Electronic Journal of Differential Equations, Vol. 2009(2009), No. 78, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ON THE RIGIDITY OF MINIMAL MASS SOLUTIONS TO THE FOCUSING MASS-CRITICAL NLS FOR ROUGH INITIAL DATA

DONG LI, XIAOYI ZHANG

Abstract. For the focusing mass-critical nonlinear Schrödinger equation $iu_t + \Delta u = -|u|^{4/d}u$, an important problem is to establish Liouville type results for solutions with ground state mass. Here the ground state is the positive solution to elliptic equation $\Delta Q - Q + Q^{1+\frac{4}{d}} = 0$. Previous results in this direction were established in [12, 16, 17, 29] and they all require $u_0 \in H_x^1(\mathbb{R}^d)$. In this paper, we consider the rigidity results for rough initial data $u_0 \in H_x^s(\mathbb{R}^d)$ for any $s > 0$. We show that in dimensions $d \geq 4$ and under the radial assumption, the only solution that does not scatter in both time directions (including the finite time blowup case) must be global and coincide with the solitary wave $e^{it}Q$ up to symmetries of the equation. The proof relies on a non-uniform local iteration scheme, the refined estimate involving the $P^{\pm}$ operator and a new smoothing estimate for spherically symmetric solutions.

1. Introduction

1.1. Background and main results.

We consider the focusing mass-critical nonlinear Schrödinger equation

$$iu_t + \Delta u = -|u|^{4/d}u \tag{1.1}$$

in dimensions $d \geq 4$; here $u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$. The name "mass critical" refers to the fact that the scaling symmetry

$$u(t,x) \mapsto \lambda^{d/2} u(\lambda^2 t, \lambda x), \quad \forall \lambda > 0 \tag{1.2}$$

leaves both the equation and the mass invariant. Here the mass is defined as

$$\text{Mass: } M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0).$$

For the initial value problem of (1.1), the local theory was established by Cazenave and Weissler in [2]. To summarize, for any initial data $u_0 \in L_x^2(\mathbb{R}^d)$, they constructed the unique local solution $u(t,x) \in C_t([-T,T]; L_x^2) \cap L_{t,x}^{2(d+2)/d}([-T,T] \times \mathbb{R}^d)$. Moreover, when the mass of the initial data is small enough, the solution is global and obeys the global spacetime estimate

$$\|u\|_{L_{t,x}^{2(d+2)/d}(\mathbb{R} \times \mathbb{R}^d)} \leq C(\|u_0\|_{L_x^2}).$$

2000 *Mathematics Subject Classification.* 35Q55.
Key words and phrases. Mass-critical; nonlinear Schrodinger equation.

Submitted April 15, 2009. Published June 16, 2009.

This estimate implies that the solution scatters in both time directions asymptotically: there exist $u_\pm \in L_x^2(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{L_x^2} = 0.$$

When the solution has large mass, blowup may occur at finite time. The existence of finite blowup solutions was proved by Glassey [7], basing on the virial argument. On the other hand, the equation (1.1) also admits solitary wave solutions of the form $e^{it}R$, where R solves the elliptic equation

$$\Delta R - R + |R|^{4/d}R = 0. \tag{1.3}$$

There are infinite many solutions to this equation, but only one positive solution which is spherically symmetric and whose mass is minimal among all these $R's$. This solution is usually called the

Definition 1.1 (Ground state)**.** The ground state Q refers to the unique positive solution to the equation (1.3). According to [1, 15], Q is spherically symmetric and decays exponentially fast as $|x| \to \infty$.

It is believed that the mass of Q is the minimal mass among all the non-scattering solutions. The precise statement of this general belief is the following scattering conjecture:

Conjecture 1.2. *Let $u_0 \in L_x^2(\mathbb{R}^d)$ be such that $M(u_0) < M(Q)$. Then the corresponding solution exists globally and scatters in both time directions.*

So far, this conjecture has been proved in dimensions $d \geq 2$ when the initial data u_0 is spherically symmetric, see [13, 14].

At the level of minimal mass, there are two explicit examples of non-scattering solutions: the solitary wave SW and the pseudo-conformal ground state $Pc(Q)$.

$$SW = e^{it}Q(x),$$
$$Pc(Q) = |t|^{-\frac{d}{2}} e^{\frac{i|x|^2-4}{4t}} Q(\frac{x}{t}).$$

It is conjectured that these are the only two threshold solutions for scattering at the level of minimal mass. The associated is the following rigidity conjecture which identify the solutions with ground state mass as either SW or $Pc(Q)$ if they do not scatter. Since both mass and the equation are invariant under a couple of symmetries, the coincidence of the solutions with the examples hold after quotienting out these symmetries. Specifically, the symmetries are: translation, phase rotation, scaling and the Galilean boost.

Conjecture 1.3. *Let $u_0 \in L_x^2(\mathbb{R}^d)$ satisfy $M(u_0) = M(Q)$. Then only the three scenarios can occur*

(1) *The solution $u(t,x)$ scatters in both time directions;*
(2) *u blows up at finite time, then u must coincide with $Pc(Q)$ up to symmetries of the equation.*
(3) *u is a global solution and u coincide with SW up to symmetries of the equation.*

Equivalently to say, the rigidity result identifies all the non-scattering solutions with minimal mass as either the pseudo-conformal ground state or the solitary wave. Therefore, the proof of the rigidity conjecture is divided into the two parts:

one is concerned about the behavior of finite blowup solutions, and the other is concerned with the asymptotics of global solutions.

The first result toward the rigidity conjecture is about finite time blowup solutions, and they were established by Weinstein and Merle. In [29], Weinstein showed that if an H_x^1 solution blows up at finite time with minimal mass, then there exist $\theta(t), x(t), \lambda(t)$ such that

$$e^{i\theta(t)}\lambda(t)^{d/2}u(t,\lambda(t)x+x(t)) \to Q \quad \text{in } H_x^1.$$

Later, Merle extended this result to show that if an H_x^1 solution with minimal mass blows up at finite time, then it must be equal to $Pc(Q)$ up to symmetries. One can also see [8] for a simpler proof of this result. The requirement that $u_0 \in H_x^1$ is essentially needed since it is the natural space to carry out the spectral analysis and to well define the energy:

$$\text{Energy: } E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L_x^2}^2 - \frac{d}{2(d+2)}\|u(t)\|_{L_x^{2(d+2)/d}}^{2(d+2)/d} = E(u_0).$$

It is also worth pointing out that their results work for all dimensions $d \geq 1$ and there is no symmetry assumption on the initial data. But the proof relies heavily on the finiteness of the blowup time.

From Merle's result and the pseudo-conformal invariance for mass critical NLS, one easily sees that if $u_0 \in \Sigma = \{v \in H_x^1, xv \in L_x^2\}$ and the corresponding solution exists globally but does not scatter in at least one time direction, then it must be the solitary wave SW up to symmetries. This is the first result toward the rigidity result if the solution is global.

Without the additional decay assumption, it is not obvious at all if the conjecture still holds. Recently in [12], [16], we established the rigidity result for global solutions under the radial assumptions.

Theorem 1.4 ([12, 16])**.** *Let $d \geq 2$. Let $u_0 \in H_x^1(\mathbb{R}^d)$ be spherically symmetric and satisfy $M(u_0) = M(Q)$. Then only the following two scenarios can occur:*

1. *The solution exists globally and scatters in both time directions;*
2. *There exist θ_0, λ_0 such that*

$$u(t,x) = e^{i\theta_0}\lambda_0^{d/2}e^{i\lambda_0^2 t}Q(\lambda_0 x).$$

In dimensions $d \geq 4$, we can relax the spherical symmetry to certain splitting-spherical symmetry, see [16] for more details.

As indicated by the statement of the theorem, the rigidity conjecture concerning the global solution holds under several additional conditions: the spherical symmetry(or the splitting-spherical symmetry) on the initial data; the dimension $d \geq 2$; and the H_x^1 regularity on the initial data. Each of them is used heavily in the proof. To give a brief explanation, the symmetry assumption is used to freeze the center of mass and provide enough decay as $|x| \to \infty$. The one dimension function does not have decay as $|x| \to \infty$, this is why the restriction on the dimension comes in. Finally, since the proof relies on the virial argument, the H_x^1 regularity is naturally needed to define the energy. Meanwhile, in low dimensions $d = 2, 3$, the H_x^1 regularity is also used to get the weak compactness for the kinetic energy, see [16] for more details.

