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REMARKS ON A 2-D NONLINEAR BACKWARD HEAT PROBLEM USING A TRUNCATED FOURIER SERIES METHOD

DANG DUC TRONG, NGUYEN HUY TUAN

Abstract. The inverse conduction problem arises when experimental measurements are taken in the interior of a body, and it is desired to calculate temperature on the surface. We consider the problem of finding, from the final data $u(x,y,T)=\varphi(x,y)$, the initial data $u(x,y,0)$ of the temperature function $u(x,y,t)$, $(x,y)\in U\equiv(0,\pi)\times(0,\pi)$, $t\in[0,T]$ satisfying the nonlinear system

$$u_t-\Delta u=f(x,y,t,u(x,y,t)),\quad (x,y,t)\in U\times(0,T),$$
$$u(0,y,t)=u(\pi,y,t)=u(x,0,t)=u(x,\pi,t)=0,\quad (x,y,t)\in U\times(0,T).$$

This problem is known to be ill-posed, as the solution exhibits unstable dependence on the given data functions. Using the Fourier series method, we regularize the problem and to get some new error estimates. A numerical experiment is given.

1. Introduction

In this paper, we consider the following two dimensional problem in an rectangle $U=(0,\pi)\times(0,\pi)$

$$u_t-\Delta u=g(x,y,t,u(x,y,t))\quad (x,y,t)\in U\times(0,T),\ U=(0,\pi)\times(0,\pi) \tag{1.1}$$
$$u(0,y,t)=u(\pi,y,t)=u(x,0,t)=u(x,\pi,t)=0\quad (x,y,t)\in U\times[0,T] \tag{1.2}$$
$$u(x,y,T)=\varphi(x,y)\quad x,y\in U. \tag{1.3}$$

where we want to determine the temperature distribution $u(.,.,t)$ for $0\le t<T$ from the data $\varphi(x,y)$. The problem is called the backward heat problem (BHP), the backward Cauchy problem or the final value problem. This is a typical ill-posed problem. In general no solution which satisfies the heat conduction equation with final data and the boundary conditions exists. Even if the solution exists, it will not be continuously dependent on the final data such that the numerical simulations are very difficult and some special regularization methods are required. In the context of approximation method for this problem, many approaches have been investigated. In the mathematical literature various methods have been proposed for solving backward Cauchy problems. Such authors as Lattes and Lions, Miller

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have approximated BHP by quasi-reversibility methods (QR method for short). In 1983, Showalter, in [16], presented a different method called the quasiboundary value (QBV method) method to regularize that linear homogeneous problem which gave a stability estimate better than the one of disscused method. The main ideas of the method is of adding an appropriate "corrector" to the final data. Using the method, Clark and Oppenheimer, in [5], and Denche-Bessila, very recently in [7], regularized the backward problem by replacing the final condition by $u(T)+\epsilon u(0) = g$ and $u(T) - \epsilon u'(0) = g$ respectively.

Although there are many papers on the linear homogeneous case of the backward problem, we only find a few papers on the nonhomogeneous and nonlinear cases of BHP. We can notably mention the method of QBV and modified quasi-reversibility to solve the one dimensional nonlinear backward heat problem (NBHP), such as [20]. Moreover, the two dimensional case of NBHP is very scarce and it is not considered by QR or QBV methods. To the authors's knowledge, in some recent papers on the nonlinear backward heat, the error estimates of most regularization methods are the form $C\epsilon^{t/T}$. It makes difficult to solve the error in the time $t = 0$. To improve this, we develop a new regularization method which is called Fourier method for solving the Problem (1.1)-(1.3). As far as we know, there are not any results of Fourier series method for treating NBHP until now. Meanwhile, we will establish faster convergence results via improved error estimates. Especially, the convergence of the approximate solution at $t = 0$ is also proved. This is an improvement of known results in [15, 20, 22].

Informally, problem (1.1)-(1.3) can be transformed to an integral equation

$$\begin{aligned} u(x,y,t) = \sum_{n,m=1}^{\infty} \Big(& e^{-(t-T)(n^2+m^2)} \varphi_{nm} \\ & - \int_t^T e^{-(t-s)(n^2+m^2)} g_{nm}(u)(s) ds \Big) \sin(nx)\sin(my), \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \varphi(x,y) &= \sum_{n,m=1}^{\infty} \varphi_{nm} \sin(nx)\sin(my), \\ g(u)(x,t) &= \sum_{n,m=1}^{\infty} g_{nm}(u)(t) \sin(nx)\sin(my), \end{aligned}$$

are the expansion of φ and $g(u)$, respectively. Since $t < T$, we know from (1.4) that, when m^2+n^2 becomes large, $\exp\{(T-t)(m^2+n^2)\}$ increases rather quickly. Thus, the term $e^{-(t-T)(m^2+n^2)}$ is the cause of the instability. So, we hope to recover the stability of problem (1.4) by filtering the high frequencies with suitable method. The essence of our regularization method is just to eliminate all high frequencies from the solution, and instead consider (1.4) only for $m^2 + n^2 \le a_\beta$, where a_β is an appropriate positive constant satisfying $\lim_{\beta\to 0} a_\beta = \infty$. We note that a_β is a constant which will be selected appropriately as regularization parameter. Then, we get a stable and convergent iteration scheme. We have the following approximation problem

$$
\begin{aligned}
v^{\beta,a_\beta}(x,y,t) = \sum_{m^2+n^2\le a_\beta}^{m,n\ge 1} \Big(& e^{(T-t)(n^2+m^2)}\varphi_{nm} \\
& - \int_t^T e^{(s-t)(n^2+m^2)} g_{nm}(v^{\beta,a_\beta})(s)ds\Big) \sin(nx)\sin(my)
\end{aligned} \tag{1.5}
$$

where

$$
\varphi_{nm} = \frac{4}{\pi^2}\langle \varphi(x,y), \sin(nx)\sin(my)\rangle,
$$

$$
g_{nm}(u)(t) = \frac{4}{\pi^2}\langle g(x,y,t,u(x,y,t)), \sin(nx)\sin(my)\rangle,
$$

and $\langle\cdot,\cdot\rangle$ is inner product in $L^2(U)$.

