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EIGENCURVES FOR A STEKLOV PROBLEM

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Abstract. In this article, we study the existence of the eigencurves for a Steklov problem and we obtain their variational formulation. Also we prove the simplicity and the isolation results of each point of the principal eigencurve. Also we obtain the continuity and the differentiability of the principal eigencurve.

1. Introduction

Consider the two parameter Steklov eigenvalue problem

$$\begin{gathered} \triangle_p u = 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u + \mu |u|^{p-2}u \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz continuous boundary, $m \in L^\infty(\partial\Omega)$ is a weight function which may change sign, λ, μ, p be real numbers with $1 < p < \infty$. The weak solutions of (1.1) are defined by

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx = \lambda\int_{\partial\Omega} m(x)|u|^{p-2}u\varphi\,d\sigma + \mu\int_{\partial\Omega}|u|^{p-2}u\varphi\,d\sigma, \tag{1.2}$$

for $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. Let us note that all solutions of problem (1.1) are of class $C^{1,\alpha}(\Omega)$ since they are p-harmonic.

Problem (1.1) has been studied by several authors in the case $\mu = 0$ and $p = 2$; we cite in particular the works [4, 7, 9]. For the nonlinear case, the authors in [3] and [11] studied the case when $\mu = 0$ and $m \in L^q(\partial\Omega)$. A problem in which the eigencurve appears in the boundary condition has been considered recently in [6]. Assuming $m \in L^\infty(\partial\Omega)$ the authors show that for each $\lambda \in \mathbb{R}$, there is an increasing sequence of eigenvalues for the nonlinear boundary-value problem

$$\begin{gathered} \triangle_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u + \mu |u|^{p-2}u \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

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also they show that the first eigenvalue is simple and isolated. Furthermore they obtain some results about their variation, density, and continuous dependence on the parameter λ.

We set

$$\begin{aligned} \mu_1(\lambda) = \inf \Big\{ \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\partial\Omega} m(x)|u|^p \, d\sigma : u \in W^{1,p}(\Omega), \\ \frac{1}{p} \int_{\partial\Omega} |u|^p \, d\sigma = 1 \Big\}, \end{aligned} \tag{1.4}$$

We understand by the principal eigencurve of the Steklov problem (1.1), the graph of the map $\mu_1 : \lambda \to \mu_1(\lambda)$ from $\mathbb{R}$ to $\mathbb{R}$.

Our purpose of this paper is to study , as in [6], the existence of the eigencurves of the Steklov problem (1.1) and we obtain their variational formulation by using Ljusternik-Schnirelmann theory ([10]). Also we prove the simplicity and isolation results of each point of the principal eigencurve $\mu_1(\lambda)$ by applying Picone's Identity [1]. Finally, we obtain the continuity and the differentiability of this principal eigencurve.

The plan of this paper is the following. In Section 2, we use a variational method to prove the existence of a sequence of eigencurves for the problem (1.1). In Section 3, we establish the simplicity and the isolation results of each point of the principal eigencurve. Finally, in Section 4, we establish the continuity of the eigenpair $(\mu_1(\lambda), u(\lambda))$ in λ and the differentiability of the principal eigencurve.

2. Existence of eigencurves

To prove the existence of a sequence of eigencurves of (1.1), we will use a variational approach and consider the energy functional on $W^{1,p}(\Omega)$ as

$$\Phi_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\partial\Omega} m(x)|u|^p \, d\sigma,$$

Set

$$S := \{u \in W^{1,p}(\Omega); \frac{1}{p} \int_{\partial\Omega} |u|^p \, d\sigma = 1\}.$$

It is clear that for any $\lambda \in \mathbb{R}$, The solutions of (1.1) are the critical points of Φ_λ restricted to S. For any $k \in \mathbb{N}^*$, let

$$\Gamma_k = \{K \subset S : K \text{ symmetric, compact and } \gamma(K) = k\},$$

where $\gamma(K) = k$ is the genus of K; i.e., the smallest integer k such that there is an odd continuous map from K to $\mathbb{R}^k \backslash \{0\}$. Next we define

$$\mu_k(\lambda) := \inf_{K \in \Gamma_k} \max_{u \in K} \Phi_\lambda(u) \tag{2.1}$$

and

$$\|u\| := \Big(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \Big)^{1/p}$$

is the $W^{1,p}(\Omega)$-norm.

