Electronic Journal of Differential Equations, Vol. 2009(2009), No. 80, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

CYCLIC APPROXIMATION TO STASIS

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Abstract. Neighborhoods of points in $\mathbb{R}^n$ where a positive linear combination of C^1 vector fields sum to zero contain, generically, cyclic trajectories that switch between the vector fields. Such points are called stasis points, and the approximating switching cycle can be chosen so that the timing of the switches exactly matches the positive linear weighting. In the case of two vector fields, the stasis points form one-dimensional C^1 manifolds containing nearby families of two-cycles. The generic case of two flows in $\mathbb{R}^3$ can be diffeomorphed to a standard form with cubic curves as trajectories.

1. Introduction

Pairs of planar vector fields were analyzed in [5] where stasis was defined as a point where the flows were directly opposed. It was shown that generically such points form one-dimensional curves and are surrounded by small cycles that switch back and forth between the fields.

This work extends these results by showing the generic existence of small switching cycles for any number of vector fields in any number of dimensions near points where a weighted sum of the vector fields is zero. We further analyze pairs of flows in higher dimensions to show that a weaker generic condition will still imply existence and define the structure of approximating two-cycles. Finally we give a detailed analysis of a canonical example for pairs of flows in three dimensions.

Definitions and the main result for multiple vector fields in $\mathbb{R}^n$ are contained in section 2. In section 3 we apply a weaker hypothesis to pairs of flows and obtain stronger results and greater insight into the structure of stasis points and approximating cycles. Section 4 contains a detailed analysis and a normal form for the case of pairs of flows in $\mathbb{R}^3$. Section 5 concludes with some open questions. The authors thank the referee for a thorough review and many helpful comments.

2000 *Mathematics Subject Classification.* 37C10, 37C27.
Key words and phrases. Two-cycles; stasis points; switching systems; piecewise smooth; relaxed controls.

Submitted July 27, 2007. Published June 24, 2009.

2. Multiple Vector Fields

Vector fields $\mathbf{V}_j : \mathbb{R}^n \to \mathbb{R}^n$ for $j = 1, \dots, k$ induce flows $\mathbf{F}_j(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is a *stasis point* if

$$\sum_{j=1}^{k} m_j \mathbf{V}_j(\mathbf{x}_0) = \mathbf{0}$$

for some weighting $(m_1, \dots, m_k) \in \mathbb{R}^k$ with $m_j \geq 0$, and not all $m_j \mathbf{V}_j(\mathbf{x}_0) = \mathbf{0}$

The stasis point is *regular* if

$$\sum_{j=1}^{k} m_j \frac{\partial \mathbf{V}_j}{\partial \mathbf{x}}(\mathbf{x}_0)$$

is non-singular.

A *switching cycle* for the sequence of vector fields $\mathbf{V}_1, \dots, \mathbf{V}_k$ is a sequence of points $\mathbf{x}_1, \dots, \mathbf{x}_k$ in $\mathbb{R}^n$, and a sequence of times $(\delta_1, \dots, \delta_k)$ with each $\delta_j \geq 0$, such that

$$\begin{gathered}\mathbf{F}_1(\mathbf{x}_1, \delta_1) = \mathbf{x}_2 \\ \mathbf{F}_2(\mathbf{x}_2, \delta_2) = \mathbf{x}_3 \\ \dots \\ \mathbf{F}_k(\mathbf{x}_k, \delta_k) = \mathbf{x}_1.\end{gathered}$$

We have the following theorem.

Theorem 2.1. *If $\mathbf{x}_0$ is a regular stasis point with weighting $(m_1, \dots, m_k)$ then for all sufficiently small $\delta > 0$ there exists a switching cycle for the sequence $\mathbf{V}_1 \dots, \mathbf{V}_k$ with the time vector $(\delta m_1, \dots, \delta m_k)$.*

Proof. Without loss of generality take $\mathbf{x}_0 = \mathbf{0}$, and it is convenient to assume $\sum m_j = 1$. Define

$$\mathcal{F} : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \times \mathbb{R} \to \mathbb{R}^n$$

as the average velocity

$$\mathcal{F}(\mathbf{x}_1, \dots, \mathbf{x}_k, \delta) = \frac{(\mathbf{F}_1(\mathbf{x}_1, \delta m_1) - \mathbf{x}_1) + \dots + (\mathbf{F}_k(\mathbf{x}_k, \delta m_k) - \mathbf{x}_k)}{\delta}.$$