2. Fourier regularization and the main results

For clarity, we denote the solution of (1.1)-(1.3) by $u(.,.,t)$, and the solution of (1.5) by $v^{\beta,a_\beta}(.,.,t)$. The main conclusion of this article is as follows.

Theorem 2.1. *Let $\varphi \in L^2(U)$ and let $g \in L^\infty(\overline{U}\times[0,T]\times R)$ satisfy*

$$
|g(x,y,t,w) - g(x,y,t,u)| \le k|w-u|
$$

for a $k>0$ independent of x,y,t,w,u. Then (1.5) has a unique solution $v^{\beta,a_\beta} \in C([0,T];H_0^1(U)) \cap C^1((0,T);L^2(U))$.

Theorem 2.2. *The solution of (1.5) depends continuously on φ in $L^2(U)$.*

Theorem 2.3. *Let φ, g be as in Theorem 2.1. If $\frac{\partial g}{\partial z}(x,y,t,z)$ is bounded on $U\times(0,T)\times R$ then (1.1)-(1.3) has at most one solution*

$$
u \in C([0,T];H_0^1(U)) \cap C^1((0,T);L^2(U)).
$$

Theorem 2.4. *Let φ, g be as in Theorem 2.1. Suppose that (1.1)-(1.3) has a unique solution $u(x,y,t)$ in $C([0,T];H_0^1(U)) \cap C^1((0,T);L^2(U))$ which satisfies*

$$
\int_0^T \sum_{n,m=1}^{\infty} e^{2s(n^2+m^2)} g_{nm}^2(u)(s)ds < \infty. \tag{2.1}
$$

Then

$$
\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \le \sqrt{M} e^{k^2T(T-t)} e^{-ta_\beta} \tag{2.2}
$$

for every $t\in[0,T]$, where

$$
M = 4\|u(0)\|^2 + \pi^2 T \int_0^T \sum_{n,m=1}^{\infty} e^{2s(n^2+m^2)} g_{nm}^2(u)(s)ds,
$$

and v^{β,a_β} is the unique solution of (1.5) corresponding to β. Moreover, if $\frac{\partial u}{\partial t} \in L^2((0,T);L^2(U))$, then there exists a t_β such that

$$
\|u(.,.,0) - v^{\beta,a_\beta}(.,.,t_\beta)\| \le \sqrt{2} C \sqrt[4]{1/a_\beta},
$$

where

$$
N = \Big(\int_0^T \|\frac{\partial u}{\partial s}(.,.,s)\|^2 ds\Big)^{1/2}, \quad C = \max\{M,N\}.
$$

Remark 2.5. (1) In the simple case of the function $g(.,.,u) = 0$ (it follows that k=0), we have

$$\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \le 2\|u(.,.,0)\| e^{-ta_\beta}.$$

Choosing $a_\beta = \frac{1}{T}\ln(1/\beta)$, we obtain the error estimate

$$\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \le 2\|u(.,.,0)\|.\beta^{t/T}$$

This error is given in [5].

(2) In most known results, such as [15, 20, 22], the errors between the exact solution and approximate solution can be calculated in the form $C\epsilon^{t/T}$. Notice that the convergence estimate in this Theorem does not give any useful information on the continuous dependence of the solution at $t = 0$. It is easy to see that if taking $t = 0$ in (2.2), the error estimate is as follows

$$\|u(.,.,0) - v^{\beta,a_\beta}(.,.,0)\| \le \sqrt{M} e^{k^2T^2}$$

does not tend to zero when $\beta \to 0$. So, the convergence of the approximate is large when $t \to 0$. In next Theorem, we will give a good estimate in which the error in the case $t = 0$ is considered.

(3) In this Theorem, we ask for a condition on the expansion coefficient g_{nm} in (2.1). We note that the solution u depends on the nonlinear term g and therefore $g_{nm}, g_{nm}(u)$ is very difficult to be valued. Such a obscurity makes this Theorem hard to be used for numerical computations. Hence, we ask the condition as follows

$$\sum_{n,m=1}^{\infty} e^{2t(n^2+m^2)} |< u(.,.,t), \sin(nx)\sin(my) >|^2 < \infty. \tag{2.3}$$

In this case, we only require the assumption of u, not need to compute the function $g_{nm}(u)$. In the homogeneous case of problem (1.1)-(1.3),i.e., $g = 0$, then the right hand side of (2.3) is equal to $\|u(.,.,0)\|^2$. Hence, the condition (2.3) is natural and acceptable.