The following theorem is the main result of this section.

Theorem 2.1. *For each $\lambda \in \mathbb{R}$, $\mu_k(\lambda)$ given by (2.1) is a nondecreasing sequence of positive eigenvalues of the problem (1.1). Moreover $\mu_k(\lambda) \to +\infty$ as $k \to +\infty$.*

We will use Ljusternick-Schnirelmann theory on C^1-manifolds. It is clear that for any $\lambda \in \mathbb{R}$, the functional Φ_λ is even and bounded from below on S. Indeed, let $u \in S$, then

$$\Phi_\lambda(u) \geq -|\lambda|\|m\|_{\infty,\partial\Omega} > -\infty, \tag{2.2}$$

where $\|\cdot\|_{\infty,\partial\Omega}$ denotes the $L^\infty(\partial\Omega)$-norm. Letting

$$A(u) := -\frac{\lambda}{p}\int_{\partial\Omega} m|u|^p\,d\sigma, \quad B(u) := \frac{1}{p}\int_{\partial\Omega}|u|^p\,d\sigma.$$

By employing the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, we deduce that A, B are weakly continuous and A', B' are compact, where A' and B' are respectively the derivative of A and B.

We are now ready to prove the Palais-Smale condition.

Lemma 2.2. *The functional Φ_λ satisfies the Palais-Smale condition on S; i.e., for each sequence $(u_n)_n \subset S$, if $\Phi_\lambda(u_n)$ is bounded and*

$$(\Phi_\lambda)'(u_n) - c_n B'(u_n) \to 0, \tag{2.3}$$

with $c_n = \frac{\langle(\Phi_\lambda)'(u_n),u_n\rangle}{\langle B'(u_n),u_n\rangle}$. Then, $(u_n)_n$ has a convergent subsequence in $W^{1,p}(\Omega)$.

Let define the property (S_+). We shall deal with operators F acting from $W^{1,p}(\Omega)$ to $(W^{1,p}(\Omega))'$. F satisfies the condition (S_+), if for any sequence v_n weakly convergent to v in $W^{1,p}(\Omega)$, and $\limsup_{n\to+\infty}\langle F(v_n), v_n - v\rangle \leq 0$ it follows that $v_n \to v$ strongly in $W^{1,p}(\Omega)$, where $(W^{1,p}(\Omega))'$ is the dual of $W^{1,p}(\Omega)$ with respect to the pairing $\langle .,.\rangle$.

Proof of Lemma 2.2. Let us first show that the sequence u_n is bounded in $W^{1,p}(\Omega)$. Assume by contradiction that, for a subsequence of $(u_n)_n$, $\|u_n\| \to +\infty$ and let $v_n := u_n/\|u_n\|$, for a subsequence, $v_n \to v$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and strongly in $L^p(\partial\Omega)$. Since $\Phi_\lambda(u_n)$ is bounded, then $\int_\Omega |\nabla u_n|^p\,dx$ remains bounded, one has $\int_\Omega |\nabla v_n|^p\,dx \to 0$. Thus v is a nonzero constant, indeed; the weak convergence of v_n imply that

$$\int_\Omega |\nabla v|^p\,dx + \int_\Omega |v|^p\,dx \leq \liminf_{n\to+\infty}\Big(\int_\Omega |\nabla v_n|^p\,dx + \int_\Omega |v_n|^p\,dx\Big).$$

Thus $\int_\Omega |\nabla v|^p\,dx = 0$, hence v is a constant. Moreover $v_n \to v$ strongly in $W^{1,p}(\Omega)$, thus v is a nonzero constant. But $B(u_n) = 1$ and so, dividing by $\|u_n\|^p$ and passing to the limit, one obtains $\int_{\partial\Omega}|v|^p\,d\sigma = 0$. This is a contradiction (since v is a nonzero constant). Thus u_n is bounded in $W^{1,p}(\Omega)$. For a subsequence of $(u_n)_n$, $u_n \to u$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. On the other hand, by (2.3), $(\Phi_\lambda)'(u_n)$ being a convergent sequence strongly to some $f \in (W^{1,p}(\Omega))'$. By calculation, we have

$$\langle F(u_n), u_n - u\rangle = \langle(\Phi_\lambda)'(u_n), (u_n - u)\rangle - \langle A'(u_n), (u_n - u)\rangle + \int_\Omega |u_n|^{p-2}u_n(u_n - u)\,dx,$$

where F is an operator defined from $W^{1,p}(\Omega)$ to $(W^{1,p}(\Omega))'$ by

$$\langle F(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega |u|^{p-2}uv\,dx.$$

