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SIGMA-CONVERGENCE OF STATIONARY NAVIER-STOKES TYPE EQUATIONS

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Abstract. In the framework of homogenization theory, the Σ-convergence method is carried out on stationary Navier-Stokes type equations on a fixed domain. Our main tools are the two-scale convergence concept and the so-called homogenization algebras.

1. Introduction

We study the homogenization of stationary Navier-Stokes type equations in a fixed bounded open subset of the N-dimensional numerical space. Here, the usual Laplace operator involved in the classical Navier-Stokes equations is replaced by an elliptic linear differential operator of order two, in divergence form, with variable coefficients. Let us give a detailed description of our object.

Let Ω be a smooth bounded open set in $\mathbb{R}^N_x$ (the N-dimensional numerical space $\mathbb{R}^N$ of variables $x = (x_1, \dots, x_N)$), where N is a given positive integer; and let ε be a real number with $0 < \varepsilon < 1$. We consider the partial differential operator

$$P^\varepsilon = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a^\varepsilon_{ij}\frac{\partial}{\partial x_j}\Big)$$

in Ω, where $a^\varepsilon_{ij}(x) = a_{ij}(\frac{x}{\varepsilon})$ $(x \in \Omega)$, $a_{ij} \in L^\infty(\mathbb{R}^N_y; \mathbb{R})$ $(1 \le i, j \le N)$ with

$$a_{ij} = a_{ji}, \tag{1.1}$$

and the assumption that there is a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(y)\xi_j\xi_i \ge \alpha|\xi|^2 \quad \text{for all } \xi = (\xi_i) \in \mathbb{R}^N \text{ and for almost all } y \in \mathbb{R}^N, \tag{1.2}$$

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. The operator P^ε acts on scalar functions, say $\varphi \in H^1(\Omega) = W^{1,2}(\Omega)$. However, we may as well view P^ε as acting on vector functions $\mathbf{u} = (u^i) \in H^1(\Omega)^N$ in a *diagonal way*, i.e.,

$$(P^\varepsilon\mathbf{u})^i = P^\varepsilon u^i \quad (i = 1, \dots, N).$$

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Remark 1.1. For any Roman character such as i, j (with $1 \leq i, j \leq N$), u^i (resp. u^j) denotes the i-th (resp. j-th) component of a vector function $\mathbf{u}$ in $L^1_{loc}(\Omega)^N$ or in $L^1_{loc}(\mathbb{R}^N_y)^N$. On the other hand, for any real $0 < \varepsilon < 1$, we define u^ε as

$$u^\varepsilon(x) = u(\frac{x}{\varepsilon}) \quad (x \in \Omega)$$

for $u \in L^1_{loc}(\mathbb{R}^N_y)$, as is customary in homogenization theory. More generally, for $u \in L^1_{loc}(\Omega \times \mathbb{R}^N_y)$, it is customary to put

$$u^\varepsilon(x) = u(x, \frac{x}{\varepsilon}) \quad (x \in \Omega)$$

whenever the right-hand side makes sense (see, e.g., [7, 8]). There is no danger of confusion between the preceding notation.

Having made these preliminaries, let $\mathbf{f} = (f^i) \in H^{-1}(\Omega; \mathbb{R})^N$. For any fixed $0 < \varepsilon < 1$, we consider the boundary value problem

$$P^\varepsilon \mathbf{u}_\varepsilon + \sum_{j=1}^N u^j_\varepsilon \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} + \operatorname{grad} p_\varepsilon = \mathbf{f} \quad \text{in } \Omega, \tag{1.3}$$

$$\operatorname{div} \mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}_\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} = \Big(\frac{\partial u^1}{\partial x_j}, \dots, \frac{\partial u^N}{\partial x_j}\Big).$$

We will later see that if N is either 2 or 3, and if $\mathbf{f}$ is "small enough", then (1.3)-(1.5) uniquely define $(\mathbf{u}_\varepsilon, p_\varepsilon)$ with $\mathbf{u}_\varepsilon = (u^i_\varepsilon) \in H^1_0(\Omega; \mathbb{R})^N$ and $p_\varepsilon \in L^2(\Omega; \mathbb{R})/\mathbb{R}$, where

$$L^2(\Omega; \mathbb{R})/\mathbb{R} = \big\{v \in L^2(\Omega; \mathbb{R}) : \int_\Omega v dx = 0\big\}.$$

Our main goal is to investigate the limiting behavior, as $\varepsilon \to 0$, of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ under an abstract assumption on a_{ij} $(1 \leq i, j \leq N)$ covering a wide range of concrete behaviour beyond the classical periodicity hypothesis. The linear version of this problem (i.e., the homogenization of (1.3)-(1.5) without the term $\sum_{j=1}^N u^j_\varepsilon \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}$) was first studied by Bensoussan, Lions and Papanicolaou [2] under the periodicity hypothesis on the coefficients a_{ij}. These authors presented a detailed mathematical analysis of the problem by the well-known approach combining the use of asymptotic expansions with Tartar's energy method.

The present study deals with a more general situation involving two major difficulties: 1) the equations are nonlinear; 2) the homogenization problem for (1.3)-(1.5) is considered not under the periodicity hypothesis, as is classical, but in the general setting characterized by an abstract assumption on $a_{ij}(y)$ covering a wide range of behaviours with respect to y, such as the periodicity, the almost periodicity, the convergence at infinity, and others.

The motivation of the present study lies in the fact that the homogenization problem for (1.3)-(1.5) is connected with the modelling of heterogeneous fluid flows, in particular multi-phase flows, fluids with spatially varying viscosities, and others; see, e.g., [16] for more details about such heterogeneous media.

Our approach is the Σ-convergence method derived from two-scale convergence ideas [1], [11] by means of so-called homogenization algebras [9], [10].

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F)$ for the continuous mappings of X into F, and $\mathcal{B}(X;F)$ for those mappings in $\mathcal{C}(X;F)$ that are bounded. We shall assume $\mathcal{B}(X;F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X;F)$ and $L^p_{\rm loc}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\rm loc}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1 \dots dx_N$.

The rest of the study is organized as follows. In Section 2 we discuss the homogenization of (1.3)-(1.5) under the periodicity hypothesis on the coefficients a_{ij}. In Section 3 we reconsider the homogenization of problem (1.3)-(1.5) in a more general setting. The periodicity hypothesis on the coefficients a_{ij} is here replaced by an abstract assumption covering a variety of concrete behaviour including the periodicity as a particular case. A few concrete examples are worked out.

2. Periodic homogenization of stationary Navier-Stokes type equations

We assume once for all that N is either 2 or 3. We set $Y = (-\frac{1}{2}, \frac{1}{2})^N$, Y considered as a subset of $\mathbb{R}^N_y$ (the space $\mathbb{R}^N$ of variables $y = (y_1,\dots,y_N)$). Our purpose is to study the homogenization of (1.3)-(1.5) under the periodicity hypothesis on a_{ij}, i.e., under the assumption that a_{ij} is Y-periodic.

2.1. Preliminaries.

Let us first recall that a function $u \in L^1_{\rm loc}(\mathbb{R}^N_y)$ is said to be Y-periodic if for each $k \in \mathbb{Z}^N$ ($\mathbb{Z}$ denotes the integers), we have $u(y+k) = u(y)$ almost everywhere (a.e.) in $y \in \mathbb{R}^N$. If in addition u is continuous, then the preceding equality holds for every $y \in \mathbb{R}^N$, of course. The space of all Y-periodic continuous complex functions on $\mathbb{R}^N_y$ is denoted by $\mathcal{C}_{\rm per}(Y)$; that of all Y-periodic functions in $L^p_{\rm loc}(\mathbb{R}^N_y)$ ($1 \le p < \infty$) is denoted by $L^p_{\rm per}(Y)$. $\mathcal{C}_{\rm per}(Y)$ is a Banach space under the supremum norm on $\mathbb{R}^N$, whereas $L^p_{\rm per}(Y)$ is a Banach space under the norm

$$\|u\|_{L^p(Y)} = \Big(\int_Y |u(y)|^p dy\Big)^{1/p} \quad (u \in L^p_{\rm per}(Y)).$$

We need the space $H^1_\#(Y)$ of Y-periodic functions $u \in H^1_{\rm loc}(\mathbb{R}^N_y) = W^{1,2}_{\rm loc}(\mathbb{R}^N_y)$ such that $\int_Y u(y)dy = 0$. Provided with the gradient norm,

$$\|u\|_{H^1_\#(Y)} = \Big(\int_Y |\nabla_y u|^2 dy\Big)^{1/2} \quad (u \in H^1_\#(Y)),$$

where $\nabla_y u = (\frac{\partial u}{\partial y_1},\dots,\frac{\partial u}{\partial y_N})$, $H^1_\#(Y)$ is a Hilbert space.

