.10}$$

Furthermore, we shall require ξ_1 to satisfy

$$\frac{\Delta\xi_1}{\xi_1^{1/2}} \in L^\infty(\Omega), \quad \frac{\nabla\xi_1}{\xi_1^{1/2}} \in L^\infty(\Omega). \tag{3.11}$$

Observe that condition (3.11) is easy to obtain: it suffices to take $\xi \in C_0^\infty(\Omega)$ satisfying (3.10), and define $\xi_1 = \xi^4$. Then ξ_1 will satisfy both (3.10) and (3.11).

Let us multiply (1.9) by $\xi_1 p e^{2s\Phi}$. To simplify notation, set $u = e^{2s\Phi}$. Then

$$\begin{aligned}
&\iint_Q z_t\xi_1 up\ dx\, dt \iint_Q (x^\alpha z_x)_x \xi_1 up\ dx\, dt - \iint_Q c(t,x) z\xi_1 up\ dx\, dt\\
&= \int_0^T\int_{\mathcal{O}} \xi_1 p^2 u\ dx\, dt\,.
\end{aligned} \tag{3.12}$$

We observe that $u(T) = u(0) = 0$. Integrating by parts in (3.12), we obtain

$$\begin{aligned}
&\iint_Q zu\xi_1\left[p_t - (x^\alpha p_x)_x + d(t,x)p\right]\ dx\, dt - \iint_Q (c+d) z\xi_1 up\ dx\, dt\\
&+ \iint_Q z\left[p(x^\alpha u\xi_1)_x + 2p_x x^\alpha (u\xi_1)_x\right]\ dx\, dt + \iint_Q zp\xi_1 u_t\ dx\, dt\\
&= \int_0^T\int_{\mathcal{O}} \xi_1 p^2 u\ dx\, dt.
\end{aligned} \tag{3.13}$$

Let us rewrite (3.13) as $I_1 + I_2 + I_3 + I_4 = \int_0^T \int_{\mathcal{O}} \xi_1 p^2 u$. We observe that $I_1 = 0$ since p satisfies (3.4). By Hölder's and Young's inequalities, we get

$$I_2 \le \frac{\delta_1}{2} \iint_Q \xi_1 p^2 u \, dx\, dt + \frac{1}{\delta_1}(\|c\|_\infty^2 + \|d\|_\infty^2) \int_0^T \int_\Omega \xi_1 z^2 u \, dx\, dt$$

with δ_1 to be chosen later.

Let us estimate I_3. First, we have

$$\begin{aligned} I_3^1 &:= \iint_Q zp(x^\alpha u \xi_1)_x \, dx\, dt \\ &= \iint_Q z \left[p\alpha x^{\alpha-1} u\xi_1 + p x^\alpha u \xi_{1,x} + p x^\alpha u_x \xi_1\right] \, dx\, dt \\ &\le \frac{\delta_2}{2} \iint_Q \xi_1 p^2 u \, dx\, dt \\ &\quad + \frac{1}{2\delta_2} \iint_Q z^2 \Big(x^{2(\alpha-1)} u \xi_1 + x^{2\alpha} \frac{|\xi_{1,x}|^2}{\xi_1} u + x^{2\alpha} \frac{|u_x|^2}{u} \xi_1 \Big) \, dx\, dt\,. \end{aligned}$$

Observe that $\frac{|u_x|^2}{u} = 4s^2 u \Phi_x^2$. Then

$$\iint_Q z^2 \Big(x^{2(\alpha-1)} u \xi_1 + x^{2\alpha} \frac{|\xi_{1,x}|^2}{\xi_1} u + x^{2\alpha} \frac{|u_x|^2}{u} \xi_1 \Big) \, dx\, dt \le C \int_0^T \int_{\omega'} u z^2 \, dx\, dt\,.$$