Therefore, removing the reliance on any of these conditions makes a very challenging problem. In this paper, we try to remove the reliance on the H_x^1 regularity in the rigidity conjecture. For some technical reasons which will be clear in the proof, we shall consider solutions which have the minimal mass and do not scatter in both time directions. Our result is the following:

Theorem 1.5 (Rigidity of SW for rough solutions)**.** *Let $d \geq 4$, $s > 0$. Let $u_0 \in H^s_x(\mathbb{R}^d)$ be spherically symmetric and such that $M(u_0) = M(Q)$. Suppose that the corresponding maximal lifespan solution $u(t,x) : (-T_*, T^*) \times \mathbb{R}^d \to \mathbb{C}$ does not scatter in both time directions:*

$$\|u\|_{L^{2(d+2)/d}_{t,x}((-T_*,0]\times\mathbb{R}^d)} = \|u\|_{L^{2(d+2)/d}_{t,x}([0,T^*)\times\mathbb{R}^d)} = \infty.$$

Then the solution must be global

$$T_* = T^* = +\infty.$$

And there exist λ_0, θ_0 such that

$$u(t,x) = e^{i\theta_0} e^{i\lambda_0^2 t} \lambda_0^{d/2} Q(\lambda_0 x).$$

As expected, the main part of the proof is devoted to upgrading the H^s_x regularity of the solution to H^1_x, when the result for H^1_x solution Theorem 1.4 can be applied. The possibility that we can upgrade the regularity of the solution comes ultimately from the fact that the solution we are considering has the *minimal mass* and does not scatter *in both time directions*.

Our strategy for upgrading the regularity is the following: Firstly, since u has the minimal mass and does not scatter on both sides, applying the same argument as in [12], one easily gets that

$$\|\phi_{>1} \nabla u(t)\|_{L^2_x} \lesssim 1.$$

This means that away from the origin, the solution is regular uniformly in time, thus it suffices for us to examine the solution near the origin. There we carefully design a local iteration scheme enabling us to go from H^t_x to $H^{t+\epsilon}_x$ for any $t < 1$ and an ϵ increasing in t. After finite many times of iteration, we get the desired H^1_x regularity. Here by "local", we mean that the scheme is designed to upgrade the regularity of the solution at some fixed time t, for example $t = 0$, not uniformly in time. More precisely, the quantity we will look at is

$$\|\phi_{\leq 1} P_N u_0\|_{L^2_x}, \ N \geq 1. \tag{1.4}$$

(Not that the piece $\|\phi_{>1} P_N u_0\|_{L^2_x}$ already gives us $N^{-1-\epsilon(d)}$ decay following the argument in [12], which is already very good). Now we split (1.4) into two parts by introducing a spatial cutoff

$$\|\phi_{\leq N^{-1-\gamma}} P_N u_0\|_{L^2_x}, \tag{1.5}$$

$$\|\phi_{N^{-1-\gamma} < \cdot \leq 1} P_N u_0\|_{L^2_x}. \tag{1.6}$$

By Hölder and Bernstein, the first quantity gives us the bound: $N^{-s-\frac{d}{2}\gamma}$ which is good for the iteration. To estimate the second piece, we project it into the incoming and outgoing wave, for the incoming wave, we use the Duhamel formula backward in time; for the outgoing wave, we use the Duhamel formula forward in time. The assumption that the solution does not scatter on both sides forbids the scattering wave, for which there is no hope to upgrade the regularity, to participate in the estimates.

The first issue when we estimate these two pieces comes from the fact that in (1.6), the spatial cutoff and the frequency cutoff does not obey the scaling like in

[13], [14], there we have good estimates for $P_N^{\pm}$ with a natural spatial cutoff $\phi_{>\frac{1}{N}}$. Indeed, when approaching the origin, the operator $P^{\pm}$ have strong singularities. To get around this problem, we refine the estimates for the operator $P_N^{\pm}$ with spatial cutoff of the form $\phi_{N^{-1-\gamma}<\cdot\le 1}$. It turns out that there will be a loss of N to some power related to γ. This loss of power is not too harmful for us if we make a judicious choice of γ and other relevant parameters in the iteration scheme. We give the detailed discussion on the properties of the operators in Section 3.

Having the operator estimate in hand, we then estimate the contribution from the in-out wave by chopping the t-integration into different pieces. Since the stationary phase point moves with time t at speed N, the contribution from the large time piece is presumably fine due to the decay property of radial functions.

It turns out that the most troublesome term is the following local piece

$$\Big\|\phi_{N^{-1-\gamma}<\cdot\le 1}\int_0^{\frac{1}{N^{2-\sigma}}} P_N^{+}e^{-i\tau\Delta}\phi_{\le 1}F(u)(\tau)d\tau\Big\|_{L_x^2}. \tag{1.7}$$

Here $0<\sigma<2$ is a small constant to be specified later in the proof. One observation from the expression (1.7) is that it is spatially localized, which suggests that the additional regularity should come from some sort of smoothing estimate.

The classical smoothing estimate [18, 10, 5] asserts that the linear propagation gain half derivative locally. This is crucial to study the NLS containing first order derivatives. In our setting, since we are considering the spherically symmetric functions, we develop the following global smoothing estimate:

$$\||x|^{(d-1)/2}|\nabla|^{1/2}e^{it\Delta}u_0\|_{L_x^\infty L_t^2(\mathbb{R}^d\times\mathbb{R})}\lesssim\|u_0\|_{L_x^2}. \tag{1.8}$$

Using the dual form of this estimate, we can successfully control the term (1.7) and close the argument. The proof of the estimate (1.8) can be found in Section 3.

We make two remarks here. First of all, it is worth pointing out that the strategy here for upgrading the regularity is quite different from the one in [13, 14]. There the argument relies on the fact that the solution is almost periodic modulo scaling, and the solution is uniformly flat. Namely, there exists $N(t)>0$ such that $\{N(t)^{-\frac{d}{2}}u(t,\frac{x}{N(t)})\}$ is precompact in $L_x^2(\mathbb{R}^d)$ *and* $N(t)\le 1$. In our setting, the solution also enjoys such compactness, but there is no a priori control on $N(t)$. Actually, the most difficult case is that $N(t)$ can fluctuate out of any control. Secondly, like in [12], our proof needs the assumption $d\ge 4$ since the nonlinearity $|u|^{4/d}u$ can easily be controlled without knowing other information than $M(u)$ being finite. It is certainly an interesting problem to prove the theorem in lower dimensions.

2. Preliminaries

2.1. **Some notation.** We write $X\lesssim Y$ or $Y\gtrsim X$ to indicate $X\le CY$ for some constant $C>0$. We use $O(Y)$ to denote any quantity X such that $|X|\lesssim Y$. We use the notation $X\sim Y$ whenever $X\lesssim Y\lesssim X$. The fact that these constants depend upon the dimension d will be suppressed. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X\lesssim_u Y$ denotes the assertion that $X\le C_uY$ for some C_u depending on u. We denote by $X\pm$ any quantity of the form $X\pm\epsilon$ for any $\epsilon>0$.

We use the 'Japanese bracket' convention $\langle x\rangle:=(1+|x|^2)^{1/2}$.

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\Big)^{q/r} \, dt\Big)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I\times\mathbb{R}^d$. When $q=r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

Throughout this paper, we will use $\phi \in C^\infty(\mathbb{R}^d)$ be a radial bump function supported in the ball $\{x\in\mathbb{R}^d : |x| \le \frac{25}{24}\}$ and equal to one on the ball $\{x\in\mathbb{R}^d : |x|\le 1\}$. For any constant $C>0$, we denote $\phi_{\le C}(x) := \phi\big(\frac{x}{C}\big)$ and $\phi_{>C} := 1-\phi_{\le C}$.

2.2. **Basic harmonic analysis.** For each number $N>0$, we define the Fourier multipliers

$$\widehat{P_{\le N} f}(\xi) := \phi_{\le N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_{>N} f}(\xi) := \phi_{>N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_N f}(\xi) := (\phi_{\le N} - \phi_{\le N/2})(\xi)\hat{f}(\xi)$$

and similarly $P_{<N}$ and $P_{\ge N}$. We also define

$$P_{M<\cdot\le N} := P_{\le N} - P_{\le M} = \sum_{M<N'\le N} P_{N'}$$

whenever $M<N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. As P_N is not truly a projection, $P_N^2 \ne P_N$, we will occasionally need to use fattened Littlewood-Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{2.1}$$

These obey $P_N\tilde{P}_N = \tilde{P}_N P_N = P_N$.

Like all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many many times, including

Lemma 2.1 (Bernstein estimates)**.** *For* $1\le p\le q\le\infty$*,*

$$\big\||\nabla|^{\pm s} P_N f\big\|_{L_x^p} \sim N^{\pm s}\|P_N f\|_{L_x^p},$$
$$\|P_{\le N} f\|_{L_x^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|P_{\le N} f\|_{L_x^p},$$
$$\|P_N f\|_{L_x^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|P_N f\|_{L_x^p}.$$

While it is true that spatial cutoffs do not commute with Littlewood-Paley operators, we still have the following result.