Using the compactness of A', we get

$$\limsup_{n\to+\infty}\langle F(u_n), u_n - u\rangle \geq 0.$$

Since the operator F satisfies the condition (S_+), $u_n \to u$ strongly in $W^{1,p}(\Omega)$. This achieves the proof of lemma. $\square$

Proof of Theorem 2.1. This theorem is proved by applying a general result from infinite dimensional Ljusternik-Schnirelmann theory. We need only to prove that for any $\lambda \in \mathbb{R}$, $\mu_k(\lambda) \to +\infty$ as $k \to +\infty$. The proof adopts the scheme in [6]. Let $(e_n, e_j^*)_{n,j}$ be a biorthogonal system such that $e_n \in W^{1,p}(\Omega)$, $e_j^* \in (W^{1,p}(\Omega))'$, the $(e_n)_n$ are dense in $W^{1,p}(\Omega)$; and the $(e_j^*)_j$ are total in $(W^{1,p}(\Omega))'$. Set for any $k \in \mathbb{N}^*$

$$\mathcal{F}_{k-1}^{\perp} = \overline{\operatorname{span}(e_{k+1}, e_{k+2}, e_{k+3}, \dots)}.$$

Observe that for any $K \in \Gamma_k$, $K \cap \mathcal{F}_{k-1}^{\perp} \neq \emptyset$ (by [10, (g) of Proposition 2.3]). Now, we claim that

$$t_k := \inf_{K \in \Gamma_k} \sup_{K \cap \mathcal{F}_{k-1}^{\perp}} \Phi_\lambda(u) \to +\infty, \quad \text{as } k \to +\infty.$$

Indeed, to obtain the contradiction, assume for k large enough that there is $u_k \in \mathcal{F}_{k-1}^{\perp}$ with $\frac{1}{p}\int_{\partial\Omega} |u_k|^p \, d\sigma = 1$ such that

$$t_k \le \Phi_\lambda(u_k) \le M,$$

for some $M > 0$ independent of k. Therefore,

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p \, dx - \frac{\lambda}{p}\int_{\partial\Omega} m(x)|u_k|^p \, d\sigma \le M.$$

Hence

$$\int_\Omega |\nabla u_k|^p \, dx \le pM + \lambda \|m\|_{\infty,\partial\Omega} < \infty. \tag{2.4}$$

To prove that $(u_k)_k$ is bounded in $W^{1,p}(\Omega)$, we follow the same method in the proof of Lemma 2.2. Thus for a subsequence of $(u_k)_k$ if necessary, we can suppose that (u_k) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. By our choice of $\mathcal{F}_{k-1}^{\perp}$, we have $u_k \to 0$ weakly in $W^{1,p}(\Omega)$. Because $\langle e_n^*, e_k \rangle = 0$, for all $k \ge n$. This contradicts the fact that $\frac{1}{p}\int_{\partial\Omega} |u_k|^p \, d\sigma = 1$, for all k and the the claim is proved.

Finally, since $\mu_k(\lambda) \ge t_k$ we conclude that $\mu_k(\lambda) \to +\infty$, as $k \to +\infty$ and the proof is complete. $\square$

3. Qualitative properties of the principal eigencurve

Now we consider qualitative properties of the principal eigencurve . Several authors studied the simplicity result in Dirichlet p-Laplacian case by using $C^{1,\alpha}$-regularity and L^∞-estimation of the first eigenfunction, we cite in particular the works [1], [2] and [8].

Let us note that all solutions of problem (1.1) are of class $C^{1,\alpha}(\Omega)$ since they are p-harmonic. Moreover, following the procedure outlined in [12] one may show that all solutions of problem (1.1) belongs to $L^\infty(\Omega)$.