Note that $\mathcal{F}$ can be C^1 extended to include $\delta = 0$ with

$$\frac{\partial}{\partial \mathbf{x}_j} \mathcal{F}(\mathbf{x}_1, \dots, \mathbf{x}_k, 0) = m_j \frac{\partial \mathbf{V}_j}{\partial \mathbf{x}}(\mathbf{x}_j)$$

Now

$$\begin{pmatrix} \mathcal{F}(\mathbf{x}_1, \dots, \mathbf{x}_k, \delta) \\ \mathbf{F}_1(\mathbf{x}_1, \delta m_1) - \mathbf{x}_2 \\ \mathbf{F}_2(\mathbf{x}_2, \delta m_2) - \mathbf{x}_3 \\ \dots \\ \mathbf{F}_{k-1}(\mathbf{x}_{k-1}, \delta m_{k-1}) - \mathbf{x}_k \end{pmatrix} : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \times \mathbb{R} \to \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k}$$

with

$$\begin{pmatrix} \mathcal{F}(\mathbf{x}_1, \dots, \mathbf{x}_k, \delta) \\ \mathbf{F}_1(\mathbf{x}_1, \delta m_1) - \mathbf{x}_2 \\ \dots \\ \mathbf{F}_{k-1}(\mathbf{x}_{k-1}, \delta m_{k-1}) - \mathbf{x}_k \end{pmatrix} |_{(\mathbf{0}, \dots, \mathbf{0}, 0)} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

By the implicit function theorem (see Theorem 5.1),

$$\begin{pmatrix} \mathcal{F}(\mathbf{x}_1,\dots,\mathbf{x}_k,\delta) \\ \mathbf{F}_1(\mathbf{x}_1,\delta m_1)-\mathbf{x}_2 \\ \vdots \\ \mathbf{F}_{k-1}(\mathbf{x}_{k-1},\delta m_{k-1})-\mathbf{x}_k \end{pmatrix} |_{(\mathbf{x}_1,\dots,\mathbf{x}_k,\delta)} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

will have solutions $\mathbf{x}_1(\delta),\dots,\mathbf{x}_k(\delta)$ for small non-zero δ provided that the $nk\times nk$ matrix

$$\begin{bmatrix} \frac{\partial \mathcal{F}(\mathbf{x}_1,\dots,\mathbf{x}_k,\delta)}{\partial \mathbf{x}_1,\dots,\mathbf{x}_k} \\ \frac{\partial(\mathbf{F}_1(\mathbf{x}_1,\delta m_1)-\mathbf{x}_2)}{\partial \mathbf{x}_1,\dots,\mathbf{x}_k} \\ \dots \\ \frac{\partial(\mathbf{F}_{k-1}(\mathbf{x}_{k-1},\delta m_{k-1})-\mathbf{x}_k)}{\partial \mathbf{x}_1,\dots,\mathbf{x}_k} \end{bmatrix}_{(\mathbf{0},\dots,\mathbf{0},0)}$$

is non-singular. This evaluates to

$$\begin{bmatrix} m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}} & m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}} & m_3\frac{\partial \mathbf{V}_3}{\partial \mathbf{x}} & \cdots & m_k\frac{\partial \mathbf{V}_k}{\partial \mathbf{x}} \\ \mathbf{I} & -\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} \end{bmatrix}_{(\mathbf{0},\dots,\mathbf{0},0)}$$

We claim that this matrix is singular if and only if $\sum_{j=1}^k m_j\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}}(\mathbf{0})$ is singular. If some nontrivial weighting of columns of this matrix were to sum to zero, the weighting would have to be equal on column numbers congruent mod n in order to make rows $n+1$ through nk sum to zero. Under such a weighting, the matrices $m_j\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}}$ on the first row will sum column-wise. Hence this weighting applied to the columns of $\sum_{j=1}^k m_j\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}}(\mathbf{0})$ would sum to the zero vector. $\square$

3. Pairs of Vector Fields and Families of Two-Cycles

For the case of pairs of vector fields we obtain a stronger result under a weaker hypothesis. The weaker hypothesis is due to the pair of vector fields travelling in opposite directions at the stasis point and so a differential singularity in that direction is inconsequential.

It is easier to analyze pairs of vector fields, and we can get a clearer picture of the structure of the set of approximating switching cycles. The degree to which these structures can be generalized to larger sets of vector fields is left as an open question.

Section 3.1 sets up the analysis for pairs of vector fields, 3.2 defines families of cycles, and 3.3 contains the main for theorem families of approximating two-cycles.