So, for I_3^1 we conclude that

$$|I_3^1| \le \frac{\delta_2}{2} \iint_Q \xi_1 p^2 u \, dx\, dt + C \int_0^T \int_{\omega'} u z^2 \, dx\, dt\,.$$

We now proceed to estimate the other term in I_3:

$$\begin{aligned} I_3^2 &:= 2 \iint_Q z p_x x^\alpha (u \xi_{1,x} + u_x \xi_1) \, dx\, dt \\ &\le \frac{\delta_3}{2} \iint_Q s\theta x^\alpha p_x^2 u \, dx\, dt + \frac{1}{2\delta_3} \iint_Q z^2 x^\alpha \Big(\frac{u_x^2 \xi_1^2}{u\theta} + \frac{u \xi_{1,x}^2}{\theta} \Big) \, dx\, dt\,. \end{aligned}$$

Observe that the term in p_x^2 can be estimated using Carleman's inequality for p, while the coefficient of z^2 in the other integral is bounded above. Thus,

$$I_3^2 \le \frac{\delta_3}{2} \int_0^T \int_{\mathcal{O}'} p^2 u \ dx\, dt + C \int_0^T \int_{\omega'} z^2 e^{2s\Phi} \ dx\, dt\,.$$

Finally, we get for I_4,

$$I_4 = \iint_Q z p \xi_1 u_t \, dx\, dt \le \frac{\delta_4}{2} \iint_Q \xi_1 p^2 u \, dx\, dt + \frac{1}{2\delta_4} \iint_Q z^2 \xi_1 \frac{|u_t|^2}{u} \, dx\, dt\,.$$

Observe that $\frac{|u_t|^2}{u} = 4s^2 \Phi_t^2 e^{2s\Phi}$ to conclude that

$$I_4 \le \frac{\delta_4}{2} \iint_Q \xi_1 p^2 u + C \int_0^T \int_{\omega'} z^2 e^{2s\Phi} \, dx\, dt\,.$$

Putting the above estimates together and choosing convenient δ_i's, we obtain, since the support of ξ_1 is contained in $\mathcal{O}$,

$$\int_0^T \int_{\mathcal{O}'} e^{2s\Phi} p^2 \, dx\, dt \le C \int_0^T \int_{\omega'} z^2 e^{2s\Phi} \, dx\, dt.$$

The last inequality together with (3.9) completes the proof. □

4. Proof of the main results

Proof of Proposition 1.4. Multiplying equation (1.8) by v_t and integrating on $(0,1)$, we obtain

$$\begin{aligned} &\int_0^1 v_t^2(t,x)dx - \frac{1}{2}\frac{d}{dt}\int_0^1 x^\alpha v_x^2(t,x)dx \\ &\le \frac{\|d\|_\infty^2}{2}\int_0^1 v^2(t,x)dx + \frac{1}{2}\int_0^1 v_t^2(t,x)dx \quad \forall t\in[0,T]\,. \end{aligned} \tag{4.1}$$

By Hardy's inequality,

$$\int_0^1 v^2(t,x)dx \le \int_0^1 x^{\alpha-2}v^2(t,x)dx \le C\int_0^1 x^\alpha v_x^2(t,x)dx\,. \tag{4.2}$$

Then, combining (4.1) and (4.2), we get

$$0 \le \frac{d}{dt}\Big(e^{Ct}\int_0^1 x^\alpha v_x^2(t,x)dx\Big) \quad \forall t\in[0,t]\,.$$

The above estimate implies that, for all $0\le t\le T/2$,

$$\frac{T}{4}\int_0^1 x^\alpha v_x^2(t,x)dx \le C\int_{T/2}^{3T/4}\int_0^1 x^\alpha v_x^2(\tau,x)dxd\tau\,.$$