Theorem 3.1. *For any $\lambda \in \mathbb{R}$, the eigenvalue $\mu_1(\lambda)$ defined by (1.4) is simple and the eigenfunctions associated to $\mu_1(\lambda)$ are either positive or negative in $\overline{\Omega}$.*

The next lemma follows from Picone's identity.

Lemma 3.2. *Let u and v be two nonnegative eigenfunction associated to some eigenvalues μ and $\tilde{\mu}$, respectively. Then*

$$0 \leq (\mu - \tilde{\mu}) \int_{\partial\Omega} u^p \, d\sigma \tag{3.1}$$

and equality holds if and only if v is multiple of u.

Proof. We first show that the trace of v satisfies $v > 0$ on $\partial\Omega$. Let $\epsilon > 0$. By the maximum principle of Vazquez (see [13]) $v > 0$ in Ω so that $\frac{v}{v+\epsilon} \to 1_\Omega$ in $L^p(\Omega)$ as $\epsilon \to 0$. On the other hand $\nabla(\frac{v}{v+\epsilon}) \to 0$ a.e. as $\epsilon \to 0$. Taking $\varphi = \frac{1}{(v+\epsilon)^{p-1}}$ as testing function in equation (1.2) satisfied by v, we have

$$(p-1) \int_\Omega \frac{|\nabla v|^p}{(v+\epsilon)^p} \, dx = \int_{\partial\Omega} (\lambda m + \mu)(\frac{v}{v+\epsilon})^p \, d\sigma$$

so that

$$|\nabla(\frac{v}{v+\epsilon})|^p = (\frac{\epsilon}{v+\epsilon})^p \frac{|\nabla v|^p}{(v+\epsilon)^p} \leq \frac{|\nabla v|^p}{v^p} \in L^1(\Omega).$$

By the dominated convergence theorem, we have that $\frac{v}{v+\epsilon} \to 1_\Omega$ in $W^{1,p}(\Omega)$. By continuity of the trace mapping, we have that $\frac{v}{v+\epsilon} \to 1_{\partial\Omega}$ in $L^1(\partial\Omega)$ as $\epsilon \to 0$ and it follows that $v > 0$ on $\partial\Omega$. Now let $\epsilon > 0$. By Picone's identity, we have

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla v|^{p-2} \nabla v \nabla (\frac{u^p}{(v+\epsilon)^{p-1}}) \, dx \\ &= \int_{\partial\Omega} (\lambda m + \mu) u^p \, d\sigma - \int_{\partial\Omega} (\lambda m + \tilde{\mu})(\frac{v}{v+\epsilon})^{p-1} u^p \, d\sigma \end{aligned}$$

and equality holds if and only if v is multiple of u. Going to the limit $\epsilon \to 0$ and using the fact that $v > 0$ on $\partial\Omega$, we get the desired inequality. □

Proof of Theorem 3.1. By Theorem 2.1, it is clear that $\mu_1(\lambda)$ is an eigenvalue of the problem (1.1) for any $\lambda \in \mathbb{R}$. Let u be an eigenfunction associated to $\mu_1(\lambda)$ so that $|u|$ is a minimiser for (1.4) and is thus an eigenfunction associated to $\mu_1(\lambda)$. It follows from the maximum principle of Vazquez that $|u| > 0$ in Ω and we conclude that u has constant sign.

Taking $\mu = \tilde{\mu} = \mu_1(\lambda)$ in (3.1), we see that any eigenfunction v associated of $\mu_1(\lambda)$ must be a multiple of u, so that $\mu_1(\lambda)$ is simple. □

To prove the isolation of $\mu_1(\lambda)$, we need the following two lemmas.

Lemma 3.3. *Let $(k, q) \in \mathbb{N}^* \times \mathbb{N}$ and let $\lambda \in \mathbb{R}$. If $\mu_k(\lambda) = \mu_{k+1}(\lambda) = \cdots = \mu_{k+q}(\lambda)$, then $\gamma(K) \geq q + 1$ where*

$$K := \{u \in S; u \text{ is an eigenfunction associated to } \mu_k(\lambda)\}.$$

The above lemma is proved by applying a general result from infinite dimensional Ljusternik-Schnirelmann theory.