Lemma 2.2 (Mismatch estimates in real space)**.** *Let* $R, N>0$*. Then*

$$\big\|\phi_{>R}\nabla P_{\le N}\phi_{\le\frac{R}{2}} f\big\|_{L_x^p} \lesssim_m N^{1-m}R^{-m}\|f\|_{L_x^p}$$
$$\big\|\phi_{>R} P_{\le N}\phi_{\le\frac{R}{2}} f\big\|_{L_x^p} \lesssim_m N^{-m}R^{-m}\|f\|_{L_x^p}$$

for any $1\le p\le\infty$ *and* $m\ge 0$*.*

Proof. We will only prove the first inequality; the second follows similarly.

It is not hard to obtain kernel estimates for the operator $\phi_{>R}\nabla P_{\leq N}\phi_{\leq \frac{R}{2}}$. Indeed, an exercise in non-stationary phase shows

$$\left|\phi_{>R}\nabla P_{\leq N}\phi_{\leq R/2}(x,y)\right| \lesssim N^{d+1-2k}|x-y|^{-2k}\phi_{|x-y|>\frac{R}{2}}$$

for any $k \geq 0$. An application of Young's inequality yields the claim. □

Similar estimates hold when the roles of the frequency and physical spaces are interchanged. The proof is easiest when working on L^2_x, which is the case we will need; nevertheless, the following statement holds on L^p_x for any $1 \leq p \leq \infty$.

Lemma 2.3 (Mismatch estimates in frequency space). *For $R > 0$ and $N, M > 0$ such that* $\max\{N, M\} \geq 4\min\{N, M\}$,

$$\left\|P_N\phi_{\leq R}P_M f\right\|_{L^2_x} \lesssim_m \max\{N,M\}^{-m}R^{-m}\|f\|_{L^2_x}$$
$$\left\|P_N\phi_{\leq R}\nabla P_M f\right\|_{L^2_x} \lesssim_m M\max\{N,M\}^{-m}R^{-m}\|f\|_{L^2_x}.$$

for any $m \geq 0$. The same estimates hold if we replace $\phi_{\leq R}$ by $\phi_{>R}$.

Proof. The first claim follows from Plancherel's Theorem and Lemma 2.2 and its adjoint. To obtain the second claim from this, we write

$$P_N\phi_{\leq R}\nabla P_M = P_N\phi_{\leq R}P_M\nabla\tilde{P}_M$$

and note that $\|\nabla\tilde{P}_M\|_{L^2_x\to L^2_x} \lesssim M$. □

We will need the following radial Sobolev embedding to exploit the decay property of a radial function. For the proof and the more complete version, one refers to see [21].

Lemma 2.4 (Radial Sobolev embedding, [21]). *Let dimension $d \geq 2$. Let $s > 0$, $\alpha > 0$, $1 < p, q < \infty$ obeys the scaling restriction: $\alpha + s = d(\frac{1}{q} - \frac{1}{p})$. Then the following holds:*

$$\||x|^\alpha f\|_{L^p_x} \lesssim \||\nabla|^s f\|_{L^q_x},$$

where the implicit constant depends on s, α, p, q. When $p = \infty$, we have

$$\||x|^{(d-1)/2}P_N f\|_{L^\infty_x} \lesssim N^{1/2}\|P_N f\|_{L^2_x}.$$

We will need the following fractional chain rule lemma.

Lemma 2.5 (Fractional chain rule for a C^1 function, [4][19][24]). *Let $F \in C^1(\mathbb{C})$, $\sigma \in (0,1)$, and $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then we have*

$$\||\nabla|^\sigma F(u)\|_{L^r_x} \lesssim \|F'(u)\|_{L^{r_1}_x}\||\nabla|^\sigma u\|_{L^{r_2}_x}.$$

For a proof of the above lemma, see [4, 19] and [24].

2.3. **Strichartz estimates.** The free Schrödinger flow has the explicit expression

$$e^{it\Delta}f(x) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}f(y)dy,$$

We will frequently use the standard Strichartz estimate.

Lemma 2.6 (Strichartz). *Let $d \geq 2$. Let I be an interval, $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^d)$ and $F \in L^{2(d+2)/(d+4)}_{t,x}(I \times \mathbb{R}^d)$. Then, the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} F(t')\, dt'$$

obeys the estimate

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^{2(d+2)/d}_{t,x}} \lesssim \|u_0\|_{L^2_x} + \|F\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. See, for example, [6, 20]. For the endpoint see [9]. □

We will also need a weighted Strichartz estimate, which exploits heavily the spherical symmetry in order to obtain spatial decay.

Lemma 2.7 (Weighted Strichartz, [13, 14]). *Let $d \geq 2$. Let I be an interval, $t_0 \in I$, and let $F : I \times \mathbb{R}^d \to \mathbb{C}$ be spherically symmetric. Then,*

$$\Big\| \int_{t_0}^{t} e^{i(t-t')\Delta} F(t')\, dt' \Big\|_{L^2_x} \lesssim \big\| |x|^{-\frac{2(d-1)}{q}} F \big\|_{L_t^{\frac{q}{q-1}} L_x^{\frac{2q}{q+4}} (I \times \mathbb{R}^d)}$$

for $4 \leq q \leq \infty$.

3. Smoothing estimate and the refined operator estimates

3.1. **Kato smoothing for radial solutions.** Kato smoothing estimate [18, 10, 5] plays an important role in studying the wellposedness for nonlinear Schrödinger equation with derivative. In one spatial dimension, the typical Kato smoothing takes the form:

$$\| |\nabla|^{1/2} e^{it\partial_{xx}} u_0 \|_{L^\infty_x L^2_t(\mathbb{R}\times\mathbb{R})} \lesssim \|u_0\|_{L^2_x}. \tag{3.1}$$

The smoothing estimate in high dimensions involves the spatial localization or a decay weight. We will not discuss in detail here. In this paper, we will need the following smoothing estimate for radial functions, which can be viewed as an extension of the one dimensional estimate (3.1).

Lemma 3.1 (Kato smoothing for radial functions with $d \geq 2$). *Let the dimension $d \geq 2$. Then for any radial function f we have*

$$\big\| |x|^{\frac{d-1}{2}} |\nabla|^{1/2} e^{it\Delta} f \big\|_{L^\infty_x L^2_t(\mathbb{R}^d \times \mathbb{R})} \lesssim \|f\|_2.$$

Proof. By passing to radial coordinates, we can write

$$|x|^{\frac{d-1}{2}} |\nabla|^{1/2} e^{it\Delta} f = |x|^{\frac{d-1}{2}} \int_0^\infty k^{1/2} e^{-itk^2} \hat{f}(k) k^{d-1} \Big(\int_{|\omega|=1} e^{ik|x|\omega_1} d\sigma(\omega) \Big) dk, \tag{3.2}$$

where $\hat{f}$ is the Fourier transform of the function f and $d\sigma(\omega)$ is the the surface measure on S^{d-1}. Since the Fourier transform of a radial function is still radial, we can slightly abuse the notation $\hat{f}(k)$ to denote the Fourier transform of f. Now consider the function

$$h(\rho) := \int_{|\omega|=1} e^{i\rho\omega_1} d\sigma(\omega).$$

It is clear that

$$\rho^{\frac{d-2}{2}} h(\rho) = J_{(d-2)/2}(\rho),$$

where $J_{(d-2)/2}$ is the usual Bessel function of order $\frac{d-2}{2}$. Then by using the asymptotics for Bessel functions, it is not difficult to see that

$$\sup_{\rho>0} \rho^{(d-1)/2}|h(\rho)| \le C_1 < \infty, \tag{3.3}$$

where C_1 is a constant depending only on the dimension d. By Plancherel, one can show that for any one-dimensional function F, we have

$$\Big\| \int_0^\infty e^{-itk^2} F(k)dk \Big\|_{L_t^2}^2 = \frac{1}{2}\int_0^\infty |F(k)|^2 \frac{dk}{k}. \tag{3.4}$$

Now by (3.2), (3.3), (3.4), we obtain

$$\begin{aligned}
\big\||x|^{(d-1)/2}|\nabla|^{1/2}e^{it\Delta}f\big\|_{L_t^2}^2 &= \frac{1}{2}\int_0^\infty |\hat f(k)|^2 \cdot |x|^{d-1}\cdot k^{2(d-1)}\cdot |h(k|x|)|^2 dk \\
&\le \frac{1}{2}\int_0^\infty |\hat f(k)|^2 k^{d-1} dk \cdot \Big(\sup_{\rho>0}\rho^{\frac{d-1}{2}}|h(\rho)|\Big)^2 \\
&\le C_1^2 \|f\|_2^2.
\end{aligned}$$

The lemma is proved. □

3.2. **The in-out decomposition and refined operator estimates.** We will need an incoming/outgoing decomposition; we will use the one developed in [13, 14]. As there, we define operators $P^\pm$ by

$$[P^\pm f](r) := \tfrac{1}{2}f(r) \pm \tfrac{i}{\pi}\int_0^\infty \frac{r^{2-d}\, f(\rho)\,\rho^{d-1}\,d\rho}{r^2-\rho^2},$$

where the radial function $f : \mathbb{R}^d \to \mathbb{C}$ is written as a function of radius only. We will refer to P^+ is the projection onto outgoing spherical waves; however, it is not a true projection as it is neither idempotent nor self-adjoint. Similarly, P^- plays the role of a projection onto incoming spherical waves; its kernel is the complex conjugate of the kernel of P^+ as required by time-reversal symmetry.