The latter inequality, combined with Hardy's inequality and (3.5), yields

$$\begin{aligned} \int_0^1 v^2(t,x)dx &\le C(T)\int_{T/2}^{3T/4}\int_0^1 x^\alpha v_x^2(\tau,x)dxd\tau \\ &\le C\iint_Q s\theta x^\alpha v_x^2(\tau,x)e^{2s\Phi}dxd\tau \\ &\le C\int_0^T\int_\omega z^2(\tau,x)dxd\tau \end{aligned} \tag{4.3}$$

for all $0\le t\le T/2$. Now, multiplying (1.7) by z_t we get

$$\begin{aligned} &\int_0^1 z_t^2(t,x)dx - \frac{d}{dt}\int_0^1 x^\alpha z_x^2(t,x)dx \\ &\le 2\|c\|_\infty^2\int_0^1 z^2(t,x)dx + 2\int_0^1 v^2(t,x)dx \quad \forall t\in[0,T]\,. \end{aligned} \tag{4.4}$$

Combining the latter with (4.3) and Hardy's inequality, we obtain

$$\begin{aligned} &\int_0^1 z_t^2(t,x)dx - \frac{d}{dt}\int_0^1 x^\alpha z_x^2(t,x)dx \\ &\le C\int_0^1 x^\alpha z_x^2(t,x)dx + C\int_0^T\int_\omega z^2(t,x)\,dx\,dt \quad \forall t\in[0,T/2]\,. \end{aligned}$$

Hence,

$$-\frac{d}{dt}\Big(e^{Ct}\int_0^1 x^\alpha z_x^2(t,x)dx\Big) \le Ce^{Ct}\int_0^T\int_\omega z^2(t,x)\,dx\,dt \quad \forall t\in[0,T/2]\,.$$

Thus, for every $0 \le s \le t \le T/2$,

$$\int_0^1 x^\alpha z_x^2(s,x)dx \le C\int_0^1 x^\alpha z_x^2(t,x)dx + C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

So, integrating in t over $[T/4, T/2]$ we get, for every $s \le T/4$,

$$\begin{aligned}\frac{T}{4}\int_0^1 x^\alpha z_x^2(s,x)dx &\le C\int_{T/4}^{T/2}\!\!\int_0^1 x^\alpha z_x^2(t,x)\,dx\,dt + C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\\ &\le C\iint_Q s\theta x^\alpha z_x^2(t,x)e^{2s\Phi}\,dx\,dt + C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\\ &\le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.\end{aligned}$$

By Hardy's inequality we conclude that, for every $s \le T/4$,

$$\begin{aligned}\int_0^1 z^2(s,x)dx &\le \int_0^1 x^{\alpha-2}z^2(s,x)dx\\ &\le C\int_0^1 x^\alpha z_x^2(s,x)dx\\ &\le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.\end{aligned}\tag{4.5}$$

Combining this result with (4.3), for $s = 0 = t$, we obtain

$$\int_0^1 (v^2(x,0)+z^2(x,0))dx \le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.\tag{4.6}$$

On the other hand, (4.5) and Carleman's inequality also yield

$$\int_0^{T/4}\!\!\int_0^1 x^\alpha z_x^2(t,x)\,dx\,dt + \iint_Q \theta x^\alpha z_x^2(t,x)e^{2s\Phi}\,dx\,dt \le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

Therefore, by Hardy's inequality and the definition of Φ, we conclude that there exists $M > 0$ such that

$$\iint_Q e^{-M/(T-t)^4}z^2(t,x)\,dx\,dt \le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

The above estimate, together with (4.6), implies (1.11).