Theorem 2.6. *Let φ, g be as in Theorem 2.1. Suppose* (1.1)-(1.3) *has a unique solution $u(x,y,t)$ satisfying* (2.3)*. Then we have*

$$\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \le Q(\beta,t,u)e^{-ta_\beta} \tag{2.4}$$

for every $t \in [0,T]$, where

$$Q(\beta,t,u) = \Big(2k^2Te^{2k^2T(T-t)} \int_0^T P(\beta,s,u)ds + \frac{\pi^2}{2}P(\beta,t,u)\Big)^{1/2} \tag{2.5}$$

and

$$\begin{aligned} P(\beta,t,u) &= \sum_{m,n\ge1, m^2+n^2\ge a_\beta} \Big(e^{T(n^2+m^2)}\varphi_{nm} - \int_t^T e^{s(n^2+m^2)} g_{nm}(u)(s)ds\Big)^2 \\ &= \sum_{m,n\ge1, m^2+n^2\ge a_\beta} e^{2t(n^2+m^2)} u_{nm}^2 \end{aligned} \tag{2.6}$$

and v^{β,a_β} is the unique solution of Problem (1.5).

Remark 2.7. If we let $t = 0$ in (2.4), we get the error at the original time,

$$\|u(.,.,0) - v^{\beta,a_\beta}(.,.,0)\|^2 \le 2k^2Te^{2k^2T^2} \int_0^T P(\beta,s,u)ds + \frac{\pi^2}{2}P(\beta,0,u). \tag{2.7}$$

Noting that the right hand side of (2.7) tends to zero when $\beta \to 0$.

For non-exact data, we have the following result.

Theorem 2.8. *Let φ, g be as in Theorem 2.1. Assume that the exact solution u of (1.1)-(1.3) corresponding to φ be defined as in Theorem 2.4. Let $\varphi_\beta \in L^2(U)$ be a measured data such that*

$$\|\varphi_\beta - \varphi\| \le \beta.$$

Suppose the problem (1.1)-(1.3) has a unique solution $u \in C([0,T]; H_0^1(U)) \cap C^1((0,T); L^2(U))$. Let us select $a_\beta = \ln\big((\frac{1}{\beta})^{1/T}(\ln\frac{1}{\beta})^{-\alpha/(2T)}\big)$.

(i) If u satisfies (2.1) then for $t \in (0,T)$, there exists a function v^{β,a_β} satisfying

$$\|v^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \le (M+1)e^{k^2T(T-t)}\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}}\Big(1 + (\ln\frac{1}{\beta})^{\frac{\alpha}{2}}\Big), \quad (2.8)$$

and

$$\|v^{\beta,a_\beta}(.,.,0) - u(.,.,0)\| \le \sqrt[4]{1/a_\beta}\Big(2\exp(k^2T^2) + \sqrt{2}C\Big)$$

where

$$M = 3\|u(.,.,0)\|^2 + 3\pi^2 T\int_0^T \sum_{n,m=1}^{\infty} e^{2s(n^2+m^2)} g_{nm}^2(u)(s)ds$$

and C is defined in Theorem 2.4.

(ii) If u such that the condition (2.3) then for all $t \in [0,T]$

$$\begin{aligned} &\|w^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le \beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}}\Big(\exp(k^2(T-t)^2) + Q(\beta,t,u)(\ln\frac{1}{\beta})^{\frac{\alpha}{2}}\Big), \end{aligned} \quad (2.9)$$

where w^{β,a_β} be the solution of problem (1.5) corresponding to φ_β.

Remark 2.9. *(1) If we let $\alpha = 0$ in (2.8), we have the simple error*

$$\|v^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \le (M+1)e^{k^2T(T-t)}\beta^{t/T}, \quad \forall t \in (0,T). \quad (2.10)$$

This error is similar to the one given in [20]. Notice that the right hand side of (2.10) does not converges to 0. This is disadvantage point of the error (2.8).

(2) In the error (2.9), if we let $t = 0$, we get

$$\|w^{\beta,a_\beta}(.,.,0) - u(.,.,0)\| \le \exp(k^2T^2)(\ln\frac{1}{\beta})^{\frac{-\alpha}{2}} + Q(\beta,0,u). \quad (2.11)$$

Notice that if $\alpha > 0$ then the right hand side of (2.11) converges to 0 and the Theorem 2.8(ii) is a generalization of the result given in [20].

3. Proof of the main results

Proof of theorem 2.1. Put

$$\begin{aligned} &G(v^{\beta,a_\beta})(x,y,t) \\ &= \Psi(x,y,t) - \sum_{m,n\ge 1, m^2+n^2 \le a_\beta}\Big(\int_t^T e^{(s-t)(n^2+m^2)} g_{nm}(v^{\beta,a_\beta})(s)ds\Big)\sin(nx)\sin(my) \end{aligned}$$

where

$$\Psi(x,y,t) = \sum_{m,n\ge 1, m^2+n^2\le a_\beta} e^{(T-t)(n^2+m^2)}\varphi_{nm}\sin(nx)\sin(my).$$

We claim that

$$\begin{aligned} &\|G^p(v^{\beta,a_\beta})(.,.,t) - G^p(w^{\beta,a_\beta})(.,.,t)\|^2 \\ &\le k^{2p} e^{2Tpa_\beta} \frac{(T-t)^p C^p}{p!} |||v^{\beta,a_\beta} - w^{\beta,a_\beta}|||^2 \end{aligned} \tag{3.1}$$

for every $p \ge 1$, where $C = \max\{T, 1\}$ and $|||\cdot|||$ is sup norm in $C([0,T];L^2(U))$.