Before we can recall the concept of Σ-convergence in the present periodic setting, let us introduce one further notation. The letter E throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting 0 as an accumulation point. For example, E may be the whole interval $(0,1)$; E may also be an ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case E will be referred to as a *fundamental sequence*. Let us observe that E may be neither $(0,1)$ nor a fundamental sequence, of course. Let Ω be a bounded open set in $\mathbb{R}^N_x$ and let $1 \leq p < \infty$.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega)$ is said to be:

(i) weakly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; L^p_{per}(Y))$ if as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\psi^\varepsilon(x)dx \to \iint_{\Omega\times Y} u_0(x,y)\psi(x,y)\,dx\,dy \tag{2.1}$$

for all $\psi \in L^{p'}(\Omega; \mathcal{C}_{per}(Y))$ ($\frac{1}{p'} = 1 - \frac{1}{p}$), where $\psi^\varepsilon(x) = \psi(x, \frac{x}{\varepsilon})$ ($x \in \Omega$);

(ii) strongly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; L^p_{per}(Y))$ if the following property is verified: Given $\eta > 0$ and $v \in L^p(\Omega; \mathcal{C}_{per}(Y))$ with $\|u_0 - v\|_{L^p(\Omega\times Y)} \leq \frac{\eta}{2}$, there is some $\alpha > 0$ such that $\|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \eta$ provided $E \ni \varepsilon \leq \alpha$.

We will briefly express weak and strong Σ-convergence by writing $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ and $u_\varepsilon \to u_0$ in $L^p(\Omega)$-strong Σ, respectively.

Remark 2.2. It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ, then (2.1) holds for $\psi \in \mathcal{C}(\overline{\Omega}; L^\infty_{per}(Y))$. See [8, Proposition 10] for the proof.

In the present context the concept of Σ-convergence coincides with the well-known one of two-scale convergence. Consequently, instead of repeating here the main results underlying Σ-convergence theory for periodic structures, we find it more convenient to draw the reader's attention to a few references regarding two-scale convergence, e.g., [1], [6], [8] and [17].

However, we recall below two fundamental results which constitute the corner stone of the two-scale convergence theory.

Theorem 2.3. *Assume that $1 < p < \infty$ and further E is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon\in E}$ be bounded in $L^p(\Omega)$. Then, a subsequence E' can be extracted from E such that $(u_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges in $L^p(\Omega)$.*

Theorem 2.4. *Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $H^1(\Omega) = W^{1,2}(\Omega)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$,*

$$\begin{gathered} u_\varepsilon \to u_0 \quad \text{in } H^1(\Omega)\text{-weak}, \\ u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega)\text{-weak } \Sigma, \\ \frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad \text{in } L^2(\Omega)\text{-weak } \Sigma \quad (1 \leq j \leq N), \end{gathered}$$

where $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\#(Y))$.

The proofs of the above theorems can be found in, e.g., [1, 6, 8] .

Now, it is not apparent that the boundary value problem (1.3)- (1.5) has a solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$, and that the latter is unique. With a view to elucidating this, we introduce, for fixed $0 < \varepsilon < 1$, the bilinear form a^ε on $H^1_0(\Omega;\mathbb{R})^N \times H^1_0(\Omega;\mathbb{R})^N$ defined by

$$a^\varepsilon(\mathbf{u},\mathbf{v}) = \sum_{k=1}^N \sum_{i,j=1}^N \int_\Omega a^\varepsilon_{ij} \frac{\partial u^k}{\partial x_j}\frac{\partial v^k}{\partial x_i} dx$$

for $\mathbf{u} = (u^k)$ and $\mathbf{v} =(v^k)$ in $H_0^1(\Omega;\mathbb{R})^N$. According to (1.1), the form a^ε is symmetric. On the other hand, in view of (1.2),

$$a^\varepsilon(\mathbf{v},\mathbf{v}) \geq \alpha\|\mathbf{v}\|^2_{H_0^1(\Omega)^N} \tag{2.2}$$

for every $\mathbf{v} =(v^k) \in H_0^1(\Omega;\mathbb{R})^N$ and $0 < \varepsilon < 1$, where

$$\|\mathbf{v}\|_{H_0^1(\Omega)^N} = \Big(\sum_{k=1}^N \int_\Omega |\nabla v^k|^2 dx\Big)^{1/2}$$

with $\nabla v^k = (\frac{\partial v^k}{\partial x_1}, \dots, \frac{\partial v^k}{\partial x_N})$. Furthermore, it is clear that a constant $c_0 > 0$ exists such that

$$|a^\varepsilon(\mathbf{u},\mathbf{v})| \leq c_0\|\mathbf{u}\|_{H_0^1(\Omega)^N}\|\mathbf{v}\|_{H_0^1(\Omega)^N} \tag{2.3}$$

for all $\mathbf{u}$, $\mathbf{v} \in H_0^1(\Omega;\mathbb{R})^N$ and all $0 < \varepsilon < 1$.

We also need the trilinear form b on $H_0^1(\Omega;\mathbb{R})^N \times H_0^1(\Omega;\mathbb{R})^N \times H_0^1(\Omega;\mathbb{R})^N$ given by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{k=1}^N\sum_{j=1}^N \int_\Omega u^j \frac{\partial v^k}{\partial x_j} w^k dx$$

for $\mathbf{u} =(u^k)$, $\mathbf{v} =(v^k)$ and $\mathbf{w} =(w^k)$ in $H_0^1(\Omega;\mathbb{R})^N$. The trilinear form b has some nice properties. Let

$$V = \big\{\mathbf{u} \in H_0^1(\Omega;\mathbb{R})^N : \operatorname{div}\mathbf{u} = 0\big\}.$$

Then

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \text{for } \mathbf{u} \in V,\ \mathbf{v} \in H_0^1(\Omega;\mathbb{R})^N, \tag{2.4}$$

and further there exists a constant $c(N) > 0$ such that

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c(N)\|\mathbf{u}\|_{H_0^1(\Omega)^N}\|\mathbf{v}\|_{H_0^1(\Omega)^N}\|\mathbf{w}\|_{H_0^1(\Omega)^N} \tag{2.5}$$

for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H_0^1(\Omega;\mathbb{R})^N$ (see [5, 15] for the proofs of these classical results).

We are now in a position to verify the following result.

Proposition 2.5. *Suppose* $\mathbf{f}$ *(the right-hand side of* (1.3)*) is "small enough" so that*

$$c(N)\|\mathbf{f}\|_{H^{-1}(\Omega)^N} < \alpha^2, \tag{2.6}$$

where α *(resp.* $c(N)$*) is that constant in* (1.2) *(resp.* (2.5)*). Then, the boundary value problem* (1.3)-(1.5) *determines a unique pair* $(\mathbf{u}_\varepsilon, p_\varepsilon)$ *with* $\mathbf{u}_\varepsilon \in H_0^1(\Omega;\mathbb{R})^N$, $p_\varepsilon \in L^2(\Omega;\mathbb{R})/\mathbb{R}$.

Proof. For fixed $0 < \varepsilon < 1$, consider the variational problem

$$\begin{gathered}\mathbf{u}_\varepsilon \in V : \\ a^\varepsilon(\mathbf{u}_\varepsilon,\mathbf{v}) + b(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} = (v^k) \in V\end{gathered} \tag{2.7}$$

with

$$(\mathbf{f},\mathbf{v}) = \sum_{k=1}^N (f^k, v^k),$$

where $(,)$ denotes the duality pairing between $H^{-1}(\Omega;\mathbb{R})$ and $H_0^1(\Omega;\mathbb{R})$ as well as between $H^{-1}(\Omega;\mathbb{R})^N$ and $H_0^1(\Omega;\mathbb{R})^N$. Thanks to (2.2)-(2.5), this variational problem admits at least one solution, as is easily seen by following [5, p.99] or [15, p.164]. Let us check that (2.7) has at most one solution. To begin, observe that any $\mathbf{u}^*$ satisfying (2.7) (i.e., with $\mathbf{u}^*$ in place of $\mathbf{u}_\varepsilon$) verifies

$$\|\mathbf{u}^*\|_{H_0^1(\Omega)^N} \leq \frac{1}{\alpha}\|\mathbf{f}\|_{H^{-1}(\Omega)^N}, \tag{2.8}$$

as is straightforward by (2.2). Now, suppose $\mathbf{u}^*$ and $\mathbf{u}^{**}$ are two solutions of (2.7). Then, letting $\mathbf{u} = \mathbf{u}^* - \mathbf{u}^{**}$, we have in an obvious manner

$$a^\varepsilon(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}^*, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}^*, \mathbf{v}) - b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = 0$$

and that for any $\mathbf{v} \in V$. By choosing in particular $\mathbf{v} = \mathbf{u}$ and recalling (2.4), it follows by (2.2),

$$\alpha\|\mathbf{u}\|^2_{H_0^1(\Omega)^N} + b(\mathbf{u}, \mathbf{u}^*, \mathbf{u}) \leq 0.$$

Hence, in view of (2.5),

$$\alpha\|\mathbf{u}\|^2_{H_0^1(\Omega)^N} \leq c(N)\|\mathbf{u}\|^2_{H_0^1(\Omega)^N}\|\mathbf{u}^*\|_{H_0^1(\Omega)^N}.$$

By (2.8) this gives

$$\Big(\alpha - \frac{c(N)}{\alpha}\|\mathbf{f}\|_{H^{-1}(\Omega)^N}\Big)\|\mathbf{u}\|^2_{H_0^1(\Omega)^N} \leq 0.$$