We now briefly describe how to prove (1.12). Proceeding as in the proof of (1.11) it is not difficult to see that for all $3T/4 \le s \le T$ we have that

$$\frac{T}{4}\int_0^1 x^\alpha p_x^2(s,x)dx \le C\int_{T/2}^{3T/4}\!\!\int_0^1 x^\alpha p_x^2(\tau,x)dxd\tau\,.$$

Then, for all $s \in [3T/4, T]$,

$$\int_0^1 z_t^2(s,x)dx - \frac{d}{dt}\int_0^1 x^\alpha z_x^2(s,x)dx \le C\int_0^1 x^\alpha z_x^2(s,x)dx + C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

Following the steps of the above proof, since $z(T,\cdot) = 0$ we easily get that

$$\int_{\frac{3T}{4}}^T\!\!\int_0^1 x^{\alpha-2}z^2(t,x)\,dx\,dt \le C\int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

Combining this result with the Carleman inequality for cascade systems we obtain, for M large enough,

$$\iint_Q e^{-M/t^4} z^2(t,x)\,dx\,dt \le C_T \int_0^T\!\!\int_\omega z^2(t,x)\,dx\,dt\,.$$

The proof is thus complete. □

Proof of Theorem 1.1. The fact that Proposition 1.4 implies Theorem 1.1 can be proved in several ways. The most direct argument is the following.

Let $H = L^2(\Omega) \times L^2(\Omega) \times L^2(e_M(T-t))$, and let M and L be the following linear mappings:

$$\begin{gathered} L : L^2(Q) \to L^2(0,1) \times L^2(0,1) \\ h \mapsto (y(T), u(T)) \end{gathered}$$

where $(y(\cdot), u(\cdot))$ is the solution corresponding to (1.1)-(1.2) with $(y^0, u^0, \xi) = (0,0,0)$, and

$$\begin{gathered} M : H \to L^2(0,1) \times L^2(0,1) \\ (y^0, u^0, \xi) \mapsto (y(T), u(T)) \end{gathered}$$

where $(y(\cdot), u(\cdot))$ now solves (1.1)-(1.2) with $h = 0$. Then Theorem 1.1 is equivalent to the inclusion

$$R(M) \subset R(L). \tag{4.7}$$

Both M and L are $L^2(0,1) \times L^2(0,1)$-valued, bounded linear operators. Consequently (4.7) holds if and only, for every $(z^0, v^0) \in L^2(0,1) \times L^2(0,1)$,

$$\|M^*(z^0, v^0)\|_H \le C\|L^*(z^0, v^0)\|_{L^2(Q)} \tag{4.8}$$

for some constant $C > 0$. Now, a simple computation shows that

$$M^*(z^0, v^0) = (z(x,0), v(x,0), z(t,x)), \quad L^*(z^0, v^0) = z1_\omega$$

where z and v solve the adjoint system (1.8)-(1.7). Hence (4.8) is just (1.11) and Theorem 1.1 is proved. □

Remark 4.1.

- The results of this paper can be generalized to systems with more general (degenerate) coefficients than $a(x) = x^\alpha$ (see for example [1] and [8]).
- The null controllability problem when $\mathcal{O} \cap \omega = \emptyset$ is open even in the non-degenerate case. Approximate controllability results for the linear case ($c(t,x) = d(t,x) = 0$) can be found in [18].
- Another interesting problem is to dispense with the condition $0 \notin \overline{\mathcal{O}}$. However, it is not difficult to see that the controllability results of this paper are valid for any open $\mathcal{O}$ such that $\mathcal{O} \cap \omega \neq \emptyset$ when the coupling term $y\mathbb{I}_{\mathcal{O}}$ in (1.2) and (1.4) is replaced by $x^{\beta/2} y \mathbb{I}_{\mathcal{O}}$ with $\beta > 2 - \alpha$. Observe that the fact that $0 \notin \overline{\mathcal{O}}$ is used only in (3.8). Under the conditions given for β, such an estimate reduces to

$$\int_0^T\!\!\int_{\mathcal{O}} e^{2s\Phi} x^\beta p^2\,dx\,dt \le C \iint_Q s^3\theta^3 x^{2-\alpha} p^2 e^{2s\Phi}\,dx\,dt.$$