3.3. **The two-dimensional case.** For $N > 0$ let $P_N^\pm$ denote the product $P^\pm P_N$ where P_N is the Littlewood-Paley projection. We record the following properties of $P^\pm$ from [13, 14]:

Proposition 3.2 (Properties of $P^\pm$, [13, 14])**.**

(i) *$P^+ + P^-$ represents the projection from L^2 onto $L^2_{\rm rad}$.*
(ii) *$P^\pm$ are bounded on $L^2(\mathbb{R}^2)$.*
(iii) *For $|x| \gtrsim N^{-1}$ and $t \gtrsim N^{-2}$, the integral kernel obeys*

$$\big|[P_N^\pm e^{\mp it\Delta}](x,y)\big| \lesssim \begin{cases} (|x||y|)^{-1/2}|t|^{-1/2} & |y|-|x|\sim Nt \\ \frac{N^2}{(N|x|)^{1/2}\langle N|y|\rangle^{1/2}}\big\langle N^2t + N|x| - N|y|\big\rangle^{-m} & \text{otherwise} \end{cases}$$

for all $m \ge 0$.

(iv) *For $|x| \gtrsim N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys*

$$\big|[P_N^\pm e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^2}{(N|x|)^{1/2}\langle N|y|\rangle^{1/2}}\big\langle N|x| - N|y|\big\rangle^{-m}$$

for any $m \ge 0$.

For a proof of the above proposition, see [13, 14].

We will also need the following Proposition concerning the properties of $P^{\pm}$ in the small x regime (i.e. $|x| \lesssim N^{-1}$) where Bessel functions have logarithmic singularities. More precisely, we have the following result.

Proposition 3.3 (Properties of $P^{\pm}$, small x regime, [16]). *Let dimension $d = 2$.*

(i) *For $t \gtrsim N^{-2}$, $N^{-3} \lesssim |x| \lesssim N^{-1}$, $|y| \ll Nt$ or $|y| \gg Nt$, the integral kernel satisfies*

$$\big|[P_N^{\pm} e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^2 \log N}{\langle N|y|\rangle^{1/2}} \langle N^2 t + N|y|\rangle^{-m}, \quad \forall\, m \geq 0.$$

(ii) *For $t \gtrsim N^{-2}$, $N^{-3} \lesssim |x| \lesssim N^{-1}$, $|y| \sim Nt$, the integral kernel satisfies*

$$\big|[P_N^{\pm} e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^2 \log N}{\langle N|y|\rangle^{1/2}}.$$

For a proof of the above proposition, see [16].

3.4. **The case $d \geq 3$.** The next lemma allows us to bound the operator $P_N^{\pm}$ slightly below the $|x| \sim 1/N$ barrier, i.e. in the regime $\frac{1}{N^{1+\gamma}} \leq |x| \leq \frac{1}{N}$ for some $\gamma > 0$. The price to pay is a polynomial growth factor in N.

Lemma 3.4. *Let the dimension $d \geq 3$. Fix $N \gtrsim 1$ and $\gamma > 0$. For any spherically symmetric function $f \in L_x^2(\mathbb{R}^d)$,*

$$\big\|P^{\pm} P_N f\big\|_{L_x^2(\frac{1}{N^{1+\gamma}} \leq |x| \leq \frac{1}{N})} \lesssim \begin{cases} N^{\frac{(d-4)\gamma}{2}} \cdot \big\|f\big\|_{L_x^2(\mathbb{R}^d)}, & \text{if } d \geq 5, \\ \langle \log N\rangle^{1/2} \cdot \big\|f\big\|_{L_x^2(\mathbb{R}^d)}, & \text{if } d = 4, \\ \big\|f\big\|_{L_x^2(\mathbb{R}^d)}, & \text{if } d = 3, \end{cases}$$

where the implied constant depends only on γ and d. Here $\langle\cdot\rangle$ is the Japanese bracket.

Proof. We shall only prove the inequality for P^+. The result for P^- is similar (or one can use the fact $P^+ + P^-$ acts as an identity on $L^2_{\rm rad}(\mathbb{R}^d)$). By the definition of P^+, we have

$$\big\|P^{\pm} P_N f\big\|^2_{L_x^2(\frac{1}{N^{1+\gamma}} \leq |x| \leq \frac{1}{N})} = \int_{\frac{1}{N^{1+\gamma}}}^{\frac{1}{N}} \Big| \int_0^\infty H^{(1)}_{\frac{d-2}{2}}(kr) \hat{f}(k) k^{d/2} \psi(\tfrac{k}{N}) dk\Big|^2 r dr. \quad (3.5)$$

Since $k \sim N$, $\frac{1}{N^{1+\gamma}} \leq r \leq \frac{1}{N}$, $\frac{1}{N^\gamma} \leq kr \lesssim 1$, we have

$$\big|H^{(1)}_{(d-2)/2}(kr)\big| \lesssim (kr)^{-(d-2)/2}.$$

Therefore, by Cauchy-Schwartz, we obtain

$$\begin{aligned} \text{RHS of (3.5)} &\lesssim \int_{\frac{1}{N^{1+\gamma}}}^{\frac{1}{N}} r^{3-d} dr \cdot N^{2-d} \cdot \int_0^\infty |\hat{f}(k)|^2 k^{d-1} dk \cdot \int_0^\infty |\psi(\frac{k}{N})|^2 k dk \\ &\lesssim \int_{\frac{1}{N^{1+\gamma}}}^{\frac{1}{N}} r^{3-d} dr \cdot N^{4-d} \cdot \|f\|^2_{L_x^2(\mathbb{R}^d)}. \end{aligned} \quad (3.6)$$

Now if $d \geq 5$, then

$$\int_{\frac{1}{N^{1+\gamma}}}^{\frac{1}{N}} r^{3-d} dr \lesssim N^{(1+\gamma)(d-4)}$$

and RHS of (3.6) $\lesssim N^{(d-4)\gamma}\|f\|^2_{L^2_x(\mathbb{R}^d)}$.

If $d=4$, then

$$\int_{\frac{1}{N^{1+\gamma}}}^{\frac{1}{N}} r^{3-d}dr \lesssim \log N$$

and RHS of (3.6) $\lesssim (\log N)\cdot\|f\|^2_{L^2_x(\mathbb{R}^d)}$.

If $d=3$, then clearly

$$\text{RHS of (3.6) } \lesssim \|f\|^2_{L^2_x(\mathbb{R}^d)}.$$

The lemma is proved. □

In the next lemma we shall give bounds of some integrals needed later in the kernel estimates. To fix notations, we assume $\tilde g_1$, $\tilde g_2$ are one-dimensional functions such that

$$\Big|\frac{d^m \tilde g_i(r)}{dr^m}\Big| \lesssim 1, \quad \forall 0<r\lesssim 1,\ m\ge 0,\ i=1,2, \tag{3.7}$$

and $a(\cdot)$ is a one-dimensional function such that

$$\Big|\frac{d^m a(r)}{dr^m}\Big| \lesssim \langle r\rangle^{-m}, \quad \forall r\ge 0,\ m\ge 0, \tag{3.8}$$

where $\langle\cdot\rangle$ is the Japanese bracket. We shall denote by ψ the multiplier function from the Littlewood-Paley projection. With these notations, we state the following lemma.