Hence $\mathbf{u} = 0$, by virtue of (2.6). This shows the unicity in (2.7), and so (2.7) determines a unique vector function $\mathbf{u}_\varepsilon$. Now, by taking in (2.7) the particular test functions $\mathbf{v} \in \mathcal{V}$ with

$$\mathcal{V} = \big\{\varphi \in \mathcal{D}(\Omega; \mathbb{R})^N : \operatorname{div}\varphi = 0\big\}$$

and using a classical argument (see, e.g., [15, p.14]), we get a distribution $p_\varepsilon \in \mathcal{D}'(\Omega)$ such that (1.3) holds (in the distribution sense on Ω), with in addition (1.4)-(1.5), of course. Let us show that p_ε lies in $L^2(\Omega; \mathbb{R})$. First of all, since $N = 2$ or 3, we have $H_0^1(\Omega; \mathbb{R}) \subset L^4(\Omega; \mathbb{R})$ (see, e.g., [15, pp.291, 296]). Thus, $\mathbf{u}_\varepsilon \in L^4(\Omega; \mathbb{R})^N$. Consequently, $u_\varepsilon^i u_\varepsilon^j \in L^2(\Omega; \mathbb{R})$ $(1 \leq i, j \leq N)$. Observing that

$$\sum_{j=1}^N u_\varepsilon^j \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} = \sum_{j=1}^N \frac{\partial}{\partial x_j}(u_\varepsilon^j \mathbf{u}_\varepsilon) \quad \text{(use (1.4))},$$

it follows that $\sum_{j=1}^N u_\varepsilon^j \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \in H^{-1}(\Omega; \mathbb{R})^N$. By (1.3), we deduce that $\operatorname{grad} p_\varepsilon \in H^{-1}(\Omega; \mathbb{R})^N$. Therefore, thanks to a well-known result (see, e.g., [15, p.14, Proposition 1.2]), the distribution p_ε is actually a function in $L^2(\Omega; \mathbb{R})$, and further the said function is unique up to an additive constant; in other words, p_ε is unique in $L^2(\Omega; \mathbb{R})/\mathbb{R}$. Conversely, it is an easy exercise to verify that if $(\mathbf{u}_\varepsilon, p_\varepsilon)$ lies in $H^1(\Omega; \mathbb{R})^N \times L^2(\Omega; \mathbb{R})$ and is a solution of (1.3)- (1.5), then $\mathbf{u}_\varepsilon$ satisfies (2.7). This completes the proof. □

2.2. **A global homogenization theorem.** Before we can establish a so-called global homogenization theorem for (1.3)-(1.5), we require a few basic notation and results. To begin, let

$$\mathcal{V}_Y = \big\{\psi \in \mathcal{C}^\infty_{\mathrm{per}}(Y; \mathbb{R})^N : \int_Y \psi(y)dy = 0,\ div_y\psi = 0\big\},$$
$$V_Y = \{\mathbf{w} \in H^1_\#(Y; \mathbb{R})^N : div_y \mathbf{w} = 0\},$$

where: $\mathcal{C}^\infty_{\mathrm{per}}(Y; \mathbb{R}) = \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{R}) \cap \mathcal{C}_{\mathrm{per}}(Y)$, div_y denotes the divergence operator in $\mathbb{R}^N_y$. We provide V_Y with the $H^1_\#(Y)^N$-norm, which makes it a Hilbert space. There is no difficulty in verifying that $\mathcal{V}_Y$ is dense in V_Y (proceed as in [13, Proposition 3.2]). With this in mind, set

$$\mathbb{F}_0^1 = V \times L^2(\Omega; V_Y).$$

This is a Hilbert space with norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \Big(\|\mathbf{v}_0\|^2_{H_0^1(\Omega)^N} + \|\mathbf{v}_1\|^2_{L^2(\Omega;V_Y)} \Big)^{1/2}, \quad \mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{F}_0^1.$$

On the other hand, put

$$\mathcal{F}_0^\infty = \mathcal{V} \times [\mathcal{D}(\Omega;\mathbb{R}) \otimes \mathcal{V}_Y],$$

where $\mathcal{D}(\Omega;\mathbb{R}) \otimes \mathcal{V}_Y$ stands for the space of vector functions ψ on $\Omega \times \mathbb{R}_y^N$ of the form

$$\psi(x,y) = \sum \varphi_i(x)\mathbf{w}_i(y) \quad (x \in \Omega, \quad y \in \mathbb{R}^N)$$

with a summation of finitely many terms, $\varphi_i \in \mathcal{D}(\Omega;\mathbb{R})$, $\mathbf{w}_i \in \mathcal{V}_Y$. It is clear that $\mathcal{F}_0^\infty$ is dense in $\mathbb{F}_0^1$ (see [15, p.18]). Now, let

$$\widehat{a}_\Omega(\mathbf{u},\mathbf{v}) = \sum_{i,j,k=1}^{N} \iint_{\Omega\times Y} a_{ij} \Big(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j} \Big) \Big(\frac{\partial v_0^k}{\partial x_i} + \frac{\partial v_1^k}{\partial y_i} \Big)\, dx\, dy$$

for $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ and $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1)$ in $\mathbb{F}_0^1$. This defines a symmetric continuous bilinear form $\widehat{a}_\Omega$ on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$. Furthermore, $\widehat{a}_\Omega$ is $\mathbb{F}_0^1$-elliptic. Specifically,

$$\widehat{a}_\Omega(\mathbf{u},\mathbf{u}) \ge \alpha \|\mathbf{u}\|^2_{\mathbb{F}_0^1} \quad (\mathbf{u} \in \mathbb{F}_0^1)$$

as is easily checked using (1.2) and the fact that $\int_Y \frac{\partial u_1^k}{\partial y_j}(x,y)dy = 0$.

In the sequel we put

$$b_\Omega(\mathbf{u},\mathbf{v},\mathbf{w}) = b(\mathbf{u}_0,\mathbf{v}_0,\mathbf{w}_0),$$
$$L(\mathbf{v}) = (\mathbf{f}, \mathbf{v}_0)$$

for $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$, $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1)$ and $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1)$ in $\mathbb{F}_0^1$, which defines a continuous trilinear form on $\mathbb{F}_0^1 \times \mathbb{F}_0^1 \times \mathbb{F}_0^1$ and a continuous linear form on $\mathbb{F}_0^1$, respectively, with further $b_\Omega(\mathbf{u},\ \mathbf{v},\mathbf{v}) = 0$ for $\mathbf{u}$, $\mathbf{v} \in \mathbb{F}_0^1$.

Here is one fundamental lemma.

Lemma 2.6. *Suppose* (2.6) *holds. Then the variational problem*

$$\begin{gathered} \mathbf{u} \in \mathbb{F}_0^1 : \\ \widehat{a}_\Omega(\mathbf{u},\mathbf{v}) + b_\Omega(\mathbf{u},\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbb{F}_0^1 \end{gathered} \tag{2.9}$$

has at most one solution.

The proof of the above lemma follows by the same line of argument as in the proof of Proposition 2.5; so we omit it. We are now able to prove the desired theorem. Throughout the remainder of the present section, it is assumed that a_{ij} is Y-periodic for any $1 \le i,j \le N$.

Theorem 2.7. *Suppose* (2.6) *holds. For each real* $0 < \varepsilon < 1$, *let* $\mathbf{u}_\varepsilon = (u_\varepsilon^k) \in H_0^1(\Omega;\mathbb{R})^N$ *be defined by* (1.3)-(1.5) *(or equivalently by* (2.7)*). Then, as* $\varepsilon \to 0$,

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \quad \text{in } H_0^1(\Omega)^N\text{-weak}, \tag{2.10}$$

$$\frac{\partial u_\varepsilon^k}{\partial x_j} \to \frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j} \quad \text{in } L^2(\Omega)\text{-weak } \Sigma \ (1 \le j,k \le N), \tag{2.11}$$

where $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ *is the (unique) solution of* (2.9).

Proof. Let $0<\varepsilon<1$. It is clear that

$$a^{\varepsilon}(\mathbf{u}_\varepsilon,\mathbf{v})+b(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v})-\int_\Omega p_\varepsilon \operatorname{div}\mathbf{v}dx=(\mathbf{f},\mathbf{v}) \tag{2.12}$$

for all $\mathbf{v}=(v^k)\in H_0^1(\Omega;\mathbb{R})^N$. Taking in particular $\mathbf{v}=\mathbf{u}_\varepsilon$ and using (2.2) and (2.4), it follows immediately that the sequence $(\mathbf{u}_\varepsilon)_{0<\varepsilon<1}$ is bounded in $H_0^1(\Omega;\mathbb{R})^N$. On the other hand, starting from (2.12) and recalling (2.3) and (2.5), we see that

$$|(\operatorname{grad}p_\varepsilon,\mathbf{v})|\leq(\|\mathbf{f}\|_{H^{-1}(\Omega)^N}+c(N)\|\mathbf{u}_\varepsilon\|^2_{H_0^1(\Omega)^N}+c_0\|\mathbf{u}_\varepsilon\|_{H_0^1(\Omega)^N})\|\mathbf{v}\|_{H_0^1(\Omega)^N}$$