The rest of the proof of the Carleman inequality remains the same. The energy estimates are easily checked just noting that the term

$$\int_0^1 x^\beta v^2 \, dx \, dt \,,$$

that now replaces $\int_0^1 v^2 \, dx \, dt$ in (4.4), can be easily bounded as follows

$$\int_0^1 x^\beta v^2 \, dx \, dt \le C \int_0^1 x^{\alpha-2} v^2 dx \le C \int_0^1 x^\alpha v_x^2 dx.$$

5. Appendix

In this appendix we give a sketch of the proof of Lemma 2.11 (Caccioppoli's inequality). Let us set $\omega = (a,b)$ and $\omega' = (a',b')$ with $a < a' < b' < b$. We can suppose, without loss of generality, that $a \neq 0$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\eta_x^2/\eta \in L^\infty(\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on (a',b'), and $\eta \equiv 0$ on $[0,a) \cup (b,1]$. Then, in view of (2.5),

$$\begin{aligned}
0 &= \int_0^T \frac{d}{dt} \int_0^1 \eta \, v^2 e^{2s\Phi} \, dx \, dt \\
&= 2 \iint_Q \eta \, v v_t e^{2s\Phi} \, dx \, dt + 2s \iint_Q \Phi_t \eta v^2 e^{2s\Phi} \, dx \, dt \\
&= 2 \iint_Q \left(\eta \, x^\alpha v_x^2 + \eta_x x^\alpha v_x v + 2s \Phi_x \eta \, x^\alpha v_x v \right) e^{2s\Phi} \, dx \, dt \\
&\quad + 2 \iint_Q F \, \eta \, v e^{2s\Phi} \, dx \, dt + 2s \iint_Q \Phi_t \eta \, v^2 e^{2s\Phi} \, dx \, dt \,.
\end{aligned}$$

Now, observe that, for every $\varepsilon > 0$,

$$\iint_Q \eta_x x^\alpha v_x v e^{2s\Phi} \, dx \, dt \le \frac{\varepsilon}{2} \iint_Q \eta \, x^\alpha v_x^2 e^{2s\Phi} \, dx \, dt + \frac{1}{2\varepsilon} \iint_Q \frac{\eta_x^2}{\eta} \, x^\alpha v^2 e^{2s\Phi} \, dx \, dt \,,$$

and

$$\iint_Q \Phi_x \eta \, x^\alpha v_x v e^{2s\Phi} \, dx \, dt \le \frac{\varepsilon}{2} \iint_Q \eta \, x^\alpha v_x^2 e^{2s\Phi} \, dx \, dt + \frac{1}{2\varepsilon} \iint_Q \Phi_x^2 \eta \, x^\alpha v^2 e^{2s\Phi} \, dx \, dt \,.$$

Proceeding in the same way with the other terms, and choosing ε small enough, we obtain that

$$\iint_Q \eta \, x^\alpha v_x^2 e^{2s\Phi} \, dx \, dt \le C \Big(\iint_Q \lambda_\eta v^2 e^{2s\Phi} \, dx \, dt + \iint_Q \eta \, F^2 \, dx \, dt \Big),$$

where λ_η is a bounded function with support in $\omega = (a,b)$, defined in terms of η. Since $a \neq 0$ and $a' \neq 0$, Caccioppoli's inequality follows.

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6. Addendum and corrigendum posted on October 10, 2016

For the main results proved in this paper we gave a Carleman inequality (2.4) that is incorrect. The aim of this addendum is to give a correct one, very similar to the one presented in the paper. The controllability results Theorems 1.1 and 1.2 remain valid and the proof we are giving is useful in other contexts, see, e.g.,

[25]. Another correct version of (2.4) has been obtained in [24]. However, in their case, the authors use different weights on each side of the inequality and this is not of use in some particular situations as in [25]. We thank the authors of [24] that informed us of our mistake. Also we thank J. Carmelo Flores, from Universidad Autónoma de la Ciudad de México, for collaborating in the proof of the correct Carleman inequality.