Lemma 3.4. *For each $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is the only positive eigenvalue associated with λ, having an eigenfunction that does not change sign on the boundary $\partial\Omega$.*

Proof. For the proof, we use the Lemma 3.2. Taking $\mu = \mu_1(\lambda)$ in (3.1), we see that no eigenvalue $\tilde{\mu} > \mu_1(\lambda)$ can be associated to a positive eigenfunction. Thus $\mu_1(\lambda)$ is the only positive eigenvalue associated to an eigenfunction of definite sign. □

Theorem 3.5. *For each $\lambda \in \mathbb{R}$, $\mu_1(\lambda)$ is isolated.*

Proof. It suffices to prove that $\mu_2(\lambda)$ is indeed the second positive eigenvalue of the problem (1.1), i.e. $\mu_1(\lambda) < \mu_2(\lambda)$ for all $\lambda \in \mathbb{R}$ and if $\mu_1(\lambda) < \mu < \mu_2(\lambda)$, then μ is not an eigenvalue of problem (1.1). By Theorem 3.1, $\gamma(K_1) = 1$ where $K_1 := \{u \in S; u \text{ is an eigenfunction associated to } \mu_1(\lambda)\}$. Thus, by Lemma 3.3, $\mu_1(\lambda) < \mu_2(\lambda)$. By contradiction, we suppose that μ is an eigenvalue of problem (1.1). Let u be an eigenfunction associated to μ. Since $\mu \neq \mu_1(\lambda)$, we deduce by Lemma 3.4 that $u^+ = \max(u, 0) \neq 0$ and $u^- = \min(u, 0) \neq 0$. It follows from (1.2) that

$$\int_\Omega |\nabla u^+|^p\,dx - \lambda \int_{\partial\Omega} m(x)|u^+|^p\,d\sigma = \mu \int_{\partial\Omega} |u^+|^p\,d\sigma,$$
$$\int_\Omega |\nabla u^-|^p\,dx - \lambda \int_{\partial\Omega} m(x)|u^-|^p\,d\sigma = \mu \int_{\partial\Omega} |u^-|^p\,d\sigma$$

Assume that u is normalized in such a way that

$$\frac{1}{p}\int_{\partial\Omega} |u^+|^p\,d\sigma = \frac{1}{p}\int_{\partial\Omega} |u^-|^p\,d\sigma = 1.$$

The set $K_2 = \{\alpha u^+ + \beta u^-; \alpha, \beta \in \mathbf{R} \text{ such that } |\alpha|^p + |\beta|^p = 1\}$ is in Γ_2. Thus

$$\begin{aligned}\mu_2(\lambda) &\leq \max_{|\alpha|^p+|\beta|^p=1} \Big(\frac{1}{p}\int_\Omega |\nabla(\alpha u^+ + \beta u^-)|^p\,dx - \frac{\lambda}{p}\int_{\partial\Omega} m(x)|\alpha u^+ + \beta u^-|^p\,d\sigma\Big)\\ &= \mu.\end{aligned}$$

This is a contradiction. The proof of the isolation of $\mu_1(\lambda)$ is complete. □

4. Continuity and differentiability in λ

In this section, we extend the results of continuity and differentiability for the first eigencurve of the Dirichlet p-Laplacian shown by Binding and Huang in [5].

Let $\lambda \in \mathbb{R}$ and $(\mu_1(\lambda), u(\lambda))$ be the corresponding eigenpair. Henceforth we normalize the eigenfunction $u(\lambda)$ to $u(\lambda) \in S$ with $u(\lambda) > 0$. In the following theorem, we consider continuity of the eigenpair in λ and differentiability of the principal eigencurve $\mu_1(\lambda)$ in λ.