for all $\mathbf{v}\in H_0^1(\Omega;\mathbb{R})^N$. In view of the preceding result, it follows that the sequence $(\operatorname{grad}p_\varepsilon)_{0<\varepsilon<1}$ is bounded in $H^{-1}(\Omega;\mathbb{R})^N$. Thanks to a classical argument [15, p.15], we deduce that the sequence $(p_\varepsilon)_{0<\varepsilon<1}$ is bounded in $L^2(\Omega;\mathbb{R})$. Thus, given any arbitrary fundamental sequence E, appeal to Theorems 2.3-2.4 yields a subsequence E' from E and functions $\mathbf{u}_0=(u_0^k)\in H_0^1(\Omega;\mathbb{R})^N$, $\mathbf{u}_1=(u_1^k)\in L^2(\Omega;H^1_\#(Y;\mathbb{R})^N)$, $p\in L^2(\Omega;L^2_{\mathrm{per}}(Y;\mathbb{R}))$ such that as $E'\ni\varepsilon\to0$, we have (2.10)-(2.11) and

$$p_\varepsilon\to p\quad\text{in } L^2(\Omega)\text{-weak }\Sigma. \tag{2.13}$$

Let us note at once that, according to (1.4), we have $\operatorname{div}\mathbf{u}_0=0$ and $\operatorname{div}_y\mathbf{u}_1=0$. Therefore $\mathbf{u}=(\mathbf{u}_0,\mathbf{u}_1)\in\mathbb{F}_0^1$. Now, for each real $0<\varepsilon<1$, let

$$\mathbf{\Phi}_\varepsilon=\psi_0+\varepsilon\psi_1^\varepsilon\quad\text{with } \psi_0\in\mathcal{D}(\Omega;\mathbb{R})^N,\ \psi_1\in\mathcal{D}(\Omega;\mathbb{R})\otimes\mathcal{V}_Y, \tag{2.14}$$

i.e., $\mathbf{\Phi}_\varepsilon(x)=\psi_0(x)+\varepsilon\psi_1(x,\frac{x}{\varepsilon})$ for $x\in\Omega$. We have $\mathbf{\Phi}_\varepsilon\in\mathcal{D}(\Omega;\mathbb{R})^N$. Thus, in view of (2.12),

$$a^{\varepsilon}(\mathbf{u}_\varepsilon,\mathbf{\Phi}_\varepsilon)+b(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{\Phi}_\varepsilon)-\int_\Omega p_\varepsilon\operatorname{div}\mathbf{\Phi}_\varepsilon dx=(\mathbf{f},\mathbf{\Phi}_\varepsilon). \tag{2.15}$$

The next point is to pass to the limit in (2.15) as $E'\ni\varepsilon\to0$. To this end, we note that as $E'\ni\varepsilon\to0$,

$$a^{\varepsilon}(\mathbf{u}_\varepsilon,\mathbf{\Phi}_\varepsilon)\to\widehat{a}_\Omega(\mathbf{u},\mathbf{\Phi}),$$

where $\mathbf{\Phi}=(\psi_0,\psi_1)$ (proceed as in the proof of the analogous result in [12, p.179]). On the other hand, thanks to the Rellich theorem, we have from (2.10) that $\mathbf{u}_\varepsilon\to\mathbf{u}_0$ in $L^2(\Omega)^N$. Combining this with (2.11), it follows by [9, Proposition 4.7] (see also [8, Proposition 8]) that as $E'\ni\varepsilon\to0$,

$$b(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{\Phi}_\varepsilon)\to b_\Omega(\mathbf{u},\mathbf{u},\mathbf{\Phi}),$$

where $\mathbf{u}$ and $\mathbf{\Phi}$ are defined above. Now, based on (2.13), there is no difficulty in showing that as $E'\ni\varepsilon\to0$,

$$\int_\Omega p_\varepsilon\operatorname{div}\mathbf{\Phi}_\varepsilon dx\to\iint_{\Omega\times Y}p\operatorname{div}\psi_0\,dx\,dy.$$

Finally, it is an easy exercise to check that $\mathbf{\Phi}_\varepsilon\to\psi_0$ in $H_0^1(\Omega)^N$-weak as $\varepsilon\to0$ (this is a classical result).

Having made this point, we can pass to the limit in (2.15) when $E'\ni\varepsilon\to0$, and the result is that

$$\widehat{a}_\Omega(\mathbf{u},\mathbf{\Phi})+b_\Omega(\mathbf{u},\mathbf{u},\mathbf{\Phi})-\int_\Omega p_0\operatorname{div}\psi_0 dx=(\mathbf{f},\psi_0), \tag{2.16}$$

where p_0 denotes the mean of p, i.e., $p_0\in L^2(\Omega;\mathbb{R})$ and $p_0(x)=\int_Y p(x,y)dy$ a.e. in $x\in\Omega$; and where $\mathbf{\Phi}=(\psi_0,\psi_1)$, ψ_0 ranging over $\mathcal{D}(\Omega;\mathbb{R})^N$ and ψ_1 ranging over $\mathcal{D}(\Omega;\mathbb{R})\otimes\mathcal{V}_Y$. Taking in particular ψ_0 in $\mathcal{V}$ and using the density of $\mathcal{F}_0^\infty$ in $\mathbb{F}_0^1$, one quickly arrives at (2.9). The unicity of $\mathbf{u}=(\mathbf{u}_0,\mathbf{u}_1)$ follows by Lemma 2.6.

Consequently, (2.10) and (2.11) still hold when $E \ni \varepsilon \to 0$ (instead of $E' \ni \varepsilon \to 0$), hence when $0 < \varepsilon \to 0$, by virtue of the arbitrariness of E. The theorem is proved. □

For further needs, we wish to give a simple representation of the vector function $\mathbf{u}_1$ in Theorem 2.7 (or Lemma 2.6). For this purpose we introduce the bilinear form $\widehat{a}$ on $V_Y \times V_Y$ defined by

$$\widehat{a}(\mathbf{v}, \mathbf{w}) = \sum_{i,j,k=1}^{N} \int_Y a_{ij} \frac{\partial v^k}{\partial y_j} \frac{\partial w^k}{\partial y_i} dy$$

for $\mathbf{v} = (v^k)$ and $\mathbf{w} = (w^k)$ in V_Y. Next, for each pair of indices $1 \leq i, k \leq N$, we consider the variational problem

$$\begin{gathered} \chi_{ik} \in V_Y : \\ \widehat{a}(\chi_{ik}, \mathbf{w}) = \sum_{l=1}^{N} \int_Y a_{li} \frac{\partial w^k}{\partial y_l} dy \quad \text{for all } \mathbf{w} = (w^j) \text{ in } V_Y, \end{gathered} \tag{2.17}$$

which determines χ_{ik} in a unique manner.

Lemma 2.8. *Under the hypothesis and notation of Theorem 2.7, we have*

$$\mathbf{u}_1(x, y) = -\sum_{i,k=1}^{N} \frac{\partial u_0^k}{\partial x_i}(x) \chi_{ik}(y) \tag{2.18}$$

almost everywhere in $(x, y) \in \Omega \times \mathbb{R}^N$.

Proof. In (2.9), choose the test functions $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1)$ such that $\mathbf{v}_0 = 0$, $\mathbf{v}_1(x, y) = \varphi(x)\mathbf{w}(y)$ for $(x, y) \in \Omega \times \mathbb{R}^N$, where $\varphi \in \mathcal{D}(\Omega; \mathbb{R})$ and $\mathbf{w} \in V_Y$. Then, almost everywhere in $x \in \Omega$, we have

$$\widehat{a}(\mathbf{u}_1(x, .), \mathbf{w}) = -\sum_{l,j,k=1}^{N} \frac{\partial u_0^k}{\partial x_j}(x) \int_Y a_{lj} \frac{\partial w^k}{\partial y_l} dy \quad \forall \mathbf{w} = (w^k) \in V_Y. \tag{2.19}$$

But it is clear that $\mathbf{u}_1(x, .)$ (for fixed $x \in \Omega$) is the sole function in V_Y solving the variational equation (2.19). On the other hand, it is an easy matter to check that the function of y on the right of (2.18) solves the same variational equation. Hence the lemma follows immediately. □

2.3. **Macroscopic homogenized equations.** Our goal here is to derive a well-posed boundary value problem for $(\mathbf{u}_0, p_0)$. To begin, for $1 \leq i, j, k, h \leq N$, let

$$q_{ijkh} = \delta_{kh} \int_Y a_{ij}(y) dy - \sum_{l=1}^{N} \int_Y a_{il}(y) \frac{\partial \chi_{jh}^k}{\partial y_l}(y) dy,$$

where: δ_{kh} is the Kronecker symbol, $\chi_{jh} = (\chi_{jh}^k)$ is defined exactly as in (2.17). To the coefficients q_{ijkh} we attach the differential operator $\mathcal{Q}$ on Ω mapping $\mathcal{D}'(\Omega)^N$ into $\mathcal{D}'(\Omega)^N$ ($\mathcal{D}'(\Omega)$ is the usual space of complex distributions on Ω) as

$$(\mathcal{Q}\mathbf{z})^k = -\sum_{i,j,h=1}^{N} q_{ijkh} \frac{\partial^2 z^h}{\partial x_i \partial x_j} \quad (1 \leq k \leq N) \quad \text{for } \mathbf{z} = (z^h), z^h \in \mathcal{D}'(\Omega). \tag{2.20}$$

$\mathcal{Q}$ is the so-called homogenized operator associated to P^ε $(0 < \varepsilon < 1)$.