In fact, the proof of inequality (2.4) is incorrect because of the choice of the weight function Φ. In this corrigendum, we use a slightly different weight function and correct the proof of the Carleman inequality.

6.1. **Carleman inequalities.** For $\omega=(a,b)$ let us call $\kappa=\frac{2a+b}{3}$, $\kappa^+=\frac{a+2b}{3}$, and let $\xi\in C^2(\mathbb{R})$ be such that $0\le\xi\le 1$ and

$$\xi(x)=\begin{cases}1 & \text{if } x\in(0,\kappa)\\ 0 & \text{if } x\in(\kappa^+,1).\end{cases}$$

Let us define

$$\phi(x)=\begin{cases}\frac{2-x^{2-\alpha}}{(2-\alpha)^2}, & 0\le\alpha<2,\ \alpha\ne 1,\ \forall x\in[0,1]\\ (c_1-e^x), & \alpha=1,\ \forall x\in[0,1]\end{cases}$$

where c_1 is such that $\phi(x)>0$ for every $x\in[0,1]$. Now, take

$$\rho(x)=\frac{1-x^{1-\alpha/2}}{1-\alpha/2},\quad x\in[0,1]$$

and define $\psi(x)=e^{2\|\rho\|_\infty}-e^{\rho(x)}$ and

$$\eta(x)=\phi(x)\xi(x)+(1-\xi(x))\psi(x).$$

Observe that

$$\eta'(x)=\phi'(x)\xi(x)+\phi(x)\xi'(x)-\xi'(x)\psi(x)+(1-\xi(x))\psi'(x).$$

We also define $\beta(x,t)=\eta(x)\theta(t)$ where

$$\theta(t)=\frac{1}{(t(T-t))^4}\quad\forall t\in(0,T).$$

The main result in the corrigendum, that substitutes Theorem 2.4, is:

Theorem 6.1. *Let $0\le\alpha<2$ and $T>0$ be given. Then there exists two positive constants C,s_0 such that for all $s\ge s_0$ and for every solution $v\in\mathcal{U}=\mathcal{C}^0([0,T];L^2(0,1))\cap L^2(0,T;H^1_a(0,1))$ to (2.5),*

$$\begin{aligned}&\iint_Q(s\theta x^\alpha v_x^2+s^3\theta^3x^{2-\alpha}v^2)e^{-2s\beta}\,dx\,dt\\&\le C\Big(\iint_Q e^{-2s\beta}F^2\,dx\,dt+\int_0^T\int_\omega e^{-2s\beta}v^2\,dx\,dt\Big)\end{aligned}\tag{6.1}$$

Proof. We define $w(t,x)=e^{-s\beta(t,x)}v(t,x)$. Then, w solves

$$\big(e^{s\beta}w\big)_t+\big(x^\alpha(e^{s\beta}w)_x\big)_x=F.$$

Clearly we obtain

$$s\beta_tw+w_t+s(x^\alpha\beta_x)_xw+s^2x^\alpha\beta_x^2w+2sx^\alpha\beta_xw_x+(x^\alpha w_x)_x=e^{-2s\beta}F.$$

As in [8] we define

$$P_s^+ = s\beta_t w + s^2 x^\alpha \beta_x^2 w + (x^\alpha w_x)_x,$$
$$P_s^- = w_t + s(x^\alpha \beta_x)_x w + 2sx^\alpha \beta_x w_x.$$

We want to estimate the L^2-scalar product (P_s^+, P_s^-). We define

$$Q_1 = \int_0^T\int_0^1 w_t(s\beta_t w + s^2 x^\alpha \beta_x^2 w + (x^\alpha w_x)_x),$$
$$Q_2 = \int_0^T\int_0^1 s\beta_t w(s(x^\alpha \beta_x)_x w + 2sx^\alpha \beta_x w_x),$$
$$Q_3 = s^3 \int_0^T\int_0^1 x^\alpha \beta_x^2 (x^\alpha \beta_x)_x w^2 + 2x^{2\alpha}\beta_x^3 w w_x,$$
$$Q_4 = s\int_0^T\int_0^1 (x^\alpha w_x)_x[(x^\alpha \beta_x)_x w + 2x^\alpha \beta_x w_x].$$