3.1. **Pairs of Vector Fields.** For a pair of C^1 vector fields $\{\mathbf{V}_1,\mathbf{V}_2\}$ with induced flows $\{\mathbf{F}_1,\mathbf{F}_2\}$ a *stasis point* is a point $\mathbf{x}_0$ where the vector fields are nonzero and anti-parallel:

$$\mathbf{V}_1(\mathbf{x}_0)|\mathbf{V}_2(\mathbf{x}_0)|+\mathbf{V}_2(\mathbf{x}_0)|\mathbf{V}_1(x_0)|=\mathbf{0}.$$

Setting $m_1=|\mathbf{V}_2(\mathbf{x}_0)|$ and $m_2=|\mathbf{V}_1(\mathbf{x}_0)|$ yields $(m_1\mathbf{V}_1+m_2\mathbf{V}_2)(\mathbf{x}_0)=\mathbf{0}$. As before, $\mathbf{x}_0$ is a regular stasis point if $(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}}+m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}})(\mathbf{x}_0)$ is nonsingular.

The stasis point is *pseudo-regular* if the matrix $(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}}+m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}})(\mathbf{x}_0)$ is of rank $n-1$ and $\mathbf{V}_1(\mathbf{x}_0)$ is not in the image of the matrix. That is, the direction of flow at stasis $\mathbf{V}_1(\mathbf{x}_0)$ is independent of the columns of $(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}} + m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}})(\mathbf{x}_0)$, but these columns span $\mathbf{V}_1(\mathbf{x}_0)^{\perp}$.

Pseudo-regularity is a weaker condition than regularity and allows for differential singularity in the direction of the flow at stasis. A stasis point that is neither regular nor pseudo-regular is *degenerate*.

A *two-cycle* is a pair of points $\mathbf{x}_1, \mathbf{x}_2$ in $\mathbb{R}^n$ and a pair of non-negative times δ_1, δ_2 with

$$\mathbf{F}_1(\mathbf{x}_1, \delta_1) = \mathbf{x}_2$$
$$\mathbf{F}_2(\mathbf{x}_2, \delta_2) = \mathbf{x}_1.$$

Example 3.1. Consider the saddle and center in figure 1 given by:

$$\begin{aligned} x' &= y & \qquad x' &= -y \\ y' &= x+1 & \qquad y' &= x-1 \end{aligned}$$

Points on the $y = 0$ axis with $-1 < x < 1$, $x \neq 0$ are regular stasis points. Points on the $x = 0$ axis with $y \neq 0$ are pseudo-regular stasis points. The origin $(0, 0)$ is a degenerate stasis point.