We consider now the boundary value problem

$$\mathcal{Q}\mathbf{u}_0 + \sum_{j=1}^{N} u_0^j \frac{\partial \mathbf{u}_0}{\partial x_j} + \operatorname{grad} p_0 = \mathbf{f} \quad \text{in } \Omega, \tag{2.21}$$

$$\operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega, \tag{2.22}$$

$$\mathbf{u}_0 = 0 \quad \text{on } \partial\Omega. \tag{2.23}$$

Lemma 2.9. *Suppose* (2.6) *holds. Then, the boundary value problem* (2.21)-(2.23) *admits at most one weak solution* $(\mathbf{u}_0, p_0)$ *with* $\mathbf{u}_0 \in H_0^1(\Omega;\mathbb{R})^N$*,* $p_0 \in L^2(\Omega;\mathbb{R})/\mathbb{R}$*.*

Proof. It can be proved without the slightest difficulty that if a pair $(\mathbf{u}_0, p_0) \in H_0^1(\Omega;\mathbb{R})^N \times L^2(\Omega;\mathbb{R})$ verifies (2.21)-(2.23), then the vector function $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ [with $\mathbf{u}_1$ given by (2.18)] satisfies (2.9) (use (2.16)). Hence the unicity in (2.21)-(2.23) follows by Lemma 2.6. □

This leads us to the following theorem.

Theorem 2.10. *Suppose* (2.6) *holds. For each real* $0 < \varepsilon < 1$*, let* $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H_0^1(\Omega;\mathbb{R})^N \times (L^2(\Omega;\mathbb{R})/\mathbb{R})$ *be defined by* (1.3)-(1.5)*. Then, as* $\varepsilon \to 0$*, we have* $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ *in* $H_0^1(\Omega)^N$*-weak and* $p_\varepsilon \to p_0$ *in* $L^2(\Omega)$*-weak, where the pair* $(\mathbf{u}_0, p_0)$ *lies in* $H_0^1(\Omega;\mathbb{R})^N \times (L^2(\Omega;\mathbb{R})/\mathbb{R})$ *and is the unique weak solution of* (2.21)-(2.23)*.*

Proof. A quick review of the proof of Theorem 2.7 reveals that from any given fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (2.10)-(2.11) and $p_\varepsilon \to p_0$ in $L^2(\Omega)$-weak (use (2.13) if necessary), and further (2.16) holds for all $\mathbf{\Phi} = (\psi_0, \psi_1) \in \mathcal{D}(\Omega;\mathbb{R})^N \times [\mathcal{D}(\Omega;\mathbb{R}) \otimes \mathcal{V}_Y]$, where $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}_0^1$. Now, substituting (2.18) in (2.16) and then choosing therein the $\mathbf{\Phi}$'s such that $\psi_1 = 0$, a simple computation leads to (2.21) with evidently (2.22)-(2.23). Hence the theorem follows by Lemma 2.8 and use of an obvious argument. □

Remark 2.11. The operator $\mathcal{Q}$ is elliptic, i.e., there is some $\alpha_0 > 0$ such that

$$\sum_{i,j,k,h=1}^{N} q_{ijkh}\xi_{ik}\xi_{jh} \geq \alpha_0 \sum_{k,h=1}^{N} |\xi_{kh}|^2$$

for all $\xi = (\xi_{kh})$, $\xi_{kh} \in \mathbb{R}$. Indeed, by following a classical line of argument (see, e.g., [2]), we can give a suitable expression of q_{ijkh}, viz.

$$q_{ijkh} = \widehat{a}(\chi_{ik} - \pi_{ik},\ \chi_{jh} - \pi_{jh}),$$

where, for each pair of indices $1 \leq i,k \leq N$, the vector function $\pi_{ik} = (\pi_{ik}^1, \dots, \pi_{ik}^N) : \mathbb{R}_y^N \to \mathbb{R}$ is given by $\pi_{ik}^r(y) = y_i\delta_{kr}$ $(r = 1, \dots, N)$ for $y = (y_1, \dots, y_N) \in \mathbb{R}^N$. Hence, the above ellipticity property follows in a classical fashion (see [2]).

3. General deterministic homogenization of stationary Navier-Stokes type equations

Our purpose here is to extend the results of Section 2 to a more general setting beyond the periodic framework. The basic notation and hypotheses (except the periodicity assumption) stated before are still valid. In particular N is either 2 or 3, and Ω denotes a bounded open set in $\mathbb{R}_x^N$.

3.1. **Preliminaries and statement of the homogenization problem.** We recall that $\mathcal{B}(\mathbb{R}^N_y)$ denotes the space of bounded continuous complex functions on $\mathbb{R}^N_y$. It is well known that $\mathcal{B}(\mathbb{R}^N_y)$ with the supremum norm and the usual algebra operations is a commutative $\mathcal{C}^*$-algebra with identity (the involution is here the usual one of complex conjugation).

Throughout the present Section 3, A denotes a separable closed subalgebra of the Banach algebra $\mathcal{B}(\mathbb{R}^N_y)$. Furthermore, we assume that A contains the constants, A is stable under complex conjugation (i.e., the complex conjugate, $\overline{u}$, of any $u \in A$ still lies in A), and finally, A has the following property: For any $u \in A$, we have $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N_x)$-weak $*$ as $\varepsilon \to 0$ ($\varepsilon > 0$), where:

$$u^\varepsilon(x) = u(\frac{x}{\varepsilon}) \quad (x \in \mathbb{R}^N),$$

the mapping $u \to M(u)$ of A into $\mathbb{C}$, denoted by M, being a positive continuous linear form on A with $M(1) = 1$ (see [9]).

A is called an H-algebra (H stands for *homogenization*). It is clear that A is a commutative $\mathcal{C}^*$-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. For the benefit of the reader it is worth recalling that $\Delta(A)$ is the set of all nonzero multiplicative linear forms on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle , \rangle$ denotes the duality pairing between A' (the topological dual of A) and A. The appropriate topology on $\Delta(A)$ is the relative weak $*$ topology on A'. So topologized, $\Delta(A)$ is a metrizable compact space, and the Gelfand transformation is an isometric isomorphism of the $\mathcal{C}^*$ -algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$. See, e.g., [4] for further details concerning the Banach algebras theory. The appropriate measure on $\Delta(A)$ is the so-called M-measure, namely the positive Radon measure β (of total mass 1) on $\Delta(A)$ such that $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ for $u \in A$ (see [9, Proposition 2.1]).

The partial derivative of index i ($1 \le i \le N$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ D_{y_i} \circ \mathcal{G}^{-1}$ (usual composition) of

$$\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$$

into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^N_y) : \psi, \quad D_{y_i}\psi \in A \quad (1 \le i \le N)\}$, $D_{y_i} = \frac{\partial}{\partial y_i}$. Higher order derivatives can be defined analogously (see [9]). Now, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}^N_y)$ such that

$$D^\alpha_y \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \dots \partial y_N^{\alpha_N}} \in A$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, and let

$$\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}.$$

Endowed with a suitable locally convex topology (see for example [9]), A^∞ (respectively $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

By a distribution on $\Delta(A)$ is understood any continuous linear form on $\mathcal{D}(\Delta(A))$. The space of all distributions on $\Delta(A)$ is then the dual, $\mathcal{D}'(\Delta(A))$, of $\mathcal{D}(\Delta(A))$. We endow $\mathcal{D}'(\Delta(A))$ with the strong dual topology. In the sequel it is assumed that A^∞ is dense in A (this is always verified in practice), which amounts to assuming that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$. Then $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ $(1 \leq p \leq \infty)$ with continuous embedding (see [9] for more details). Hence we may define

$$H^1(\Delta(A)) = \{u \in L^2(\Delta(A)) : \partial_i u \in L^2(\Delta(A)) \quad (1 \leq i \leq N)\},$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative is defined in the classical case). This is a Hilbert space with norm

$$\|u\|_{H^1(\Delta(A))} = \Big(\|u\|^2_{L^2(\Delta(A))} + \sum_{i=1}^{N} \|\partial_i u\|^2_{L^2(\Delta(A))}\Big)^{1/2} \quad (u \in H^1(\Delta(A))).$$

However, in practice the appropriate space is not $H^1(\Delta(A))$ but its closed subspace

$$H^1(\Delta(A))/\mathbb{C} = \big\{u \in H^1(\Delta(A)) : \int_{\Delta(A)} u(s)d\beta(s) = 0\big\}$$

equipped with the seminorm

$$\|u\|_{H^1(\Delta(A))/\mathbb{C}} = \Big(\sum_{i=1}^{N} \|\partial_i u\|^2_{L^2(\Delta(A))}\Big)^{1/2} \quad (u \in H^1(\Delta(A))/\mathbb{C}).$$

Unfortunately, the pre-Hilbert space $H^1(\Delta(A))/\mathbb{C}$ is in general nonseparated and noncomplete. We introduce the separated completion, $H^1_\#(\Delta(A))$, of $H^1(\Delta(A))/\mathbb{C}$, and the canonical mapping J of $H^1(\Delta(A))/\mathbb{C}$ into its separated completion. See [9] (and in particular Remark 2.4 and Proposition 2.6 there) for more details.

We will now recall the notion of Σ-convergence in the present context. Let $1 \leq p < \infty$, and let E be as in Section 2.