Theorem 4.1. *For any bounded domain Ω, the function $\lambda \to \mu_1(\lambda)$ is differentiable on $\mathbb{R}$ and the function $\lambda \to u(\lambda)$ is continuous from $\mathbb{R}$ to $W^{1,p}(\Omega)$. More precisely*

$$\mu_1'(\lambda_0) = -\frac{1}{p}\int_{\partial\Omega} m(x)(u(\lambda_0))^p\,d\sigma, \quad \forall \lambda_0 \in \mathbb{R}. \tag{4.1}$$

Proof. By (1.4), it is easy to see that $\lambda \to \mu_1(\lambda)$ is a concave function in $\mathbb{R}$. Continuity of $\lambda \to \mu_1(\lambda)$ follows from the concavity. To prove continuity of $\lambda \to u(\lambda)$, we proceed as follows. Let $\Lambda \subset \mathbb{R}$ be bounded. For $\lambda \in \Lambda$, since

$$\mu_1(\lambda) = \frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p\,dx - \frac{\lambda}{p}\int_{\partial\Omega} m(x)|u(\lambda)|^p\,d\sigma \leq \text{constant},$$

we have that $\int_\Omega |\nabla u(\lambda)|^p\,dx$ remains bounded. To prove that $u(\lambda)$ is bounded in $W^{1,p}(\Omega)$, we follow the same method in the proof of Lemma 2.2. Thus, for a subsequence, $u(\lambda) \to u_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and strongly in

$L^p(\partial\Omega)$ as $\lambda \to \lambda_0 \in \overline{\Lambda}$. Passing to the limit in the following equality

$$\begin{aligned}&\int_\Omega |\nabla u(\lambda)|^{p-2}\nabla u(\lambda)\nabla\varphi\,dx\\&= \lambda\int_{\partial\Omega} m(x)|u(\lambda)|^{p-2}u\varphi\,d\sigma + \mu_1(\lambda)\int_{\partial\Omega}|u(\lambda)|^{p-2}u(\lambda)\varphi\,d\sigma,\end{aligned}\tag{4.2}$$

we have

$$\begin{aligned}&\int_\Omega |\nabla u_0|^{p-2}\nabla u_0\nabla\varphi\,dx\\&= \lambda_0\int_{\partial\Omega} m(x)|u_0|^{p-2}u_0\varphi\,d\sigma + \mu_1(\lambda_0)\int_{\partial\Omega}|u_0|^{p-2}u_0\varphi\,d\sigma,\end{aligned}\tag{4.3}$$

On the other hand $u_0 \not\equiv 0$ (since $u_0 \in S$). Thus u_0 is an eigenfunction associated to $\mu_1(\lambda_0)$. By simplicity of $\mu_1(\lambda_0)$, we have $u_0 = u(\lambda_0)$. Taking $\varphi = u_0$ in (4.3), we obtain

$$\frac{1}{p}\int_\Omega |\nabla u_0|^p\,dx = \frac{\lambda_0}{p}\int_{\partial\Omega} m(x)|u_0|^p\,d\sigma + \mu_1(\lambda_0).\tag{4.4}$$

For $\varphi = u(\lambda)$ in (4.2), we get

$$\frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p\,dx = \frac{\lambda}{p}\int_{\partial\Omega} m(x)|u(\lambda)|^p\,d\sigma + \mu_1(\lambda).\tag{4.5}$$

Letting $\lambda \to \lambda_0$ in (4.5), we have

$$\lim_{\lambda\to\lambda_0}\frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p\,dx = \frac{\lambda_0}{p}\int_{\partial\Omega} m(x)|u_0|^p\,d\sigma + \mu_1(\lambda_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p\,dx.$$

Since $u(\lambda) \to u_0$ strongly in $L^p(\Omega)$, $\|u(\lambda)\| \to \|u_0\|$ as $\lambda \to \lambda_0$. Finally by the uniform convexity of $W^{1,p}(\Omega)$, we conclude that $u(\lambda) \to u_0 = u(\lambda_0)$ strongly in $W^{1,p}(\Omega)$ as $\lambda \to \lambda_0$.

For the differentiability of $\lambda \to \mu_1(\lambda)$, it suffices to use the variational characterization of $\mu_1(\lambda)$ and of $\mu_1(\lambda_0)$, so that

$$\frac{\lambda_0-\lambda}{p}\int_{\partial\Omega} m(x)(u(\lambda))^p\,d\sigma \le \mu_1(\lambda) - \mu_1(\lambda_0) \le \frac{\lambda_0-\lambda}{p}\int_{\partial\Omega} m(x)(u(\lambda_0))^p\,d\sigma,$$

for all $\lambda, \lambda_0 \in \mathbb{R}$. Dividing by $\lambda - \lambda_0$ and letting $\lambda \to \lambda_0$, we obtain (4.1). □

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