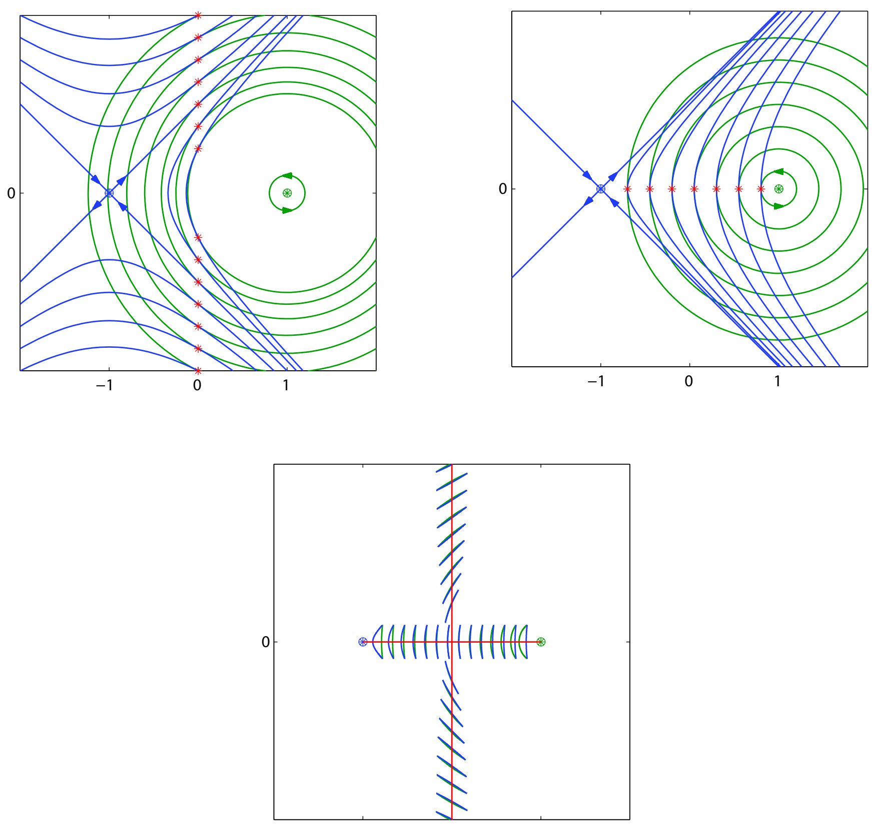

Figure 1. Vertical & horizontal stasis lines and approximating two-cycles

Example 3.2. The pair given by

$$\begin{aligned} x' &= 1 \qquad & x' &= -1 \\ y' &= y \qquad & y' &= y \end{aligned}$$

shown in figure 2 and has pseudo-regular stasis points on the $y = 0$ axis, and approximating two-cycles and confined to the axis.

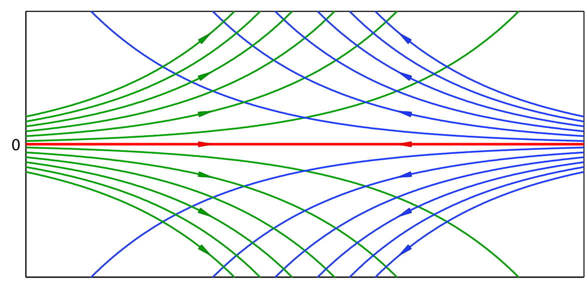

FIGURE 2. Flow aligns with stasis line

Example 3.3. The pair

$$\begin{aligned} x' &= -1 \qquad & x' &= 1 \\ y' &= 0 \qquad & y' &= x^2 \end{aligned}$$

shown in figure 3 has degenerate stasis points at $x = 0$ for all y, and no approximating two-cycles.

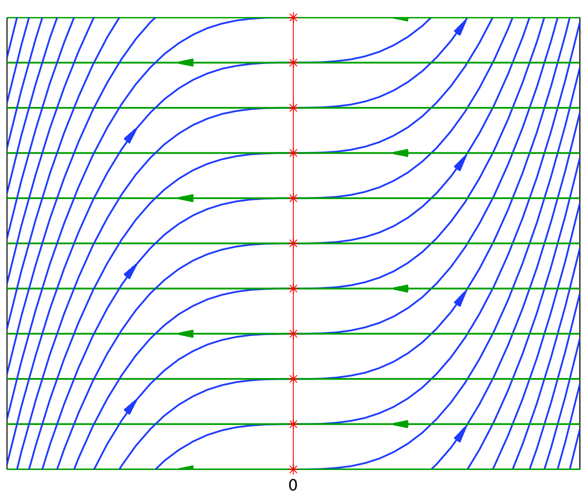

FIGURE 3. Degenerate stasis without two-cycles

3.2. **Families of Two-Cycles.** We want to understand the structure of the collection of two-cycles near a stasis point for pairs of vector fields. This structure can appear in a highly organized form modelled by a piecewise smooth system, or can be more singular and described only as a two cycle family.

In general, a *piecewise smooth* system consisting of a finite partition $R = \bigcup \overline{P_i}$ of non-overlapping open sets $P_i \in R$ with shared boundaries $\overline{P_i} \cap \overline{P_j}$, and vector fields $\mathbf{V}_i$ defined and C^1 on P_i. The vector field

$$\mathbf{V}(\mathbf{x}) = \mathbf{V}_i(\mathbf{x}) \text{ for } \mathbf{x} \in P_i$$

is piecewise continuous. Generically, the flux of two flows at a partition boundary is either (i) aligned, giving rise to switching manifolds $\Sigma_{i,j}, \Sigma_{j,i}$; (ii) opposed and convergent; or (iii) opposed and divergent. Existence and uniqueness holds for a piecewise smooth system so long as the flows $\mathbf{V}_i$ and $\mathbf{V}_j$ are transverse to $\Sigma_{i,j}$ (no grazing or fixed points), and for all i_1, i_2, i_3, $\Sigma_{i_1,i_2} \cap \Sigma_{i_1,i_3} = \emptyset$ (no ambiguous switches), and $\Sigma_{i_1,i_2} \cap \Sigma_{i_2,i_3} = \emptyset$ (no double switches). See [1, 2] for a thorough treatment of these systems.