Definition 3.1. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ is said to be:
(i) weakly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A)) = L^p(\Omega; L^p(\Delta(A)))$ if as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\psi^\varepsilon(x)dx \to \iint_{\Omega\times\Delta(A)} u_0(x,s)\widehat{\psi}(x,s)dxd\beta(s)$$

for all $\psi \in L^{p'}(\Omega; A)$ $(\frac{1}{p'} = 1 - \frac{1}{p})$, where ψ^ε is as in Definition 2.1 , and where $\widehat{\psi}(x,.) = \mathcal{G}(\psi(x,.))$ a.e. in $x \in \Omega$;
(ii) strongly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if the following property is verified: Given $\eta > 0$ and $v \in L^p(\Omega; A)$ with $\|u_0 - \widehat{v}\|_{L^p(\Omega\times\Delta(A))} \leq \frac{\eta}{2}$, there is some $\alpha > 0$ such that

$$\|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \eta \quad \text{provided } E \ni \varepsilon \leq \alpha.$$

Remark 3.2. The existence of such v's as in (ii) results from the density of $L^p(\Omega; \mathcal{C}(\Delta(A)))$ in $L^p(\Omega; L^p(\Delta(A)))$.

We will use the same notation as in Section 2 to briefly express weak and strong Σ-convergence.

Theorem 2.3 (together with its proof) carries over to the present setting. Instead of Theorem 2.4, we have here the following notion.

Definition 3.3. The H-algebra A is said to be H^1- *proper* (or simply proper when there is no risk of confusion) if the following conditions are fulfilled.
(PR1) $\mathcal{D}(\Delta(A))$ is dense in $H^1(\Delta(A))$.

(PR2) Given a fundamental sequence E, and a sequence $(u_\varepsilon)_{\varepsilon\in E}$ which is bounded in $H^1(\Omega)$, one can extract a subsequence E' from E such that as $E'\ni\varepsilon\to 0$, we have $u_\varepsilon\to u_0$ in $H^1(\Omega)$-weak and $\frac{\partial u_\varepsilon}{\partial x_j}\to\frac{\partial u_0}{\partial x_j}+\partial_j u_1$ in $L^2(\Omega)$-weak Σ $(1\le j\le N)$, where $u_0\in H^1(\Omega)$, $u_1\in L^p(\Omega;H^1_\#(\Delta(A)))$.

The H-algebra $A=\mathcal{C}_{\rm per}(Y)$ (see Section 2) is H^1-proper. Other examples of H^1-proper H-algebras can be found in [9] and [10].

Having made the above preliminaries, let us turn now to the statement of a general deterministic homogenization problem for (1.3)-(1.5). For this purpose, let Ξ^2 be the space of functions $u\in L^2_{\rm loc}(\mathbb{R}^N_y)$ such that

$$\|u\|_{\Xi^2}=\sup_{0<\varepsilon\le 1}\Big(\int_{B_N}|u(\frac{x}{\varepsilon})|^2dx\Big)^{1/2}<\infty,$$

where B_N denotes the open unit ball in $\mathbb{R}^N_y$. Ξ^2 is a complex vector space, and the mapping $u\to\|u\|_{\Xi^2}$, denoted by $\|.\|_{\Xi^2}$, is a norm on Ξ^2 which makes it a Banach space (this is a simple exercise left to the reader). We define $\mathfrak{X}^2$ to be the closure of A in Ξ^2. We provide $\mathfrak{X}^2$ with the Ξ^2-norm, which makes it a Banach space.

Our main goal in the present section is to discuss the homogenization of (1.3)-(1.5) under the assumption

$$a_{ij}\in\mathfrak{X}^2\quad(1\le i,j\le N).\tag{3.1}$$

As is pointed out in [9], [10] and [12], assumption (3.1) covers a great variety of concrete behaviors. In particular, (3.1) generalizes the usual periodicity hypothesis (see Section 2). Indeed, for $A=\mathcal{C}_{\rm per}(Y)$, we have $\mathfrak{X}^2=L^2_{\rm per}(Y)$ (use Lemma 1 of [8]).

The approach we follow here is analogous to that which was presented in Section 2. Throughout the rest of the section, it is assumed that (3.1) is satisfied, and A is H^1-proper.

3.2. **A global homogenization theorem.** We need a few preliminaries. To begin, we set

$$\mathcal{G}(\psi)=(\mathcal{G}(\psi^i))_{1\le i\le N}$$

for any $\psi=(\psi^i)$ with $\psi^i\in A$ $(1\le i\le N)$. We have $\mathcal{G}(\psi)\in\mathcal{C}(\Delta(A))^N$, and the transformation $\psi\to\mathcal{G}(\psi)$ of A^N into $\mathcal{C}(\Delta(A))^N$ maps in particular $(A^\infty_{\mathbb{R}})^N$ isomorphically onto $\mathcal{D}(\Delta(A);\mathbb{R})^N$, where we denote

$$A^\infty_{\mathbb{R}}=A^\infty\cap\mathcal{C}(\mathbb{R}^N;\mathbb{R}).$$

Likewise, letting $\mathbf{J}(\mathbf{u})=(J(u^i))_{1\le i\le N}$ for $\mathbf{u}=(u^i)$ with $u^i\in H^1(\Delta(A))/\mathbb{C}$ $(1\le i\le N)$, we have $\mathbf{J}(\mathbf{u})\in H^1_\#(\Delta(A))^N$ and the transformation $\mathbf{u}\to\mathbf{J}(\mathbf{u})$ of $[H^1(\Delta(A))/\mathbb{C}]^N$ into $H^1_\#(\Delta(A))^N$ maps in particular $[H^1(\Delta(A);\mathbb{R})/\mathbb{C}]^N$ isometrically into $H^1_\#(\Delta(A);\mathbb{R})^N$, where we denote

$$H^1_\#(\Delta(A);\mathbb{R})=\{u\in H^1_\#(\Delta(A)):\partial_i u\in L^2(\Delta(A);\mathbb{R})\quad(1\le i\le N)\}.$$

We will set

$$\mathbb{E}^1_0=H^1_0(\Omega;\mathbb{R})^N\times L^2(\Omega;H^1_\#(\Delta(A);\mathbb{R})^N),$$

$$\mathcal{E}^\infty_0=\mathcal{D}(\Omega;\mathbb{R})^N\times\Big(\mathcal{D}(\Omega;\mathbb{R})\otimes\mathbf{J}[\mathcal{D}(\Delta(A);\mathbb{R})/\mathbb{C}]^N\Big),$$

where $\mathcal{D}(\Delta(A);\mathbb{R})/\mathbb{C} = \mathcal{D}(\Delta(A);\mathbb{R}) \cap [H^1(\Delta(A))/\mathbb{C}]$. $\mathbb{E}_0^1$ is topologized in an obvious way and $\mathcal{E}_0^\infty$ is considered without topology. It is clear that $\mathcal{E}_0^\infty$ is dense in $\mathbb{E}_0^1$.

At the present time, let

$$\widehat{a}_\Omega(\mathbf{u},\mathbf{v}) = \sum_{i,j,k=1}^N \iint_{\Omega\times\Delta(A)} \widehat{a}_{ij}\Big(\frac{\partial u_0^k}{\partial x_j} + \partial_j u_1^k\Big)\Big(\frac{\partial v_0^k}{\partial x_i} + \partial_i v_1^k\Big) dx\, d\beta$$

for $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ and $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1)$ in $\mathbb{E}_0^1$ with, of course, $\mathbf{u}_0 = (u_0^k)$, $\mathbf{u}_1 = (u_1^k)$ (and analogous expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$), where $\widehat{a}_{ij} = \mathcal{G}(a_{ij})$. This gives a bilinear form $\widehat{a}_\Omega$ on $\mathbb{E}_0^1 \times \mathbb{E}_0^1$, which is symmetric, continuous, and coercive (see [9]). We also define b_Ω and L as in Subsection 2.2 but with $\mathbb{E}_0^1$ in place of $\mathbb{F}_0^1$.

Now, let

$$V_A = \{\mathbf{u} = (u^i) \in H^1_\#(\Delta(A);\mathbb{R})^N : \widehat{\operatorname{div}}\mathbf{u} = 0\},$$

where

$$\widehat{\operatorname{div}}\mathbf{u} = \sum_{i=1}^N \partial_i u^i.$$

Equipped with the $H^1_\#(\Delta(A))^N$-norm, V_A is a Hilbert space. We next put

$$\mathbb{F}_0^1 = V \times L^2(\Omega; V_A)$$

provided with an obvious norm. It is an easy exercise to check that Lemma 2.6 together with its proof can be carried over mutatis mutandis to the present setting. This leads us to the analogue of Theorem 2.7.

Theorem 3.4. *Suppose* (3.1) *holds and further A is H^1-proper. On the other hand, let* (2.6) *be satisfied. For each real $0 < \varepsilon < 1$, let $\mathbf{u}_\varepsilon = (u_\varepsilon^k) \in H_0^1(\Omega;\mathbb{R})^N$ be defined by* (1.3)-(1.5) *(or equivalently by* (2.7)*). Then, as $\varepsilon \to 0$,*

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \quad \text{in } H_0^1(\Omega)^N\text{-weak}, \tag{3.2}$$

$$\frac{\partial u_\varepsilon^k}{\partial x_j} \to \frac{\partial u_0^k}{\partial x_j} + \partial_j u_1^k \quad \text{in } L^2(\Omega)\text{-weak } \Sigma\ (1 \le j,k \le N), \tag{3.3}$$

where $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ (with $\mathbf{u}_0 = (u_0^k)$ and $\mathbf{u}_1 = (u_1^k)$) is the unique solution of (2.9).