We shall prove the latter inequality by induction. For $p = 1$, we have

$$\begin{aligned} &\|G(v^{\beta,a_\beta})(.,.,t) - G(w^{\beta,a_\beta})(.,.,t)\|_2^2 \\ &= \frac{\pi^2}{4} \sum_{m,n\ge1, m^2+n^2\le a_\beta} \Big[e^{(s-t)(n^2+m^2)}(g_{nm}(v^{\beta,a_\beta})(s) - g_{nm}(w^{\beta,a_\beta})(s))ds\Big]^2 \\ &\le \frac{\pi^2}{4} \sum_{m,n\ge1, m^2+n^2\le a_\beta} \int_t^T (e^{2(s-t)(n^2+m^2)} ds \int_t^T (g_{nm}(v^{\beta,a_\beta})(s) \\ &\quad - g_{nm}(w^{\beta,a_\beta})(s))^2 ds \\ &\le \frac{\pi^2}{4} \sum_{m,n\ge1, m^2+n^2\le a_\beta} e^{2Ta_\beta}(T-t) \int_t^T (g_{nm}(v^{\beta,a_\beta})(s) - g_{nm}(w^{\beta,a_\beta})(s))^2 ds \\ &= \frac{\pi^2}{4} e^{2Ta_\beta}(T-t) \int_t^T \sum_{m,n\ge1, m^2+n^2\le a_\beta} (g_{nm}(v^{\beta,a_\beta})(s) - g_{nm}(w^{\beta,a_\beta})(s))^2 ds \\ &\le e^{2Ta_\beta}(T-t) \int_t^T \int_0^\pi \int_0^\pi (g(x,y,s,v^{\beta,a_\beta}(x,y,s)) \\ &\quad - g(x,y,s,w^{\beta,a_\beta}(x,y,s)))^2 \,dx\,dy\,ds \\ &\le k^2 e^{2Ta_\beta}(T-t) \int_t^T \int_0^\pi \int_0^\pi |v^{\beta,a_\beta}(x,y,s) - w^{\beta,a_\beta}(x,y,s)|^2 \,dx\,dy\,ds \\ &\le Ck^2 e^{2Ta_\beta}(T-t)|||v^{\beta,a_\beta} - w^{\beta,a_\beta}|||^2. \end{aligned}$$

Thus (3.1) holds. Suppose that (3.1) holds for $p = j$. We prove that (3.1) holds for $p = j+1$. We have

$$\begin{aligned} &\|G^{j+1}(v^{\beta,a_\beta})(.,.,t) - G^{j+1}(w^{\beta,a_\beta})(.,.,t)\|^2 \\ &= \frac{\pi^2}{4} \sum_{m,n\ge1, m^2+n^2\le a_\beta} \Big[\int_t^T (e^{2(s-t)(n^2+m^2)} ds \Big(g_{nm}(G^j(v^{\beta,a_\beta}))(s) \\ &\quad - g_{nm}(G^j(w^{\beta,a_\beta}))(s)\Big) ds\Big]^2 \\ &\le \frac{\pi^2}{4} e^{2Ta_\beta} \sum_{m,n\ge1, m^2+n^2\le a_\beta} \Big[\int_t^T |g_{nm}(G^j(v^{\beta,a_\beta}))(s) - g_{nm}(G^j(w^{\beta,a_\beta}))(s)| ds\Big]^2 \\ &\le \frac{\pi^2}{4} e^{2Ta_\beta}(T-t) \int_t^T \sum_{n,m=1}^\infty |g_{nm}(G^j(v^{\beta,a_\beta}))(s) - g_{nm}(G^j(w^{\beta,a_\beta}))(s)|^2 ds \\ &\le e^{2Ta_\beta}(T-t) \int_t^T \|g(.,.,s,G^j(v^{\beta,a_\beta})(.,.,s)) - g(.,.,s,G^j(w^{\beta,a_\beta})(.,.,s))\|^2 ds \\ &\le e^{2Ta_\beta}(T-t)k^2 \int_t^T \|G^j(v^{\beta,a_\beta})(.,.,s) - G^j(w^{\beta,a_\beta})(.,.,s)\|^2 ds \end{aligned}$$

$$\leq e^{2Ta_\beta}(T-t)k^2k^{2j}e^{2Tja_\beta}\int_t^T \frac{(T-s)^j}{j!}dsC^j|||v^{\beta,a_\beta}-w^{\beta,a_\beta}|||^2$$
$$\leq k^{2(j+1)}e^{2T(j+1)a_\beta}\frac{(T-t)^{j+1}}{(j+1)!}C^{j+1}|||v^{\beta,a_\beta}-w^{\beta,a_\beta}|||^2.$$

Therefore,

$$\|G^p(v^{\beta,a_\beta})(.,.,t)-G^p(w^{\beta,a_\beta})(.,.,t)\|^2 \leq k^{2p}e^{2Tpa_\beta}\frac{(T-t)^pC^p}{p!}|||v^{\beta,a_\beta}-w^{\beta,a_\beta}|||^2$$

for all $v^{\beta,a_\beta}, w^{\beta,a_\beta} \in C([0,T];L^2(U))$. We consider

$$G : C([0,T];L^2(U)) \to C([0,T];L^2(U)).$$

Since $\lim_{p\to\infty} k^pe^{Tpa_\beta}\frac{T^{p/2}C^p}{\sqrt{p!}} = 0$, there exists a positive integer number p_0, such that G^{p_0} is a contraction. It follows that the equation $G^{p_0}(u) = u$ has a unique solution $v^{\beta,a_\beta} \in C([0,T];L^2(U))$. We claim that $G(v^{\beta,a_\beta}) = v^{\beta,a_\beta}$. In fact, one has

$$G(G^{p_0}(v^{\beta,a_\beta})) = G(v^{\beta,a_\beta}).$$

Hence

$$G^{p_0}(G(v^{\beta,a_\beta})) = G(v^{\beta,a_\beta}).$$

By the uniqueness of the fixed point of G^{p_0}, one has $G(v^{\beta,a_\beta}) = v^{\beta,a_\beta}$, i.e., the equation $G(v^{\beta,a_\beta}) = v^{\beta,a_\beta}$ has a unique solution $v^{\beta,a_\beta} \in C([0,T];L^2(U))$. □