Lemma 3.5. *Let $N\gtrsim 1$ be a dyadic number. Assume $0<c_1$, $c_2\lesssim \frac{1}{N}$ are two fixed numbers. Then for any $t\gtrsim N^{-2}$, we have*

$$\int_0^\infty \tilde g_1(kc_1)\tilde g_2(kc_2)e^{itk^2}\psi(\tfrac{k}{N})dk \lesssim N\cdot\langle N^2t+Nc_2\rangle^{-m}, \quad \forall\, m\ge 0. \tag{3.9}$$

If c_3 is a number such that $\frac{1}{N}\lesssim c_3\ll Nt$, then

$$\int_0^\infty \tilde g_1(kc_1)\frac{a(kc_3)}{\langle kc_3\rangle^{1/2}}e^{i(tk^2\pm c_3k)}dk \lesssim \frac{N}{\langle Nc_3\rangle^{1/2}}\cdot\langle N^2t+Nc_3\rangle^{-m}, \quad \forall\, m\ge 0. \tag{3.10}$$

Similarly if c_4 is such that $c_4\gg Nt$, then

$$\int_0^\infty \tilde g_1(kc_1)\frac{a(kc_4)}{\langle kc_4\rangle^{1/2}}e^{i(tk^2\pm c_4k)}dk \lesssim \frac{N}{\langle Nc_4\rangle^{1/2}}\cdot\langle N^2t+Nc_4\rangle^{-m}, \quad \forall\, m\ge 0. \tag{3.11}$$

Proof. All the estimates (3.9)–(3.11) essentially follow from integrating by parts. Let $k=N\tilde k$, then $\tilde k\sim 1$ due to the cut-off function ψ. Change k to $N\tilde k$ in (3.9)–(3.11). Note by (3.7) and the fact that $0<c_1$, $c_2\lesssim\frac{1}{N}$, $N\gtrsim 1$, we have

$$\Big|\frac{d^m}{d\tilde k^m}(\tilde g_i(\tilde k c_i N))\Big| \lesssim 1, \quad \forall\,\tilde k\sim 1,\ m\ge 0,\ i=1,2.$$

Also by (3.8) and the assumptions on c_3, c_4, we have

$$\Big|\frac{d^m}{d\tilde k^m}a(\tilde k c_3 N)\Big| + \Big|\frac{d^m}{d\tilde k^m}a(\tilde k c_4 N)\Big| \lesssim 1, \quad \forall\,\tilde k\sim 1,\ m\ge 0.$$

The desired estimates (3.9)–(3.11) now follow from integration by parts and the above derivative estimates on $\tilde g_1$, $\tilde g_2$, a. □

Proposition 3.6 (Properties of $P^{\pm}$, small x regime)**.** *Let dimension $d\ge 3$ and assume $\gamma>0$. Let $N\gtrsim 1$ be a dyadic number.*

(i) *For* $t \gtrsim N^{-2}$, $\frac{1}{N^{1+\gamma}} \lesssim |x| \lesssim N^{-1}$, $|y| \ll Nt$ *or* $|y| \gg Nt$, *the integral kernel satisfies*

$$\big|[P_N^{\pm} e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^{(1+\gamma)(d-2)+2}}{\langle N|y|\rangle^{1/2}} \langle N^2 t + N|y|\rangle^{-m}, \quad \forall\, m \geq 0.$$

(ii) *For* $t \gtrsim N^{-2}$, $\frac{1}{N^{1+\gamma}} \lesssim |x| \lesssim N^{-1}$, $|y| \sim Nt$, *the integral kernel satisfies*

$$\big|[P_N^{\pm} e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^{(1+\gamma)(d-2)+2}\log N}{\langle N|y|\rangle^{1/2}}.$$

Proof. We shall only provide the proof for $P_N^{+} e^{-it\Delta}$ since the other kernel is its complex conjugate. The first claim is an exercise in stationary phase. By Fourier transform we have the following formula for the kernel

$$\begin{aligned} &[P_N^{+} e^{-it\Delta}](x,y) \\ &= \tfrac{1}{2}\big(|x||y|\big)^{-(d-2)/2} \int_0^\infty H^{(1)}_{(d-2)/2}(k|x|) J_{(d-2)/2}(k|y|) e^{itk^2} \psi\big(\tfrac{k}{N}\big)\, k\, dk \end{aligned} \tag{3.12}$$

where ψ is the multiplier function from the Littlewood–Paley projection. First note that

$$H^{(1)}_{(d-2)/2}(r) = J_{(d-2)/2}(r) + iY_{(d-2)/2}(r). \tag{3.13}$$

Since $k \sim N$, $\frac{1}{N^{1+\gamma}} \lesssim |x| \lesssim \frac{1}{N}$, we have $r = k|x|$ satisfies $\frac{1}{N^\gamma} \lesssim r \lesssim 1$. Now using the expansion

$$J_{(d-2)/2}(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})} \cdot (\tfrac{r}{2})^{2m+\frac{d-2}{2}},$$

we can write

$$r^{-(d-2)/2} J_{(d-2)/2}(r) = \tilde{g}_1(r), \tag{3.14}$$

where

$$\big|\frac{\partial^m \tilde{g}_1(r)}{\partial r^m}\big| \lesssim 1, \quad \forall\, m \geq 0,\ r \lesssim 1.$$

Here the factor $r^{-(d-2)/2}$ in (3.14) is needed since the dimension d may possibly be a odd integer. To treat the function $Y_{\frac{d-2}{2}}$ in the regime $\frac{1}{N^\gamma} \lesssim r \lesssim 1$, we discuss two cases. If the dimension d is even, then we use the series

$$\begin{aligned} Y_{(d-2)/2}(r) = &-\frac{(\frac{r}{2})^{-(d-2)/2}}{\pi} \sum_{k=0}^{\frac{d-4}{2}} \frac{(\frac{d-4}{2}-k)!}{k!} \cdot \big(\frac{1}{4}r^2\big)^k + \frac{2}{\pi}\log(\tfrac{r}{2}) J_{(d-2)/2}(r) \\ &- \frac{(\frac{r}{2})^{\frac{d-2}{2}}}{\pi} \sum_{k=0}^{\infty} \big(\psi_0(k+1) + \psi_0(n+k+1)\big) \cdot \frac{(-\frac{1}{4}r^2)^k}{k!(\frac{d-2}{2}+k)!}, \end{aligned}$$

where ψ_0 is the digamma function defined by

$$\psi_0(n) = -\gamma_0 + \sum_{k=1}^{n-1} \frac{1}{k},$$

and γ_0 is the Euler-Masheroni constant. It follows easily that

$$Y_{(d-2)/2}(r) = r^{-(d-2)/2}\tilde{g}_2(r) + \log r \cdot r^{\frac{d-2}{2}} \tilde{g}_3(r) + r^{\frac{d-2}{2}} \cdot \tilde{g}_4(r), \tag{3.15}$$

where

$$\big|\frac{\partial^m \tilde{g}_j(r)}{\partial r^m}\big| \lesssim 1, \quad \forall\, m \geq 0,\ r \lesssim 1,\ j = 2,3,4.$$

Now if the dimension d is odd, then we use the formula

$$Y_{\frac{d-2}{2}}(r) = Y_{\frac{d-3}{2}+\frac{1}{2}}(r) = -\frac{2\cdot(\frac{r}{2})^{\frac{d-2}{2}}}{\sqrt{\pi\cdot(\frac{d-3}{2})!}}\cdot\Big(1+\frac{d^2}{dr^2}\Big)^{\frac{d-3}{2}}\big(\frac{\cos r}{r}\big).$$

It follows that we can write

$$r^{-(d-2)/2}\cdot Y_{(d-2)/2}(r) = r^{-(d-2)}\tilde{g}_5(r), \tag{3.16}$$

where

$$\big|\frac{\partial^m \tilde{g}_5(r)}{\partial r^m}\big| \lesssim 1, \quad \forall\, m\geq 0,\, r\lesssim 1.$$

Next we also use the following information about Bessel functions in the regime $r\gtrsim 1$:

$$J_{(d-2)/2}(r) = \frac{a(r)e^{ir}}{\langle r\rangle^{1/2}} + \frac{\bar{a}(r)e^{-ir}}{\langle r\rangle^{1/2}}\,, \tag{3.17}$$

where $a(r)$ obeys the symbol estimates

$$\Big|\frac{\partial^m a(r)}{\partial r^m}\Big| \lesssim \langle r\rangle^{-m} \quad \text{for all } m\geq 0,\, r\gtrsim 1$$

Finally substitute (3.13), (3.14), (3.15) (when d is even), (3.16) (when d is odd), (3.17) into (3.12). Consider three regimes of y: $1/N\lesssim |y|\ll N|t|$, $|y|\lesssim 1/N$, $|y|\gg N|t|$ and use different asymptotics of the Bessel function in these regimes. Note also that the singular part of the Hankel function near $r=0$ adds only a power of N due to our lower bound on x. It is then easy to see that a stationary phase point can only occur when $|y|\sim Nt$. Since we assume $|y|\ll Nt$ or $|y|\gg Nt$, integrating by parts and using Lemma 3.5 yield the first claim. The second claim follows from a trivial L^1 estimate. We omit the details. □

4. The proof of Theorem 1.5

We first explain why it suffices for us to show that such two way non-scattering solution with minimal mass must be regular: $u_0\in H^1_x$. Indeed, if $u_0\in H^1_x$ and the corresponding solution blows up at finite time, according to Merle's result [17], we know it must scatter one way which contradicts our assumption. Then the solution must be global, here a direct application of Theorem 1.4 immediately yields the coincidence of the solution with SW up to symmetries.