Proof. This is an adaptation of the proof of Theorem 2.7 and we will not go too deeply into details. Starting from (2.12), we see that the generalized sequences $(\mathbf{u}_\varepsilon)_{0<\varepsilon<1}$ and $(p_\varepsilon)_{0<\varepsilon<1}$ are bounded in $H_0^1(\Omega;\mathbb{R})^N$ and $L^2(\Omega;\mathbb{R})/\mathbb{R}$, respectively. Hence, from any given fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (2.13), (3.2) and (3.3), where p lies in $L^2(\Omega; L^2(\Delta(A);\mathbb{R}))$ and $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ lies in $\mathbb{F}_0^1$.

Now, for each real $0 < \varepsilon < 1$, let

$$\mathbf{\Phi}_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon \quad \text{with } \psi_0 \in \mathcal{D}(\Omega;\mathbb{R})^N, \psi_1 \in \mathcal{D}(\Omega;\mathbb{R}) \otimes (A_\mathbb{R}^\infty/\mathbb{C})^N \tag{3.4}$$

and

$$\mathbf{\Phi} = \big(\psi_0, \mathbf{J}(\widehat{\psi}_1)\big),$$

where: $A_\mathbb{R}^\infty/\mathbb{C} = \{\psi \in A_\mathbb{R}^\infty : M(\psi) = 0\}$, $\widehat{\psi}_1$ stands for the function $x \to \mathcal{G}(\psi_1(x,.))$ of Ω into $[\mathcal{D}(\Delta(A);\mathbb{R})/\mathbb{C}]^N$ (ψ_1 being viewed as a function say in $\mathcal{C}(\Omega; A^N)$), $\mathbf{J}(\widehat{\psi}_1)$ stands for the function $x \to \mathbf{J}(\widehat{\psi}_1(x,.))$ of Ω into $H^1_\#(\Delta(A);\mathbb{R})^N$. It is clear that

$\mathbf{\Phi} \in \mathcal{E}_0^\infty$. With this in mind, we can pass to the limit in (2.15) (with $\mathbf{\Phi}_\varepsilon$ given by (3.4)) as $E' \ni \varepsilon \to 0$, and we obtain

$$\widehat{a}_\Omega(\mathbf{u}, \mathbf{\Phi}) + b_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{\Phi}) - \iint_{\Omega\times\Delta(A)} p(\operatorname{div}\psi_0 + \widehat{\operatorname{div}}\widehat{\psi}_1) dx d\beta = (\mathbf{f}, \psi_0).$$

Therefore, thanks to the density of $\mathcal{E}_0^\infty$ in $\mathbb{E}_0^1$,

$$\widehat{a}_\Omega(\mathbf{u}, \mathbf{v}) + b_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \iint_{\Omega\times\Delta(A)} p(\operatorname{div}\mathbf{v}_0 + \widehat{\operatorname{div}}\mathbf{v}_1) dx d\beta = (\mathbf{f}, \mathbf{v}_0), \tag{3.5}$$

and that for all $\mathbf{v} =(\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{E}_0^1$. Taking in particular $\mathbf{v} \in \mathbb{F}_0^1$ leads us immediately to (2.9). Hence the theorem follows by the same argument as used in the proof of Theorem 2.7. □

As pointed out in Section 2, it is of interest to give a suitable representation of $\mathbf{u}_1$ (in Theorem 3.4). To this end, let

$$\widehat{a}(\mathbf{v}, \mathbf{w}) = \sum_{i,j,k=1}^N \int_{\Delta(A)} \widehat{a}_{ij} \partial_j v^k \partial_i w^k d\beta$$

for $\mathbf{v} =(v^k)$ and $\mathbf{w} =(w^k)$ in $H^1_\#(\Delta(A);\mathbb{R})^N$. This defines a bilinear form $\widehat{a}$ on $H^1_\#(\Delta(A);\mathbb{R})^N \times H^1_\#(\Delta(A);\mathbb{R})^N$, which is symmetric, continuous and coercive. For each pair of indices $1 \le i, k \le N$, we consider the variational problem

$$\begin{gathered} \chi_{ik} \in V_A : \\ \widehat{a}(\chi_{ik}, \mathbf{w}) = \sum_{l=1}^N \int_{\Delta(A)} \widehat{a}_{li} \partial_l w^k d\beta \quad \text{for all } \mathbf{w} =(w^j) \in V_A, \end{gathered} \tag{3.6}$$

which uniquely determines χ_{ik}.

Lemma 3.5. *Under the assumptions and notation of Theorem 3.4, we have*

$$\mathbf{u}_1(x, s) = -\sum_{i,k=1}^N \frac{\partial u_0^k}{\partial x_i}(x)\chi_{ik}(s) \tag{3.7}$$

almost everywhere in $(x, s) \in \Omega \times \Delta(A)$.

Proof. This is a simple adaptation of the proof of Lemma 2.8; the verification is left to the reader. □

3.3. **Macroscopic homogenized equations.** The aim here is to derive from (3.5) a well-posed boundary value problem for the pair $(\mathbf{u}_0, p_0)$, where $\mathbf{u}_0$ is the weak limit in (3.2) and p_0 is the mean of p (in (3.5)), i.e., $p_0(x) = \int_{\Delta(A)} p(x, s)d\beta(s)$ for $x \in \Omega$. We will proceed exactly as in Subsection 2.3.

First, for $1 \le i, j, k, h \le N$, let

$$q_{ijkh} = \delta_{kh} \int_{\Delta(A)} \widehat{a}_{ij}(s)d\beta(s) - \sum_{l=1}^N \int_{\Delta(A)} \widehat{a}_{il}(s)\partial_l \chi^k_{jh}(s)d\beta(s),$$

where $\chi_{jh} = (\chi^k_{jh})$ is defined as in (3.6). To these coefficients we associate the differential operator $\mathcal{Q}$ on Ω given by (2.20). Finally, we consider the boundary value problem (2.21)-(2.23).

Lemma 3.6. *Under the hypotheses of Theorem 3.4, the boundary value problem* (2.21)-(2.23) *admits at most one weak solution* $(\mathbf{u}_0, p_0)$ *with* $\mathbf{u}_0 \in H_0^1(\Omega;\mathbb{R})^N$, $p_0 \in L^2(\Omega;\mathbb{R})/\mathbb{R}$.

Proof. It is an easy exercise to show that if a pair $(\mathbf{u}_0, p_0) \in H_0^1(\Omega;\mathbb{R})^N \times L^2(\Omega;\mathbb{R})$ is a solution of (2.21)-(2.23), then the pair $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)$ (in which $\mathbf{u}_1$ is given by (3.7)) satisfies (2.9) and is therefore unique. Hence Lemma 3.6 follows at once. $\square$

We are now in a position to state and prove the next theorem.

Theorem 3.7. *Let the hypotheses of Theorem 3.4 be satisfied. For each real* $0 < \varepsilon < 1$, *let* $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H_0^1(\Omega;\mathbb{R})^N \times [L^2(\Omega;\mathbb{R})/\mathbb{R}]$ *be defined by* (1.3)-(1.5). *Then, as* $\varepsilon \to 0$, *we have* $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ *in* $H_0^1(\Omega)^N$*-weak and* $p_\varepsilon \to p_0$ *in* $L^2(\Omega)$*-weak, where the pair* $(\mathbf{u}_0, p_0)$ *lies in* $H_0^1(\Omega;\mathbb{R})^N \times [L^2(\Omega;\mathbb{R})/\mathbb{R}]$ *and is the unique weak solution of* (2.21)- (2.23).

Proof. As was pointed out above, from any arbitrarily given fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (3.2)-(3.3) and (2.13) hence $p_\varepsilon \to p_0$ in $L^2(\Omega)$-weak, where p_0 is the mean of p and thus $p_0 \in L^2(\Omega;\mathbb{R})/\mathbb{R}$, and where $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}_0^1$. Furthermore, (3.5) holds for all $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{E}_0^1$. Substituting (3.7) in (3.5) and then choosing therein the particular test functions $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{E}_0^1$ with $\mathbf{v}_1 = 0$ leads to Theorem 3.7, thanks to Lemma 3.6. $\square$

It is possible to present q_{ijkh} in a suitable form as in Remark 2.11. For this purpose, we introduce the space $\mathcal{M}$ of all $N \times N$ matrix functions with entries in $L^2(\Delta(A);\mathbb{R})$. Specifically, $\mathcal{M}$ denotes the space of $\mathbf{F} = (F^{ij})_{1 \le i,j \le N}$ with $F^{ij} \in L^2(\Delta(A);\mathbb{R})$. Provided with the norm

$$\|\mathbf{F}\|_{\mathcal{M}} = \Big(\sum_{i,j=1}^{N} \|F^{ij}\|^2_{L^2(\Delta(A))} \Big)^{1/2}, \quad \mathbf{F} = (F^{ij}) \in \mathcal{M},$$