Example 3.4. A simple example of two-cycles near a stasis point in a piecewise smooth system is to take

$$\begin{array}{cc} \mathbf{V}_1: & \mathbf{V}_2: \\ x' = y + 1 & x' = y - 2 \\ y' = -x & y' = -x \end{array}$$

and flow under $\mathbf{V}_1$ for $y > 0$ and $\mathbf{V}_2$ for $y < 0$, taking $\Sigma_{1,2}$ as the half line $x > 0$, $y = 0$, and $\Sigma_{2,1}$ as the half line $x < 0$, $y = 0$, as shown in Figure 4.

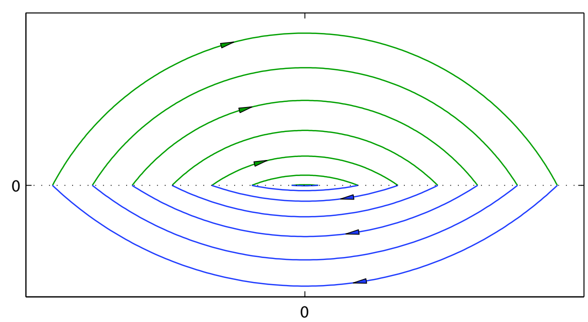

FIGURE 4. Piecewise smooth system

Closely related to piecewise smooth systems is the idea of a *switching system* consisting of a collection of C^1 vector fields $\mathbf{V}_1, \dots, \mathbf{V}_k$ defined on $\mathbb{R} \in \mathbb{R}^n$ and C^1 switching manifolds $\Sigma_{i,j}$. Trajectories $\mathbf{x}(t)$ starting under flow i satisfy $\mathbf{x}' = \mathbf{V}_i(\mathbf{x})$ until hitting $\Sigma_{i,j}$ for some j, and then they switch to satisfying $\mathbf{x}' = \mathbf{V}_j(x)$.

Example 3.5. Consider

$$\begin{array}{cc} \mathbf{V}_1: & \mathbf{V}_2: \\ x' = y + 1 & x' = -y - 2 \\ y' = -x & y' = x \end{array}$$

and take $\Sigma_{1,2}$ as the half line $x > 0$, $y = 0$, and $\Sigma_{2,1}$ as the half line $x < 0$, $y = 0$, as shown in Figure 5.

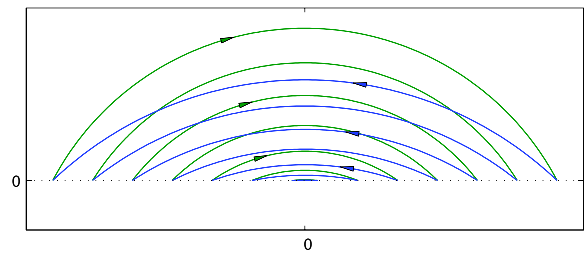

FIGURE 5. Switching system

Any switching system can be thought of as a piecewise continuous system on a branched cover of $\mathbb{R}^n$. The branches are created by opposing flux of $\mathbf{V}_i$ and $\mathbf{V}_j$ at the switching manifold $\Sigma_{i,j}$. Many of the results from piecewise continuous systems immediately apply to switching systems. See [3, 4, 5] for more information on switching systems.

In piecewise smooth and switching systems, the switch from one vector field $\mathbf{V}_i$ to another $\mathbf{V}_j$ is defined by the event of the trajectory hitting the switching manifold $\Sigma_{i,j}$. This condition need not apply in the construct of a cycle family.

A p-dimensional *two-cycle family* is constructed with a pair of manifolds $\Sigma_{1,2}(\rho)$ and $\Sigma_{2,1}(\rho)$ defined for some open set $P \subset \mathbb{R}^p$ such that for each $\rho \in P$, there exist positive $\delta_1(\rho), \delta_2(\rho)$ with

$$\mathbf{F}_1(\Sigma_{2,1}(\rho), \delta_1(\rho)) = \Sigma_{1,2}(\rho)$$
$$\mathbf{F}_2(\Sigma_{1,2}(\rho), \delta_2(\rho)) = \Sigma_{2,1}(\rho).$$

Example 3.6. Taking $\Sigma_{1,2}(\rho) = (\rho, 0)$ $\Sigma_{2,1}(\rho) = (-\rho, 0)$ for $\rho > 0$ generates a two-cycle family for the system pair

$$\begin{array}{cc} \mathbf{V}_1: & \mathbf{V}_2: \\ x' = y + 1 & x' = -1 \\ y' = -x & y' = 0 \end{array}$$

shown in figure 6.