Since the following proof of upgrading the regularity works for all two-way non-scattering solutions, for the sake of simplicity, we assume the solution is global. The discussion of the finite time blowup solutions is only notationally more complicated.

To begin with, we recall the following result. The proof of this result is implicitly contained in [12].

Lemma 4.1 (Regularity of solutions away from the origin, [12])**.** *Let $d\geq 4$. Let $u_0\in L^2_x(\mathbb{R}^d)$ be spherically symmetric and $M(u_0)=M(Q)$. Let $u(t,x)$ be the corresponding solution such that it does not scatter in both time directions:*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,0]\times\mathbb{R}^d)} = \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\infty)\times\mathbb{R}^d)} = \infty.$$

Then there exists $\epsilon=\epsilon(d)>0$ such that

$$\|\phi_{>1}P_N u(t)\|_{L^2_x}\lesssim N^{-1-\epsilon},\ \forall N\geq 1,\ t\in\mathbb{R}.$$

In particular,

$$\|\phi_{>1}\nabla u(t)\|_{L_x^2} \lesssim 1, \ \forall t \in \mathbb{R}.$$

Now we use this information to upgrade the regularity of the initial data. To this end, we seek for the refined decay estimate for single frequency $P_N u_0$ with $N \geq 1$. Let $\gamma > 0$ be a small parameter to be chosen later, we use triangle inequality to bound

$$\|P_N u_0\|_{L_x^2} \lesssim \|\phi_{\leq N^{-1-\gamma}} P_N u_0\|_{L_x^2} \tag{4.1}$$
$$+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N u_0\|_{L_x^2} \tag{4.2}$$
$$+ \|\phi_{>1} P_N u_0\|_{L_x^2}. \tag{4.3}$$

First of all, Lemma 4.1 yields that (4.3) $\lesssim N^{-1-\epsilon}$. Next, using Hölder and Bernstein, (4.1) can be controlled rather easily:

$$(4.3) \lesssim N^{\frac{d}{2}(-1-\gamma)} \|P_N u_0\|_{L_x^\infty} \lesssim N^{-s-\frac{d}{2}\gamma} \|u_0\|_{H_x^s} \lesssim N^{-s-\frac{d}{2}\gamma}.$$

The task now is to estimate (4.2), for which we will use the in-out decomposition and the improved Duhamel formula as we explain now. Since the solution u does not scatter in both time directions and has minimal mass, according to [22, 14] [1] we have

$$u(t) = \lim_{T\to\infty} -i \int_t^T e^{i(t-s)\Delta} F(u(s))ds \tag{4.4}$$
$$= \lim_{T\to-\infty} i \int_T^t e^{i(t-s)\Delta} F(u(s))ds, \tag{4.5}$$

where $F(u) = |u|^{4/d}u$ and the limit is understood in the weak L_x^2 sense. Using the in-out decomposition and (4.4), (4.5), we estimate

$$\begin{aligned}(4.2) &\leq \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ u_0\|_{L_x^2} + \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^- u_0\|_{L_x^2} \\ &\lesssim \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ \int_0^\infty e^{-i\tau\Delta} F(u(\tau))d\tau\|_{L_x^2} \quad (4.6)\\ &+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^- \int_0^\infty e^{i\tau\Delta} F(u(-\tau))d\tau\|_{L_x^2}. \quad (4.7)\end{aligned}$$

Expression (4.6) and (4.7) will give the same contribution so we only need to estimate one of them. By splitting into different time pieces and introducing spatial cutoffs, we estimate (4.6) as follows

$$(4.6) \lesssim \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ \int_{\frac{1}{N}}^\infty e^{-i\tau\Delta} \phi_{>\frac{N\tau}{2}} F(u(\tau))d\tau\|_{L_x^2} \tag{4.8}$$
$$+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ \int_{\frac{1}{N}}^\infty e^{-i\tau\Delta} \phi_{\leq\frac{N\tau}{2}} F(u(\tau))d\tau\|_{L_x^2} \tag{4.9}$$
$$+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ \int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} e^{-i\tau\Delta} \phi_{>\frac{N\tau}{2}} F(u(\tau))d\tau\|_{L_x^2} \tag{4.10}$$
$$+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} P_N^+ \int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} e^{-i\tau\Delta} \phi_{\leq\frac{N\tau}{2}} F(u(\tau))d\tau\|_{L_x^2} \tag{4.11}$$

[1] The first reference established the improved Duhamel formula for minimal-mass non-scattering solution in which the scattering wave vanish when the t approaches the maximal life time. The second one identifies $M(Q)$ as the minimal mass within all the spherically symmetric solutions.

$$+\|\phi_{N^{-1-\gamma}<\cdot\le 1}P_N^+\int_0^{\frac{1}{N^{2-\sigma}}} e^{-i\tau\Delta}F(u(\tau))d\tau\|_{L_x^2}, \tag{4.12}$$

where $0<\sigma<2$ is a small constant to be fixed later. We first look at (4.9), (4.11) where the desired decay in N comes from the kernel estimate Lemma 3.3. Let $A=(1+\gamma)(d-2)+2$, then for any $\tau>\frac{1}{N^2}$, $m>0$,

$$\begin{aligned}\Big|\Big(\phi_{N^{-1-\gamma}<\cdot\le 1}P_N^+e^{-i\tau\Delta}\phi_{\le\frac{N\tau}{2}}\Big)(x,y)\Big| &\lesssim_m N^A\langle N^2\tau+N|x|+N|y|\rangle^{-2m}\\ &\lesssim_m N^A|N^2\tau|^{-m}\langle N|x-y|\rangle^{-m}.\end{aligned}$$

Using this and Young's inequality, (4.9), (4.11) can be bounded as follows

$$\begin{aligned}(4.9) &\lesssim_m N^A\int_{\frac{1}{N}}^\infty |N^2\tau|^{-m}\|\langle N|\cdot|\rangle * F(u(\tau))\|_{L_x^2}d\tau\\ &\lesssim_m N^AN^{-2m}\int_{\frac{1}{N}}^\infty \tau^{-m}d\tau\|\langle N|\cdot|\rangle\|_{L_x^{\frac{d}{d-2}}}\|F(u)\|_{L_t^\infty L_x^{\frac{2d}{d+4}}}\\ &\lesssim_m N^{-m+1+A-d}\|u\|_{L_t^\infty L_x^2}^{1+\frac{4}{d}}\\ &\lesssim_m N^{-m+1+A-d}.\end{aligned}$$

Thus, by taking m large enough depending on d,

$$(4.9)\lesssim N^{-10}.$$

Expression (4.11) can be estimated in a similar way:

$$\begin{aligned}(4.11) &\lesssim_m N^A\int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} |N^2\tau|^{-m}\|\langle N|\cdot|\rangle * F(u(\tau))\|_{L_x^2}d\tau\\ &\lesssim_m N^{-2m+A+2-d}\int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}}\tau^{-m}d\tau\\ &\lesssim_m N^{A-(m-1)\sigma-d}.\end{aligned}$$

For (4.8) and (4.10), we will use the weighted Strichartz estimate Lemma 2.7. In what follows we shall only present the details for $d\ge 5$. The case $d=4$ is similar and will be omitted. In dimension $d\ge 5$, from the L_x^2-boundedness of the operator $\phi_{>\frac{1}{N^{1+\gamma}}}P_N^+$ Lemma 3.4, Lemma 2.7 and Lemma 2.6, we have

$$\begin{aligned}(4.8) &\lesssim N^{\frac{\gamma}{2}(d-4)}\Big\|\int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}\tilde P_N\phi_{>\frac{N\tau}{2}}F(u(\tau))d\tau\Big\|_{L_x^2}\\ &\lesssim N^{\frac{\gamma}{2}(d-4)}\Big\|\int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}\tilde P_N\phi_{>\frac{N\tau}{2}}F(u\phi_{>\frac{N\tau}{4}})(\tau)d\tau\Big\|_{L_x^2}\\ &\lesssim N^{\frac{\gamma}{2}(d-4)}\Big(\Big\|\int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}\tilde P_N\phi_{>\frac{N\tau}{2}}P_{\le N/8}F(u\phi_{>\frac{N\tau}{4}})(\tau)d\tau\Big\|_{L_x^2}\\ &\quad+\Big\|\int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}\tilde P_N\phi_{>\frac{N\tau}{2}}P_{>N/8}F(u\phi_{>\frac{N\tau}{4}})(\tau)d\tau\Big\|_{L_x^2}\Big)\\ &\lesssim N^{\frac{\gamma}{2}(d-4)}\Big(\|\tilde P_N\phi_{>\frac{N\tau}{2}}P_{\le N/8}F(u\phi_{>\frac{N\tau}{4}})\|_{L_\tau^1L_x^2([\frac{1}{N},\infty)\times\mathbb{R}^d)} \qquad (4.13)\\ &\quad+\|(N\tau)^{-\frac{2(d-1)}{d}}P_{>N/8}F(u\phi_{>\frac{N\tau}{4}})\|_{L_\tau^{\frac{d}{d-1}}L_x^{\frac{2d}{d+4}}([\frac{1}{N},\infty)\times\mathbb{R}^d)}\Big). \qquad (4.14)\end{aligned}$$