Integrating by parts and using the boundary conditions, we obtain

$$Q_1 = -\frac{1}{2}\int_0^T\int_0^1 w^2(s\theta_{tt}\eta + 2s^2 x^\alpha \theta\theta_t \eta_x^2).$$

To bound Q_2, we follow the computation of the boundary terms in [8] at $x = 0$ using the fact that $\beta_x = \phi'(x)\theta(t)$ near $x = 0$ and $w(1) = 0$ to obtain

$$\int_0^T \beta_t \beta_x x^\alpha w^2\Big|_0^1 = 0.$$

So

$$Q_2 = -s^2\int_0^T\int_0^1 \theta\theta_t \eta_x^2 x^\alpha w^2.$$

Similarly, we obtain

$$Q_3 = -s^3\int_0^T\int_0^1 \theta^3 x^{2\alpha-1}(2x\eta_{xx} + \alpha\eta_x)\eta_x^2 w^2.$$

Finally

$$\begin{aligned} Q_4 = &-s\int_0^T\int_0^1 \theta x^{2\alpha-1}(2x\eta_{xx} + \alpha\eta_x)w_x^2 \\ &- s\int_0^T\int_0^1 \theta x^\alpha (x^\alpha \eta_x)_{xx} w_x w + s\int_0^T \theta(t)\eta_x(x)x^{2\alpha}w_x^2(t,x)\Big|_0^1 dt. \end{aligned}$$

Observe that $\int_0^T \theta(t)\eta_x(x)x^{2\alpha}w_x^2(t,x)\big|_{x=0}dt = 0$. On the other hand, at $x = 1$,

$$\eta_x(1) = \psi'(1) = 1.$$

Alltogether we have

$$\begin{aligned} (P_s^+, P_s^-) \geq &-\frac{1}{2}\int_0^T\int_0^1 (s\theta_{tt}\eta + 4s^2 x^\alpha \theta\theta_t \eta_x^2)w^2 \\ &- s^3\int_0^T\int_0^1 \theta^3 x^{2\alpha-1}(2x\eta_{xx} + \alpha\eta_x)\eta_x^2 w^2 \\ &- s\int_0^T\int_0^1 \theta x^{2\alpha-1}(2x\eta_{xx} + \alpha\eta_x)w_x^2 \end{aligned}$$

$$\begin{aligned}&- s\int_0^T\int_0^1 \theta x^\alpha (x^\alpha\eta_x)_{xx} w_x w\\&=: I_1+I_2+I_3+I_4.\end{aligned}$$

Now, observe that our choice of weight functions yields

$$\begin{aligned}&2x\eta_{xx}(x)+\alpha\eta_x(x)\\&= -x^{1-\alpha}\xi(x)+(1-\xi(x))(2x\psi''(x)+\alpha\psi'(x))+f(x)\chi_{[\kappa,\kappa^+]}(x)\end{aligned}$$

where f is bounded and $\chi_{[\kappa,\kappa^+]}$ denotes the characteristic function of $[\kappa,\kappa^+]$. Moreover, $-(2x\psi''(x)+\alpha\psi'(x)) = 2x^{1-\alpha}e^{\rho(x)}$. Therefore,

$$-s\int_0^T\int_0^1 \theta x^{2\alpha-1}(2x\eta_{xx}+\alpha\eta_x)w_x^2 \ge s\int_0^T\int_0^1 \theta x^\alpha w_x^2 - Cs\int_0^T\int_\kappa^{\kappa^+}\theta w_x^2$$

for some constant $C\ge 0$. Similarly, we have

$$-s^3\int_0^T\int_0^1 \theta^3 x^{2\alpha-1}(2x\eta_{xx}+\alpha\eta_x)\eta_x^2 w^2 \ge s^3\int_0^T\int_0^1 \theta^3 x^\alpha\eta_x^2 w^2 - Cs^3\int_0^T\int_\kappa^{\kappa^+}\theta^3 w^2.$$