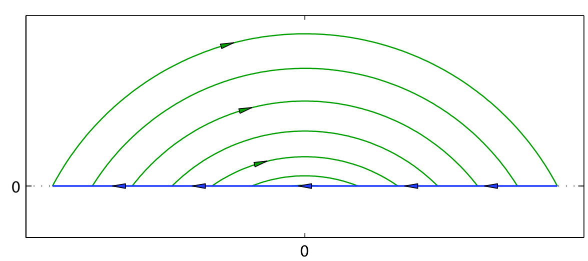

FIGURE 6. Cycle family

The manifolds $\Sigma_{1,2}(\rho)$, $\Sigma_{1,2}(\rho)$ from Example 3.6 also define two-cycle families for Examples 3.4 and 3.5.

Note that the manifolds $\Sigma_{i,j}(\rho)$ parameterized by $\rho \in \mathbb{R}^p$ need not be p-dimensional manifolds. Taking example 3.2 from the previous section, we can construct a two-parameter cycle family by taking

$$\Sigma_{1,2}(\sigma,\epsilon) = (\sigma - \epsilon, 0), \quad \Sigma_{2,1}(\sigma,\epsilon) = (\sigma + \epsilon, 0) \quad \text{for } -\infty < \rho < \infty, \epsilon > 0$$

For the general idea of a cycle family for multiple vector fields, consider a specific sequence $i_0, i_2, \dots, i_{m-1}$ of length m from a collection of flows $\{\mathbf{V}_i\}$, and for convenience take $\oplus, \ominus$ as addition and subtraction modulo m. A *cycle family* is characterized by manifolds $\Sigma_{i,j}(\rho)$ defined for each pair $i_m, i_{m\oplus 1}$ and C^0 parameterized by $\rho \in P \subset \mathbb{R}^p$, such that for all $\rho \in P$ there exists a trajectory $x(t)$ under V_{i_m} connecting the points $\Sigma_{i_{m\ominus 1}, i_m}$ and $\Sigma_{i_m, i_{m\oplus 1}}$.

3.3. **Existence of Two-Cycles.** Stasis structure and pairs of vector fields in $\mathbb{R}^2$ have been explored in [5], where the existence of a two-cycle is implied by a difference in curvature of the two flows at a stasis point. Higher dimensional cases require more extensive consideration, and the following is the main theorem for pairs of flows in $\mathbb{R}^n$.

Theorem 3.7. *For a pair of C^1 vector fields $\{\mathbf{V}_1, \mathbf{V}_2\}$ mapping $\mathbb{R}^n \to \mathbb{R}^n$, the set $\mathbf{s}$ of pseudo-regular stasis points is a one-dimensional C^1 manifold. Near any point on the stasis manifold there exists a two-dimensional family of two-cycles with manifolds $\Sigma_{1,2}(\sigma,\delta)$, $\Sigma_{2,1}(\sigma,\delta)$ defined for $|\sigma| < \epsilon$ and $0 < \delta < \epsilon$, for some $\epsilon > 0$, such that δ is the length of the cycle, and $\lim_{\delta\to 0} \Sigma_{i,j}(\sigma,\delta)$ is a point on $\mathbf{s}$ of arclength distance σ away from $\mathbf{x}_0$.*

A simple visual of possible manifolds is shown in figure 7. The left depicts a range of σ for fixed $\delta = 0.1$, and the right depicts a range of δ for fixed $\sigma = 0$.

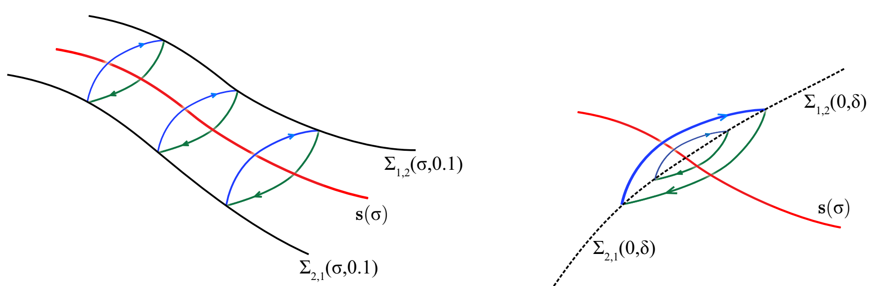

Figure 7. Stasis with two-cycles

Another example is to take

$$\Sigma_{1,2}(\sigma,\delta) = (\sigma - \delta/4, 0), \quad \Sigma_{2,1}(\sigma,\epsilon) = (\sigma + \delta/4, 0) \quad \text{for } -\infty < \rho < \infty, \delta > 0$$

for the system in Example 3.2, section 3.2.