Proof of Theorem 2.2. Let u and v be two solutions of (1.5) corresponding to the values φ and ω. We have

$$\begin{aligned}
&\|u(.,.,t)-v(.,.,t)\|^2\\
&= \frac{\pi^2}{4}\sum_{m,n\geq 1, m^2+n^2\leq a_\beta}\Big|e^{(T-t)(n^2+m^2)}(\varphi_{nm}-\omega_{nm})\\
&\quad -\int_t^T e^{(s-t)(n^2+m^2)}(g_{nm}(u)(s)-g_{nm}(v)(s)ds)\Big|^2\\
&\leq \frac{\pi^2}{2}\sum_{m,n\geq 1, m^2+n^2\leq a_\beta}(e^{(T-t)(n^2+m^2)}|\varphi_{nm}-\omega_{nm}|)^2\\
&\quad +\frac{\pi^2}{2}\sum_{m,n\geq 1, m^2+n^2\leq a_\beta}\Big(\int_t^T e^{(s-t)(n^2+m^2)}|g_{nm}(u)(s)-g_{nm}(v)(s)|ds\Big)^2
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\|u(.,.,t)-v(.,.,t)\|^2 \leq\\
&\leq 2e^{2(T-t)a_\beta}\|\varphi-\omega\|^2 + 2k^2(T-t)e^{-2ta_\beta}\int_t^T e^{2sa_\beta}\|u(.,.,s)-v(.,.,s)\|^2ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&e^{2ta_\beta}\|u(.,.,t)-v(.,.,t)\|^2\\
&\leq e^{2Ta_\beta}\|\varphi-\omega\|^2 + 2k^2(T-t)\int_t^T e^{2sa_\beta}\|u(.,.,s)-v(.,.,s)\|^2ds.
\end{aligned}$$

Using Gronwall's inequality we have

$$\|u(.,.,t)-v(.,.,t)\| \leq e^{(T-t)a_\beta}\exp(k^2(T-t)^2)\|\varphi-\omega\|.$$

This completes the proof of the theorem. □

Proof of Theorem 2.3. Let $M > 0$ be such that
$$|\frac{\partial g}{\partial z}(x,y,t,z)| \le M$$
for all $(x,y,t,z) \in U \times (0,T) \times R$. Let $u_1(x,y,t)$ and $u_2(x,y,t)$ be two solutions of Problem (1.1)-(1.3) such that $u_1, u_2 \in C([0,T]; H_0^1(U)) \cap C^1((0,T); L^2(U))$. Put $w(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$. Then w satisfies the equation
$$w_t(x,y,t) - \Delta w(x,y,t) = g(x,y,t,u_1(x,y,t)) - g(x,y,t,u_2(x,y,t)).$$
Thus
$$w_t(x,y,t) - \Delta w(x,y,t) = \frac{\partial g}{\partial z}(x,y,t,\overline{u}(x,y,t))w(x,y,t),$$
for some $\overline{u}(x,y,t)$. It follows that
$$(w_t - \Delta w)^2 \le M^2 w^2.$$
Now $w(0,y,t) = w(\pi,y,t) = w(x,0,t) = w(x,\pi,t) = 0$ and $w(x,y,T) = 0$. Hence by the Lees-Protter theorem [8, p. 373], $w = 0$ which gives $u_1(x,y,t) = u_2(x,y,t)$ for all $t \in [0,T]$. The proof is completed. □