Using the mismatch estimate Lemma 2.3, we can bound (4.13) as

$$\begin{aligned}
(4.13) &\lesssim \|(N^2\tau)^{-11}P_{\leq N}F(u\phi_{>\frac{N\tau}{4}})\|_{L^1_\tau L^2_x([\frac{1}{N},\infty)\times\mathbb{R}^d)}\\
&\lesssim N^{-20}\|\tau^{-11}F(u\phi_{>\frac{N\tau}{4}})\|_{L^1_\tau L_x^{\frac{2d}{d+4}}([\frac{1}{N},\infty)\times\mathbb{R}^d)}\\
&\lesssim N^{-20}\|\tau^{-11}\|_{L^1_\tau([\frac{1}{N},\infty))}\lesssim N^{-10}.
\end{aligned}$$

For (4.14), we use Bernstein estimate and Lemma 4.1 to get

$$\begin{aligned}
(4.14) &\lesssim N^{-\frac{2(d-1)}{d}}\|\tau^{-\frac{2(d-1)}{d}}N^{-1}\|\nabla F(u\phi_{>\frac{N\tau}{4}})\|_{L_x^{\frac{2d}{d+4}}}\|_{L_\tau^{\frac{d}{d-1}}([\frac{1}{N},\infty))}\\
&\lesssim N^{-1-\frac{2(d-1)}{d}}\|\tau^{-\frac{2(d-1)}{d}}\|u\phi_{>\frac{N\tau}{4}}\|_{L^2_x}^{4/d}\|\nabla(u\phi_{>\frac{N\tau}{4}})\|_{L^2_x}\|_{L_\tau^{\frac{d}{d-1}}([\frac{1}{N},\infty))}\\
&\lesssim N^{-1-\frac{2(d-1)}{d}}\|\tau^{-\frac{2(d-1)}{d}}\|_{L_\tau^{d/(d-1)}(\frac{1}{N},\infty)}\\
&\lesssim N^{-1-\frac{d-1}{d}}.
\end{aligned}$$

Therefore, summarizing the two pieces together we have

$$(4.8)\lesssim N^{\frac{\gamma}{2}(d-4)}N^{-1-\frac{d-1}{d}}.$$

Now we look at the piece (4.10) where the uniform kinetic energy estimate Lemma 4.1 is no longer available. Instead, we will use the fact $u_0\in H^s_x$, therefore locally we have the bound

$$\|u\|_{L^\infty_\tau H^s_x([0,1]\times\mathbb{R}^d)}\lesssim_{u_0} 1. \tag{4.15}$$

Using this information, Lemma 3.4, Lemma 2.3 and Lemma 2.7, we control (4.10) as

$$\begin{aligned}
(4.10) &\lesssim N^{\frac{\gamma}{2}(d-4)}\Big\|\tilde{P}_N\int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} e^{-i\tau\Delta}\phi_{>\frac{N\tau}{2}}F(u(\tau))d\tau\Big\|_{L^2_x}\\
&\lesssim N^{\frac{\gamma}{2}(d-4)}\Big(\Big\|\tilde{P}_N\int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} e^{-i\tau\Delta}\phi_{>\frac{N\tau}{2}}P_{\leq\frac{N}{8}}F(u(\tau))d\tau\Big\|_{L^2_x}\\
&\quad+\Big\|\tilde{P}_N\int_{\frac{1}{N^{2-\sigma}}}^{\frac{1}{N}} e^{-i\tau\Delta}\phi_{>\frac{N\tau}{2}}P_{>N/8}F(u(\tau))d\tau\Big\|_{L^2_x}\Big)\\
&\lesssim N^{\frac{\gamma}{2}(d-4)}\Big(\|\tilde{P}_N\phi_{>\frac{N\tau}{2}}P_{\leq N/8}F(u)\|_{L^1_\tau L^2_x([\frac{1}{N^{2-\sigma}},\frac{1}{N}]\times\mathbb{R}^d)}\\
&\quad+\||N\tau|^{-\frac{2(d-1)}{d}}P_{>\frac{N}{8}}F(u(\tau))\|_{L_\tau^{\frac{d}{d-1}}L_x^{\frac{2d}{d+4}}([\frac{1}{N^{2-\sigma}},\frac{1}{N}]\times\mathbb{R}^d)}\Big)\\
&\lesssim_m N^{\frac{\gamma}{2}(d-4)}\Big(\||N^2\tau|^{-m}\|P_{\leq N/8}F(u)\|_{L^2_x}\|_{L^1_\tau([\frac{1}{N^{2-\sigma}},\frac{1}{N}])}\\
&\quad+\||N\tau|^{-\frac{2(d-1)}{d}}N^{-s}\||\nabla|^sP_{>N/8}F(u)\|_{L_x^{\frac{2d}{d+4}}}\|_{L_\tau^{\frac{d}{d-1}}([\frac{1}{N^{2-\sigma}},\frac{1}{N}])}\\
&\lesssim_{m,u_0} N^{\frac{\gamma}{2}(d-4)}(N^{2-2m}\|\tau^{-m}\|_{L^1_\tau([\frac{1}{N^{2-\sigma}},\frac{1}{N}])}\\
&\quad+N^{-s-\frac{2(d-1)}{d}}\|\tau^{-\frac{2(d-1)}{d}}\|_{L_\tau^{\frac{d}{d-1}}([\frac{1}{N^{2-\sigma}},\frac{1}{N}])}\Big)\\
&\lesssim_{m,u_0} N^{\frac{\gamma}{2}(d-4)}(N^{-\sigma(m-1)}+N^{-s-\frac{d-1}{d}\sigma}).
\end{aligned}$$

Finally, we give the estimate of (4.12) which can firstly be trivially bounded as

$$(4.12) \lesssim \|\phi_{N^{-1-\gamma}<\cdot\leq 1} \int_0^{\frac{1}{N^{2-\sigma}}} P_N^+ e^{-i\tau\Delta} \phi_{>1} \tilde{P}_N F(u(\tau)) d\tau\|_{L_x^2} \tag{4.16}$$

$$+ \|\phi_{N^{-1-\gamma}<\cdot\leq 1} \int_0^{\frac{1}{N^{2-\sigma}}} P_N^+ e^{-i\tau\Delta} \phi_{\leq 1} \tilde{P}_N F(u(\tau)) d\tau\|_{L_x^2}. \tag{4.17}$$

For (4.16), we use the L_x^2 boundedness Lemma 3.4, weighted Strichartz Lemma 2.7, Bernstein and local estimate (4.15) to get

$$\begin{aligned}
(4.16) &\lesssim N^{\frac{\gamma}{2}(d-4)} \| \int_0^{\frac{1}{N^{2-\sigma}}} e^{-i\tau\Delta} \phi_{>1} \tilde{P}_N F(u(\tau)) d\tau\|_{L_x^2} \\
&\lesssim N^{\frac{\gamma}{2}(d-4)} \|\tilde{P}_N F(u)\|_{L_\tau^{\frac{d}{d-1}} L_x^{\frac{2d}{d+4}}([0,\frac{1}{N^{2-\sigma}}]\times\mathbb{R}^d)} \\
&\lesssim N^{\frac{\gamma}{2}(d-4)} N^{-(2-\sigma)\frac{d-1}{d}} N^{-s} \||\nabla|^s F(u)\|_{L_\tau^\infty L_x^{\frac{2d}{d+4}}} \\
&\lesssim_{u_0} N^{\frac{\gamma}{2}(d-4)} N^{-\frac{(2-\sigma)(d-1)}{d}} N^{-s}.
\end{aligned}$$

To estimate (4.17), we will use the duality of the smoothing estimate (1.8) as follows:

$$\| \int_{\mathbb{R}} e^{-i\tau\Delta} |\nabla|^{1/2} f(\tau) d\tau\|_{L_x^2} \lesssim \||x|^{-\frac{d-1}{2}} f\|_{L_x^1 L_t^2(\mathbb{R}^d\times\mathbb{R})}. \tag{4.18}$$