Proof of Theorem 3.7. Consider C^1 vector fields $\mathbf{V}_1$, $\mathbf{V}_2$, and a pseudo-regular stasis point $\mathbf{x}_0$ with $(m_1\mathbf{V}_1 + m_2\mathbf{V}_2)(\mathbf{x}_0) = \mathbf{0}$, $m_j > 0$. Without loss of generality we take $\mathbf{x}_0 = \mathbf{0}$.

We begin by rectifying one of the flows. Since $\mathbf{V}_2$ is C^1 and non-zero near $\mathbf{0}$, there is a local rectifying diffeomorphism $\mathbf{y} = \Phi(\mathbf{x})$ with $\mathbf{0} = \Phi(\mathbf{0})$ and

$$\frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x})\, \mathbf{V}_2(\mathbf{x}) \equiv (-1; 0; \dots; 0)$$

near $\mathbf{0}$. We denote column vectors as $(a;b) = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $\mathbf{U}$ defined by

$$\mathbf{U}(\Phi(\mathbf{x})) = \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x})\mathbf{V}_1(\mathbf{x})$$

is a C^1 field and

$$m_1\mathbf{U}(\Phi(\mathbf{x})) + m_2\,(-1;0;\dots;0) = \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x})\;(m_1\mathbf{V}_1 + m_2\mathbf{V}_2)(\mathbf{x}).$$

With $(m_1\mathbf{V}_1 + m_2\mathbf{V}_2)(\mathbf{0}) = \mathbf{0}$, we have

$$\frac{\partial \mathbf{U}}{\partial \mathbf{y}}(\mathbf{0})\;\frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{0}) = \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{0})\big(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}} + m_2\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}}\big)(\mathbf{0}).$$

If $\mathbf{0}$ is a regular stasis point then nonsingularity of $\big(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}} + m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}}\big)(\mathbf{0})$ implies nonsingularity of $\frac{\partial \mathbf{U}}{\partial \mathbf{y}}(\mathbf{0})$. If $\mathbf{0}$ is pseudo-regular then the columns of $\big(m_1\frac{\partial \mathbf{V}_1}{\partial \mathbf{x}} + m_2\frac{\partial \mathbf{V}_2}{\partial \mathbf{x}}\big)(\mathbf{0})$ span $\mathbf{V}_1(\mathbf{0})^\perp$, hence the columns of $\frac{\partial \mathbf{U}}{\partial \mathbf{y}}(\mathbf{0})$ will span $\mathbf{U}(\mathbf{0})^\perp$. We have thus diffeomorphed the pair of systems $\mathbf{V}_1$, $\mathbf{V}_2$ to the pair $\mathbf{U}$, $(-1;0;\cdots;0)$ and the properties of regularity and pseudo-regularity carry over. With

$$\mathbf{U}(\mathbf{y}) = \begin{pmatrix} u_1(\mathbf{y}) \\ u_2(\mathbf{y}) \\ \vdots \\ u_n(\mathbf{y}) \end{pmatrix}$$

stasis points are characterized by $n-1$ equations

$$\begin{gathered} 0 = u_2(\mathbf{y}) \\ \dots \\ 0 = u_n(\mathbf{y}). \end{gathered}$$

By pseudo-regularity, the $n-1$ gradients $\frac{\partial u_2}{\partial \mathbf{y}}(\mathbf{0}), \dots, \frac{\partial u_n}{\partial \mathbf{y}}(\mathbf{0})$ are independent. With $u_2(\mathbf{0}) = \cdots = u_n(\mathbf{0}) = 0$, the implicit function theorem (see Theorem 5.1) implies the local existence of a C^1 curve $\mathbf{s}(\sigma)$ of solutions $u_2(\mathbf{s}(\sigma)) = \cdots = u_n(\mathbf{s}(\sigma)) = 0$ for small $|\sigma|$. We can take σ as arclength with $\mathbf{s}(0) = \mathbf{0}$. By construction, $\mathbf{s}'(0)$ is perpendicular to each of $\frac{\partial u_2}{\partial \mathbf{y}}(\mathbf{0}), \dots, \frac{\partial u_n}{\partial \mathbf{y}}(\mathbf{0})$.