Proof of Theorem 2.4. The functions $u(.,.,t)$ can be written in the form
$$\begin{aligned} u(x,y,t) = \sum_{n,m=1}^{\infty} &(e^{-(t-T)(n^2+m^2)}\varphi_{nm} \\ &- \int_t^T e^{-(t-s)(n^2+m^2)} g_{nm}(u)(s)ds) \sin(nx)\sin(my), \end{aligned}$$
and $v^{\beta,a_\beta}(.,.,t)$ in the form
$$\begin{aligned} v^{\beta,a_\beta}(x,y,t) = \sum_{m,n\ge 1, m^2+n^2 \le a_\beta} &\Big(e^{(T-t)(n^2+m^2)}\varphi_{nm} \\ &- \int_t^T e^{(s-t)(n^2+m^2)} g_{nm}(v^{\beta,a_\beta})(s)ds\Big) \sin(nx)\sin(my) \end{aligned}$$
Hence
$$\begin{aligned} v^{\beta,a_\beta}(x,y,t) - u(x,y,t) = &\sum_{m,n\ge 1, m^2+n^2 \ge a_\beta} (e^{-(t-T)(n^2+m^2)}\varphi_{nm} \\ &- \int_t^T e^{-(t-s)(n^2+m^2)} g_{nm}(u)(s)ds) \sin(nx)\sin(my) \\ &+ \sum_{m,n\ge 1, m^2+n^2 \le a_\beta} \int_t^T \Big(e^{(s-t)(n^2+m^2)}(g_{nm}(v^{\beta,a_\beta})(s) \\ &- g_{nm}(v)(s))ds\Big) \sin(nx)\sin(my) \end{aligned}$$
Using the inequality $(a+b)^2 \le 2(a^2+b^2)$, we obtain
$$\begin{aligned} &\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \\ &\le \frac{\pi^2}{2} \sum_{m,n\ge 1, m^2+n^2 \ge a_\beta} \Big(e^{-(t-T)(n^2+m^2)}\varphi_{nm} - \int_t^T e^{-(t-s)(n^2+m^2)} g_{nm}(u)(s)ds\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\pi^2}{2} \sum_{m,n\geq 1, m^2+n^2\leq a_\beta} \Big(\int_t^T e^{(s-t)(n^2+m^2)} |g_{nm}(u)(s) - g_{nm}(v^{\beta,a_\beta})(s)| ds \Big)^2 \\
&\leq 2\frac{\pi^2}{2} e^{-2t(n^2+m^2)} \sum_{m,n\geq 1, m^2+n^2\geq a_\epsilon} \Big(e^{T(n^2+m^2)}\varphi_{nm} - \int_0^T e^{s(n^2+m^2)} g_{nm}(u)(s) ds \Big)^2 \\
&+ 2\frac{\pi^2}{2} e^{-2t(n^2+m^2)} \sum_{m,n\geq 1, m^2+n^2\geq a_\beta} \Big(\int_0^t e^{s(n^2+m^2)} g_{nm}(u)(s) ds \Big)^2 \\
&+ \frac{\pi^2}{2}(T-t) \int_t^T \sum_{n,m=1}^{\infty} e^{2(s-t)a_\beta} (g_{nm}(u)(s) - g_{nm}(v^{\beta,a_\beta})(s))^2 ds \\
&\leq 4e^{-2ta_\beta}\|u(.,.,0)\|^2 + \pi^2 T e^{-2ta_\beta} \int_0^T e^{2s(n^2+m^2)} g_{nm}^2(u)(s) ds \\
&+ 2(T-t)e^{-2ta_\beta} \int_t^T e^{2sa_\beta} \|g(.,.,s,u(.,.,s)) - g(.,.,s,v^{\beta,a_\beta}(.,.,s))\|^2 ds \\
&\leq 4e^{-2ta_\beta}\|u(.,.,0)\|^2 + \pi^2 T e^{-2ta_\beta} \int_0^T e^{2s(n^2+m^2)} g_{nm}^2(u)(s) ds \\
&+ e^{-2ta_\beta} 2k^2 T \int_t^T e^{2sa_\beta} \|u(.,.,s) - v^{\beta,a_\beta}(.,.,s)\|^2 ds.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&e^{2ta_\beta}\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \\
&\leq e^{-2ta_\beta} M + 2k^2 T \int_t^T e^{2sa_\beta} \|u(.,.,s) - v^{\beta,a_\beta}(.,.,s)\|^2 ds
\end{aligned}
$$

Using Gronwall's inequality, we obtain

$$
e^{2ta_\beta}\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \leq M e^{2k^2 T(T-t)}
$$

which implies

$$
\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \leq \sqrt{M} e^{k^2 T(T-t)} e^{-ta_\beta}. \tag{3.2}
$$

Then we have the equality

$$
u(x,y,t) - u(x,y,0) = \int_0^t \frac{\partial u}{\partial s}(x,y,s) ds\,.
$$

It follows that

$$
\|u(.,.,0) - u(.,.,t)\|^2 \leq t \int_0^t \big\|\frac{\partial u}{\partial s}(.,.,s)\big\|^2 ds \leq N^2 t.
$$

Combining this and (3.2), we have

$$
\begin{aligned}
\|u(.,.,0) - v^{\beta,a_\beta}(.,.,t)\| &\leq \|u(.,.,0) - u(.,.,t)\| + \|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\| \\
&\leq C(\sqrt{t} + e^{-ta_\beta}).
\end{aligned}
$$

For every β, there exists a t_β such that $\sqrt{t_\beta} = e^{-t_\beta a_\beta}$; i.e., $\frac{lnt_\beta}{t_\beta} = -2a_\beta$. Using the inequality $\ln t > -\frac{1}{t}$ for every $t > 0$, we obtain $t_\beta \leq 1/(2\sqrt{a_\beta})$. Hence,

$$
\|u(.,.,0) - v^{\beta,a_\beta}(.,.,t_\beta)\| \leq \sqrt{2} C \sqrt[4]{1/a_\beta}.
$$

This completes the proof of Theorem 2.4. □

Proof of Theorem 2.6. We recall that

$$\begin{aligned} P(\beta,t,u) &= \sum_{m,n\geq 1, m^2+n^2\geq a_\beta} \Big(e^{T(n^2+m^2)}\varphi_{nm} - \int_t^T e^{s(n^2+m^2)} g_{nm}(u)(s)ds\Big)^2 \\ &= \sum_{m,n\geq 1, m^2+n^2\geq a_\beta} e^{2t(n^2+m^2)} u_{nm}^2. \end{aligned} \tag{3.3}$$