$\mathcal{M}$ is a Hilbert space. Now, let

$$\mathcal{A}(\mathbf{F}, \mathbf{G}) = \sum_{i,j,k=1}^{N} \int_{\Delta(A)} \widehat{a}_{ij}(s) F^{jk}(s) G^{ik}(s) d\beta(s)$$

for $\mathbf{F} = (F^{jk})$ and $\mathbf{G} = (G^{ik})$ in $\mathcal{M}$. This gives a bilinear form $\mathcal{A}$ on $\mathcal{M} \times \mathcal{M}$, which is symmetric, continuous and coercive. Furthermore,

$$\widehat{a}(\mathbf{u}, \mathbf{v}) = \mathcal{A}\Big(\widehat{\nabla}\mathbf{u}, \widehat{\nabla}\mathbf{v}\Big), \quad \mathbf{u}, \mathbf{v} \in H^1_{\#}(\Delta(A);\mathbb{R})^N,$$

where $\widehat{\nabla}\mathbf{u} = (\partial_j u^k)$ for any $\mathbf{u} = (u^k) \in H^1_{\#}(\Delta(A);\mathbb{R})^N$. Now, by the same line of proceeding as followed in [2] (see also [8]) one can quickly show that

$$q_{ijkh} = \mathcal{A}(\widehat{\nabla}\chi_{ik} - \ \theta_{ik}, \widehat{\nabla}\chi_{jh} - \theta_{jh}),$$

where, for any pair of indices $1 \le i, k \le N$, χ_{ik} is defined by (3.6), and $\theta_{ik} = (\theta_{ik}^{lm}) \in \mathcal{M}$ with $\theta_{ik}^{lm} = \delta_{il}\delta_{km}$. Having made this point, Remark 2.11 can then be carried over to the present setting.

3.4. **Some concrete examples.** In the present subsection we consider a few examples of homogenization problems for (1.3)-(1.5) in a concrete setting (as opposed to the abstract assumption (3.1)) and we show how their study leads naturally to the abstract setting of Subsection 3.1 and so we may conclude by merely applying Theorems 3.4 and 3.7.

Example 3.8 (Almost periodic setting)**.** The aim here is to study the homogenization of (1.3)-(1.5) under the almost periodicity hypothesis

$$a_{ij} \in L^2_{AP}(\mathbb{R}^N_y) \quad (1 \leq i,j \leq N), \tag{3.8}$$

where $L^2_{AP}(\mathbb{R}^N_y)$ denotes the space of all functions $w \in L^2_{loc}(\mathbb{R}^N_y)$ that are almost periodic in the sense of Stepanoff (see, e.g., [14, Section 4]). According to [14, Proposition 4.1], the hypothesis (3.8) yields a countable subgroup $\mathcal{R}$ of $\mathbb{R}^N_y$ such that $a_{ij} \in L^2_{AP,\mathcal{R}}(\mathbb{R}^N_y)$ $(1 \leq i,j \leq N)$, where $L^2_{AP,\mathcal{R}}(\mathbb{R}^N_y) = \{u \in L^2_{AP}(\mathbb{R}^N_y) : Sp(u) \subset \mathcal{R}\}$, $Sp(u)$ being the spectrum of u, i.e., $Sp(u) = \{k \in \mathbb{R}^N : M(u\overline{\gamma}_k) \neq 0\}$ with $\gamma_k(y) = \exp(2i\pi k.y)$ $(y \in \mathbb{R}^N)$. The appropriate H-algebra is here $AP_{\mathcal{R}}(\mathbb{R}^N_y) = \{u \in AP(\mathbb{R}^N_y) : Sp(u) \subset \mathcal{R}\}$, where $AP(\mathbb{R}^N_y)$ denotes the space of almost periodic continuous complex functions on $\mathbb{R}^N_y$ (see, e.g., [3, Chapter 5] and [4, Chapter 10]). The H-algebra $A = AP_{\mathcal{R}}(\mathbb{R}^N_y)$ is H^1-proper (see [9]) and further (3.1) is satisfied, since $L^2_{AP,\mathcal{R}}(\mathbb{R}^N_y) \subset \mathfrak{X}^2$ (use [8, Lemma 1]). Hence the study of the problem under consideration reduces to the abstract analysis in Subsections 3.2 and 3.3.

Example 3.9. Let (L^2, ℓ^∞) be the space of all $u \in L^2_{loc}(\mathbb{R}^N_y)$ such that

$$\|u\|_{2,\infty} = \sup_{k \in \mathbb{Z}^N} \Big(\int_{k+Y} |u(y)|^2 dy \Big)^{1/2} < \infty,$$

where $Y = (-\frac{1}{2}, \frac{1}{2})^N$. This is a Banach space under the norm $\|\cdot\|_{2,\infty}$. We denote by $L^2_{\infty,per}(Y)$ the closure in (L^2, ℓ^∞) of the space of all finite sums

$$\sum \varphi_i u_i \quad (\varphi_i \in \mathcal{B}_\infty(\mathbb{R}^N_y), \quad u_i \in \mathcal{C}_{per}(Y)), \tag{3.9}$$

where $\mathcal{C}_{per}(Y)$ is defined in Subsection 2.1, and $\mathcal{B}_\infty(\mathbb{R}^N_y)$ is the space of all $u \in \mathcal{C}(\mathbb{R}^N_y)$ such that $\lim_{|y|\to\infty} u(y) = \xi \in \mathbb{C}$ (ξ depending on u, $|y|$ the Euclidean norm of y in $\mathbb{R}^N$). The problem to be worked out here states as in Example 3.8 except that (3.8) is replaced by

$$a_{ij} \in L^2_{\infty,per}(Y) \quad (1 \leq i,j \leq N). \tag{3.10}$$

We define A to be the closure in $\mathcal{B}(\mathbb{R}^N_y)$ of the finite sums in (3.9). This is an H^1 -proper homogenization algebra on $\mathbb{R}^N_y$ (see [9, Example 5.4]) and further (3.1) holds because the space (L^2, ℓ^∞) is continuously embedded in Ξ^2 (use [8, Lemma 1]). Therefore, we arrive at the same conclusion as above.

Example 3.10. We assume here that the coefficients a_{ij} are constant on each cell $k + Y$ ($k \in \mathbb{Z}^N$, Y as above). More precisely, we assume that there exists a family of functions $r_{ij} : \mathbb{Z}^N \to \mathbb{R}$ $(1 \leq i,j \leq N)$ such that for each $k \in \mathbb{Z}^N$, we have $a_{ij}(y) = r_{ij}(k)$ a.e. in $y \in k + Y$, and that for $1 \leq i,j \leq N$. We also assume the following behaviour: $r_{ij} \in \mathcal{B}_\infty(\mathbb{Z}^N)$ $(1 \leq i,j \leq N)$, i.e., each $r_{ij}(k)$ tends to a finite limit as $|k| \to \infty$. Under these hypotheses in place of (3.10), we consider the above homogenization problem. As is explained in detail in [10], one can find an H^1-proper homogenization algebra A on $\mathbb{R}^N_y$ such that (3.1) holds true, which leads us to the same conclusion as above.

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References

[1] G. Allaire; Homogenization and two-scale convergence, *SIAM J. Math. Anal.,* **23** (1992), 1482-1518.
[2] A. Bensoussan, J. L. Lions and G. Papanicolaou; *Asymptotic Analysis for Periodic Structures,* North-Holland, 1978.
[3] A. Guichardet; *Analyse Harmonique Commutative*, Dunod, Paris, 1968.
[4] R. Larsen; *Banach Algebras*, Dekker, New York, 1973.
[5] J. L. Lions; *Quelques méthodes de ré solution des problèmes aux limites non linéaires,* Dunod, Paris, 1969.
[6] D. Lukkassen, G. Nguetseng and P. Wall; Two-scale convergence, *Int. J. Pure and Appl. Math,* **2** (2002), 35-86.
[7] G. Nguetseng; Almost periodic homogenization: Asymptotic analysis of a second order elliptic equation (Preprint).
[8] G. Nguetseng; Sigma-convergence of parabolic differential operators (To appear).
[9] G. Nguetseng; Homogenization structures and applications I *, Z. Anal. Anw.*, **22** (2003), 73-107.
[10] G. Nguetseng; Homogenization structures and applications II*, Z. Anal. Anw.*, **23** (2004), 483-508.
[11] G. Nguetseng; A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, **20** (1989), 608-623.
[12] G. Nguetseng; Deterministic homogenization of a semilinear elliptic partial differential equation of order $2m$, *Math. Reports*, **8 (58)** (2006), 167-195.
[13] G. Nguetseng and H. Nnang; Homogenization of nonlinear monotone operators beyond the periodic setting, *Electron. J. Differ. Eqns.*, **2003** (2003), no. 36, 1-24.
[14] G. Nguetseng and J. L. Woukeng; Deterministic homogenization of parabolic monotone operators with time dependent coefficients, *Electron. J. Differ. Eqns.*, **2004** (2004), no. 82, 1-23.
[15] R. Temam; *Navier-Stokes Equations,* North-Holland, Amsterdam, 1977.
[16] P. R. Voke and M. W. Collins; Forms of the generalized Navier-Stokes equations, *J. Eng. Math.*, **18** (1984), 219-233.
[17] V. V. Zhikov; On two-scale convergence, *J. Math. Sc.*, **120** (2004), 1328-1352.

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