Following, on $[0,\kappa]$, the same reasoning used in the derivation of [8, (3.10)] and observing that

$$\Big|\int_0^T\int_{\kappa^+}^1 \theta_{tt}\eta w^2\Big| \le C\int_0^T\int_{\kappa^+}^1 \theta^{3/2}\eta w^2 \le C'\int_0^T\int_0^1 \theta^3 x^\alpha \eta_x^2 w^2\,,$$

for s large enough one can dominate the terms in I_1 by

$$s\int_0^T\int_0^1 \theta x^\alpha w_x^2 \quad\text{and}\quad s^3\int_0^T\int_0^1 \theta^3 x^\alpha\eta_x^2 w^2$$

plus integrals of w^2 and w_x^2 over $[\kappa,\kappa^+]$. Moreover, since $(x^\alpha\eta_x)_{xx}\equiv 0$ on $[0,\kappa]$, the same is true for I_4. Finally, we conclude that

$$\begin{aligned}&\iint_Q (s\theta x^\alpha w_x^2 + s^3\theta^3 x^{2-\alpha}w^2)\,dx\,dt\\&\le C\Big(\iint_Q F^2\,dx\,dt + \int_0^T\int_{\widetilde\omega}(w^2+w_x^2)\,dx\,dt\Big)\end{aligned}\tag{6.2}$$

where we have used the fact that $x^\alpha\eta_x^2\sim x^{2-\alpha}$ on $[0,\kappa]$ and is bounded below by a constant times $x^{2-\alpha}$ on the rest of $[0,1]$. Next, recalling that $w(t,x) = e^{-s\beta(t,x)}v(t,x)$, we immediately recover the zero order bound

$$\begin{aligned}&\iint_Q s^3\theta^3 x^{2-\alpha}v^2 e^{-2s\beta}\,dx\,dt\\&\le C\Big(\iint_Q e^{-2s\beta}F^2\,dx\,dt + \int_0^T\int_{\widetilde\omega} e^{-2s\beta}v^2 + \int_0^T\int_{\widetilde\omega} e^{-2s\beta}v_x^2\,dx\,dt\Big)\end{aligned}$$

Moreover, since $w_x(t,x) = e^{-s\beta(t,x)}v_x(t,x) - s\beta_x(t,x)v(t,x)$, again recalling that $x^\alpha\eta_x^2\sim x^{2-\alpha}$ on $[0,\kappa]$ and appealing to (6.2) we deduce

$$\begin{aligned}&\iint_Q (s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha}v^2)e^{-2s\beta}\,dx\,dt\\&\le C\Big(\iint_Q e^{-2s\beta}F^2\,dx\,dt + \int_0^T\int_{\widetilde\omega} e^{-2s\beta}v^2 + \int_0^T\int_{\widetilde\omega} e^{-2s\beta}v_x^2\,dx\,dt\Big).\end{aligned}$$

To obtain (6.1) we eliminate the term $\int_0^T\int_{\widetilde{\omega}} e^{-2s\beta}v_x^2\,dx\,dt$ performing local energy estimates and "growing" $\widetilde{\omega}$ to ω. That is, we use Cacciopoli's inequality, which is:

$$\int_0^T\int_{\widetilde{\omega}} e^{-2s\beta}v_x^2\,dx\,dt \le C\Big(\int_0^T\int_{\omega} e^{-2s\beta}v^2\,dx\,dt + \int_0^T\int_{\Omega} e^{-2s\beta}F^2\,dx\,dt\Big).$$

This completes the proof. □

Additional References.

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End of addendum.

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