It is easy to prove that $P(\beta,t,u)\to 0$ when $\beta\to 0$. As in the proof of Theorem 2.4, we have

$$\begin{aligned} &\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \\ &\leq \frac{\pi^2}{2} \sum_{m,n\geq 1, m^2+n^2\geq a_\beta} \Big(e^{-(t-T)(n^2+m^2)}\varphi_{nm} - \int_t^T e^{-(t-s)(n^2+m^2)} g_{nm}(u)(s)ds\Big)^2 \\ &\quad + \frac{\pi^2}{2} \sum_{m,n\geq 1, m^2+n^2\leq a_\beta} \Big(\int_t^T e^{(s-t)(n^2+m^2)} |g_{nm}(u)(s) - g_{nm}(v^{\beta,a_\beta})(s)|ds\Big)^2 \\ &\leq \frac{\pi^2}{2} e^{-2ta_\beta} P(\beta,t,u) \\ &\quad + \frac{\pi^2}{2}(T-t)\int_t^T \sum_{n,m=1}^{\infty} e^{2(s-t)a_\beta}(g_{nm}(u)(s) - g_{nm}(v^{\beta,a_\beta})(s))^2 ds \\ &\leq \frac{\pi^2}{2} e^{-2ta_\beta} P(\beta,t,u) + 2(T-t)e^{-2ta_\beta}\int_t^T e^{2sa_\beta}\big\|g(.,.,s,u(.,.,s)) \\ &\quad - g(.,.,s,v^{\beta,a_\beta}(.,.,s))\big\|^2 ds \\ &\leq \frac{\pi^2}{2} e^{-2ta_\beta} P(\beta,t,u) + e^{-2ta_\beta} 2k^2 T \int_t^T e^{2sa_\beta}\|u(.,.,s) - v^{\beta,a_\beta}(.,.,s)\|^2 ds. \end{aligned}$$

This implies that

$$\begin{aligned} &e^{2ta_\beta}\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \\ &\leq \frac{\pi^2}{2} e^{-2ta_\beta} P(\beta,t,u) + 2k^2 T\int_t^T e^{2sa_\beta}\|u(.,.,s) - v^{\beta,a_\beta}(.,.,s)\|^2 ds \end{aligned}$$

Applying Gronwall's inequality, we obtain

$$e^{2ta_\beta}\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \leq 2k^2 T e^{-2k^2 Tt}\int_t^T e^{2k^2 Ts} P(\beta,s,u)ds + \frac{\pi^2}{2}P(\beta,t,u).$$

Finally,

$$\|u(.,.,t) - v^{\beta,a_\beta}(.,.,t)\|^2 \leq \Big(2k^2 T e^{2k^2 T(T-t)}\int_0^T P(\beta,s,u)ds + \frac{\pi^2}{2}P(\beta,t,u)\Big)e^{-2ta_\beta}.$$

This completes the proof of Theorem 2.6. □

Proof of Theorem 2.8. (i) Let v_1^{β,a_β} be the solution of (1.5) corresponding to φ and let w^{β,a_β} be the solution of (1.5) corresponding to φ_β where φ,φ_β are defined in Theorem 2.8. Using Theorem 2.4, there exists a t_β such that $\sqrt{t_\beta} = e^{-2t_\beta a_\beta}$ and

$$\|v_1^{\beta,a_\beta}(.,.,t_\beta) - u(.,.,0)\| \leq \sqrt{2}C\sqrt[4]{1/a_\beta}. \tag{3.4}$$

We denote

$$v^{\beta,a_\beta}(.,.,t) = \begin{cases} w^{\beta,a_\beta}(.,.,t), & 0 < t < T, \\ w^{\beta,a_\beta}(.,.,t_\beta), & t = 0\,. \end{cases}$$

Using Theorems 2.2 and 2.4, we obtain

$$\begin{aligned} &\|v^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le \|w^{\beta,a_\beta}(.,.,t) - v_1^{\beta,a_\beta}(.,.,t)\| + \|v_1^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le e^{(T-t)a_\beta} exp(k^2(T-t)^2)\|\varphi - \varphi_\beta\| + Me^{k^2T(T-t)}e^{-ta_\beta} \\ &\le \exp(k^2(T-t)^2)\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}} + Me^{k^2T(T-t)}\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{\alpha t}{2T}}, \\ &\le (M+1)e^{k^2T(T-t)}\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}}\big(1 + (\ln\frac{1}{\beta})^{\frac{\alpha}{2}}\big) \end{aligned}$$

for every $t \in (0,T)$. Using the results in Theorem 2.4, we have the estimate

$$\begin{aligned} \|v^{\beta,a_\beta}(.,.,0) - u(.,.,0)\| &\le \|w^{\beta,a_\beta}(.,.,t_\beta) - v_1^{\beta,a_\beta}(.,.,t_\beta)\| \\ &\quad + \|v_1^{\beta,a_\beta}(.,.,t_\beta) - u(.,.,0)\| \\ &\le 2e^{-t_\beta a_\beta}\exp(k^2T^2) + \sqrt{2}C\sqrt[4]{1/a_\beta} \\ &= 2\sqrt[4]{1/a_\beta}\exp(k^2T^2) + \sqrt{2}C\sqrt[4]{1/a_\beta} \\ &= \sqrt[4]{1/a_\beta}\big(2\exp(k^2T^2) + \sqrt{2}C\big). \end{aligned}$$

This completes the proof of part (i).