Let $\eta > 0$ be a tiny number to be chosen later, using Lemma 3.4 and (4.18) we have

$$\begin{aligned}
(4.17) &\lesssim N^{\frac{\gamma}{2}(d-4)} N^{-1/2} \| \int_{\mathbb{R}} e^{-i\tau\Delta} |\nabla|^{1/2} (\phi_{\leq 1} \tilde{P}_N F(u(\tau)) \chi_{0<\tau\leq\frac{1}{N^{2-\sigma}}}) d\tau\|_{L_x^2} \\
&\lesssim N^{\frac{\gamma}{2}(d-4)} N^{-1/2} \||x|^{-\frac{d-1}{2}} \phi_{\leq 1} \tilde{P}_N F(u)\|_{L_x^1 L_\tau^2(\mathbb{R}^d\times[0,\frac{1}{N^{2-\sigma}}])} \\
&\lesssim N^{\frac{\gamma}{2}(d-4)} N^{-1/2} \||x|^{\frac{1}{2}-\eta} \tilde{P}_N F(u)\|_{L_{\tau,x}^2([0,\frac{1}{N^{2-\sigma}}]\times\mathbb{R}^d)} \||x|^{-\frac{d}{2}+\eta} \phi_{\leq 1}\|_{L_x^2}. \\
&\lesssim N^{\frac{\gamma}{2}(d-4)-\frac{1}{2}-\frac{2-\sigma}{2}} \||x|^{\frac{1}{2}-\eta} \tilde{P}_N F(u)\|_{L_\tau^\infty L_x^2([0,\frac{1}{N^{2-\sigma}}]\times\mathbb{R}^d)}.
\end{aligned}$$

Now, using the radial Sobolev embedding Lemma 2.4, Bernstein, and (4.15), we bound the $F(u)$ term as

$$\begin{aligned}
\||x|^{\frac{1}{2}-\eta} \tilde{P}_N F(u)\|_{L_x^2} &\lesssim \||\nabla|^\eta \tilde{P}_N F(u)\|_{L_x^{\frac{2d}{d+1}}} \\
&\lesssim N^\eta N^{d(\frac{d^2+4d-8s}{2d^2}-\frac{d+1}{2d})} \|\tilde{P}_N F(u)\|_{L_x^{\frac{2d^2}{d^2+4d-8s}}} \\
&\lesssim N^{\eta+\frac{3d-8s}{2d}} N^{-s} \||\nabla|^s u\|_{L_x^2} \|u\|_{L_x^{\frac{2d}{d-2s}}}^{4/d} \\
&\lesssim_{u_0} N^{\eta+\frac{3}{2}-\frac{4s}{d}-s}.
\end{aligned}$$

Plugging in this estimate back to the estimate of (4.17) we have

$$(4.17) \lesssim_{u_0} N^{\frac{\gamma}{2}(d-4)+\eta+\frac{\sigma}{2}-\frac{4s}{d}-s}.$$

Combining the estimate for (4.16) and (4.17) together gives the final estimate of (4.12):

$$(4.12) \lesssim_{u_0} N^{\frac{\gamma}{2}(d-4)} (N^{-\frac{(2-\sigma)(d-1)}{d}-s} + N^{\eta+\frac{\sigma}{2}-\frac{4s}{d}-s}).$$

Now adding all the estimate for (4.8) through (4.12), we finish estimating the term (4.6). This together with the estimate for (4.1) and (4.3) finally gives that

$$\|P_N u_0\|_{L_x^2} \lesssim_{m,u_0} N^{-1-\epsilon} + N^{-s-\frac{d}{2}\gamma} + N^{(1+\gamma)(d-2)+2-d-\sigma(m-1)} + N^{\frac{\gamma}{2}(d-4)}(N^{-\frac{d-1}{d}\sigma-s} + N^{-\sigma(m-1)} + N^{-\frac{d-1}{d}(2-\sigma)-s} + N^{\eta+\frac{\sigma}{2}-\frac{4s}{d}-s}).$$

For any $s>0$, choosing $\sigma=\frac{s}{100d}$, $\eta=\frac{s}{1000d}$, $\gamma=\frac{s}{1000d^2}$, $m=1+\frac{200d}{s}$, we finally obtain

$$\|P_N u_0\|_{L_x^2} \lesssim_{u_0} N^{-1-\epsilon} + N^{-s-\frac{s}{2000d}}.$$

It is easy to see that after finite many times of iteration we obtain

$$\|P_N u_0\|_{L_x^2} \lesssim_{u_0} N^{-1-}, \ \forall N \geq 1.$$

Therefore, $u_0 \in H_x^1$. The proof of Theorem 1.5 is complete.

Acknowledgements. Both authors were supported by the National Science Foundation under agreement No. DMS-0635607 and start-up funding from the Mathematics Department of University of Iowa. X. Zhang was also supported by NSF grant No. 10601060 and project 973 in China.

References

[1] H. Berestycki and P. L. Lions, *Existence d'ondes solitaires dans des problèmes nonlinéaires du type Klein-Gordon*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), A395–A398. MR0552061

[2] T. Cazenave and F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. **14** (1990), 807–836. MR1055532

[3] T. Cazenave, *Semilinear Schrödinger equations.* Courant Lecture Notes in Mathematics, **10**. American Mathematical Society, 2003. MR2002047

[4] M. Christ, M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), 87-109.

[5] P. Constantin, J. C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. 1,(1988), 413-446.

[6] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations,* Comm. Math. Phys. **144** (1992), 163–188. MR1151250

[7] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), 1794–1797. MR0460850

[8] T. Hmidi and S. Keraani, *Blowup theory for the critical nonlinear Schrödinger equations revisited,* Int. Math. Res. Not. **46** (2005), 2815–2828. MR2180464

[9] M. Keel and T. Tao, *Endpoint Strichartz Estimates*, Amer. J. Math. **120** (1998), 955–980. MR1646048

[10] C. E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equation*, Ann. Inst. Henri Poincaré, Sect C, 10(1993), 255-288.

[11] C. E. Kenig, G. Ponce and L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math., 134(1998), 489-545.

[12] R. Killip, D. Li, M. Visan and X. Zhang, *Characterization of minimal-mass blowup solutions to the focusing mass-critical NLS*, preprint, `arXiv:0804.1124`.

[13] R. Killip, M. Visan, and T. Tao, *The cubic nonlinear Schrödinger equation in two dimensions with radial data,* preprint, `arXiv:0707.3188`.

[14] R. Killip, M. Visan, and X. Zhang, *The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher,* preprint, `arXiv:0708.0849`.

[15] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$,* Arch. Rat. Mech. Anal. **105** (1989), 243–266. MR0969899

[16] D. Li, X. Zhang, *On the rigidity of solitary waves for the focusing mass-critical NLS in dimensions $d \geq 2$.* Preprint, `arxiv:0902.0802`.

[17] F. Merle, *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equation with critical power,* Duke Math. J. **69** (1993), 427–453. MR1203233

[18] P. Sjölin, *Regularity of solutions of Schrödinger equations*, Duke Math. J. 55(1987), 699-715.

[19] G. Staffilani, *On the generalized Korteweg-de Vries-type equations*, Differential Integral Equations **10** (1997), 777–796.

[20] R. S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–774. MR0512086

[21] T. Tao, M. Visan, X. Zhang, *Global well-posedness and scattering for the mass-critical defocusing NLS with spherical symmetry in higher dimensions*, Duke Math. J. 140 (2007), 165-202.

[22] T. Tao, M. Visan, and X. Zhang, *Minimal-mass blowup solutions of the mass-critical NLS,* to appear in Forum Math.

[23] T. Tao, *On the asymptotic behavior of large radial data for a focusing nonlinear Schrödinger equation.* Dynamics of PDE 1, (2004), 1-48.

[24] M. E. Taylor, *Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs, **81**, American Mathematical Society, Providence, RI, 2000.

[25] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. 102, 874-878.

[26] M. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates,* Comm. Math. Phys. **87** (1983), 567–576. MR0691044

[27] M.I. Weinstein, *Modulation stability of ground states of nonlinear Schrödinger equations*, Siam. J. Math. Anal. 16(1985), 472–491.

[28] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure. Appl. Math. 39(1986), 51–68.

[29] M. Weinstein, *On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations,* Comm. Partial Differential Equations **11** (1986), 545–565. MR0829596

Dong Li

Institute for Advanced Study, Princeton, NJ, 08544, USA

E-mail address: dongli@ias.edu

Xiaoyi Zhang

Academy of Mathematics and System Sciences, Beijing, China.

Institute for Advanced Study, Princeton, NJ, 08544, USA

E-mail address: xiaoyi@ias.edu