(ii) Using Theorems 2.2 and 2.6, we obtain

$$\begin{aligned} &\|w^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le \|w^{\beta,a_\beta}(.,.,t) - v_1^{\beta,a_\beta}(.,.,t)\| + \|v_1^{\beta,a_\beta}(.,.,t) - u(.,.,t)\| \\ &\le e^{(T-t)a_\beta} exp(k^2(T-t)^2)\|\varphi - \varphi_\beta\| + Q(\beta,t,u)e^{-ta_\beta} \\ &\le \exp(k^2(T-t)^2)\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}} + Q(\beta,t,u)\beta^{t/T}(\ln\frac{1}{\beta})^{\frac{\alpha t}{2T}}, \\ &\le \beta^{t/T}(\ln\frac{1}{\beta})^{\frac{-\alpha(T-t)}{2T}}\Big(\exp(k^2(T-t)^2) + Q(\beta,t,u)(\ln\frac{1}{\beta})^{\frac{\alpha}{2}}\Big) \end{aligned}$$

for every $t \in [0,T]$. □

4. Numerical experiments

Let us consider the simple two dimensional Allen-Cahn equation

$$\begin{gathered} u_t - u_{xx} - u_{yy} = u - u^3 + g(x,y,t), \quad (x,y,t) \in (0,\pi)\times(0,\pi)\times(0,1) \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad (x,y,t) \in (0,\pi)\times(0,\pi)\times[0,1], \\ u(x,y,1) = \varphi(x,y), \;\; x,y \in (0,\pi)\times(0,\pi), \end{gathered}$$

where

$$\begin{gathered} g(x,y,t) = 2e^t \sin x \sin y + e^{3t}\sin^3 x \sin^3 y, \\ u(x,y,1) = \varphi_0(x,y) \equiv e \sin x \sin y. \end{gathered}$$

The exact solution of this equation is. $u(x,y,t) = e^t \sin x \sin y$ In particular,

$$u\big(x,y,\frac{39999}{40000}\big) \equiv u(x,y) = \exp\big(\frac{39999}{40000}\big) \sin x \sin y.$$

Let $\varphi_\beta(x,y) \equiv \varphi(x,y) = (\beta+1)e \sin x \sin y$. Then we have

$$\|\varphi_\beta - \varphi\|_2 = \Big(\int_0^\pi \int_0^\pi \beta^2 e^2 \sin^2(x) \sin^2 y dx dy \Big)^{1/2} = \beta e \frac{\pi}{2}$$

Choose $a_\beta = \frac{1}{\beta}$, and let p be a natural number satisfying $p = [\sqrt{\frac{1}{2}\ln(\frac{1}{\beta})}]$. We find the regularized solution $v^{\beta,a_\beta}(x,y,\frac{39999}{40000}) \equiv u_\beta(x,y)$ having the form

$$v^{\beta,a_\beta}(x,y) = v_m(x,y) = w_{11,m} \sin x \sin y + w_{pp,m} \sin(px) \sin(py)$$

where

$$v_1(x,y) = (\beta+1)e \sin x \sin y, \quad w_{11,1} = (\beta+1)e, \quad w_{pp,1} = 0.$$

and $a = 1/400000$, $t_m = 1 - am$ for $m = 1, 2, \dots, 10$,

$$\begin{aligned} w_{11,m+1} &= e^{2(t_m - t_{m+1})} w_{ij,m} - \frac{4}{\pi^2} \int_{t_{m+1}}^{t_m} e^{2(s-t_{m+1})} \\ &\quad \times \Big(\int_0^\pi \int_0^\pi \big(v_m(x,y) - v_m^3(x,y) + g(x,y,s)\big) \sin x \sin y \, dx \, dy \Big) ds\,, \end{aligned}$$

$$\begin{aligned} w_{pp,m+1} &= e^{2p^2(t_m - t_{m+1})} w_{pp,m} - \frac{4}{\pi^2} \int_{t_{m+1}}^{t_m} e^{2p^2(s-t_{m+1})} \\ &\quad \times \Big(\int_0^\pi \int_0^\pi \big(v_m(x,y) - v_m^3(x,y) + g(x,y,s)\big) \sin px \sin py dx dy \Big) ds. \end{aligned}$$

Let $a_\beta = \|v^{\beta,a_\beta} - u\|$ be the error between the regularized solution v^{β,a_β} and the exact solution u. Let $\beta = \beta_1 = 10^{-5} (p = 2)$, $\beta = \beta_2 = 10^{-8}$, $\beta = \beta_3 = 10^{-16}$. Then we have

β	v^{β,a_β}	a_β
$\beta_1 = 10^{-5}$	$2.718241061 \sin x \sin y -$ $0.002038827910 \sin(3x) \sin(3y)$	0.002039009193
$\beta_2 = 10^{-8}$	$2.718213894 \sin x \sin y -$ $0.0002039162480 \sin(3x) \sin(3y)$	0.0002039162492
$\beta_3 = 10^{-16}$	$2.718220664 \sin x \sin y -$ $0.0001835495554 \sin(3x) \sin(3y)$	0.0001835495554

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Dang Duc Trong
HoChiMinh City National University, Department of Mathematics and Informatics, 227 Nguyen Van Cu, Q. 5, HoChiMinh City, Vietnam
E-mail address: ddtrong@math.hcmuns.edu.vn

Nguyen Huy Tuan
Department of Information Technology and Applied Mathematics Ton Duc Thang University 98 Ngo Tat To, Hochiminh City, Vietnam
E-mail address: tuanhuy_bs@yahoo.com