To find two-cycles, let $\mathbf{G}(\mathbf{y}, t)$ be the flow induced by $\mathbf{U}(\mathbf{y})$. For $\mathbf{y} \neq \mathbf{0}$ near $\mathbf{0}$ and small $|\delta| > 0$, define the average velocity

$$\mathcal{G}(\mathbf{y}, \delta) = \frac{\mathbf{G}(\mathbf{y}, \delta) - \mathbf{y}}{\delta}.$$

Note that

$$\lim_{\delta \to 0} \mathcal{G}(\mathbf{y}, \delta) = \mathbf{U}(\mathbf{y})$$

and so by extension, $\mathcal{G}$ is defined and C^1 for sufficiently small δ, and $\mathbf{y}$ in a neighborhood of $\mathbf{0}$. Writing

$$\mathcal{G}(\mathbf{y}, \delta) = \begin{pmatrix} g_1(\mathbf{y}, \delta) \\ g_2(\mathbf{y}, \delta) \\ \vdots \\ g_n(\mathbf{y}, \delta) \end{pmatrix}$$

we are interested in solutions $\mathbf{y}_\delta$ with $\delta \neq 0$ to the $n-1$ equations

$$\begin{gathered} 0 = g_2(\mathbf{y}_\delta, \delta) \\ \dots \\ 0 = g_n(\mathbf{y}_\delta, \delta). \end{gathered}$$

For any such solution, the points $\mathbf{y}_\delta$ and $\mathbf{G}(\mathbf{y}_\delta, \delta)$ differ only in their first coordinate. These two points can be joined by a segment of the rectified flow creating a two-cycle. Note that we allow δ to be positive or negative; so if $\mathbf{y}_\delta$ is a solution for some δ, then $\mathbf{G}(\mathbf{y}_\delta, \delta)$ is a solution for $-\delta$.

Define $\rho(\mathbf{y})$ as the as the arclength along the stasis curve $\mathbf{s}$ from $\mathbf{0}$ to the perpendicular projection of $\mathbf{y}$ onto $\mathbf{s}$, which is well defined sufficiently close to $\mathbf{s}$. If $\mathbf{y}$ is a point on $\mathbf{s}$, then $\rho(\mathbf{y})$ is just the arclength from $\mathbf{0}$ to $\mathbf{y}$, hence $\frac{\partial \rho}{\partial \mathbf{y}}(\mathbf{0}) = \mathbf{s}'(0)$, which is independent of $\frac{\partial u_2}{\partial \mathbf{y}}(\mathbf{0}), \dots, \frac{\partial u_n}{\partial \mathbf{y}}(\mathbf{0})$.

The equations

$$\begin{gathered} 0 = \rho(\mathbf{y}) \\ 0 = g_2(\mathbf{y}, \delta) \\ \dots \\ 0 = g_n(\mathbf{y}, \delta). \end{gathered} \tag{3.1}$$

are satisfied by $\delta = 0$ and $\mathbf{y}_0 = \mathbf{0}$. By construction of ρ and pseudo-regularity, the gradients $\frac{\partial \rho}{\partial \mathbf{y}}(\mathbf{0}), \frac{\partial g_2}{\partial \mathbf{y}}(\mathbf{0}, 0), \dots, \frac{\partial g_n}{\partial \mathbf{y}}(\mathbf{0}, 0)$ are independent. By the implicit function theorem (see Theorem 5.1), for all δ sufficiently small there is a one-dimensional manifold of solutions $\mathbf{y}_\delta$ to equations (3.1). Since $\rho(\mathbf{y}_\delta) = 0$, these solutions are perpendicular to $\mathbf{s}$ at $\mathbf{0}$.

This construction applies at any point $\mathbf{s}(\sigma)$ on the stasis curve. Take $\Sigma_{1,2}(\delta, \sigma)$ as $\mathbf{y}_\delta$ constructed at the point $\mathbf{s}(\sigma)$ for $\delta > 0$, and $\Sigma_{2,1}(\delta, \sigma)$ as $\mathbf{G}(\mathbf{y}_\delta, \delta)$. □

4. The Three Dimensional Case

Stasis structure and pairs of vector fields in $\mathbb{R}^2$ have been explored in [5], and the theorem in the previous section contains addresses the structure for flows in higher dimensions. In this section we make an examination of the structure of the three case. In section 4.1 we do a complete analysis of the structure of two-cycles for a specific pair of $\mathbb{R}^3$ systems. In section 4.2 we show that this structure is generic by renormalizing along Frenet frame dimensions.

4.1. **A Representative Example.**

Consider the pair of $\mathbb{R}^3$ systems

$$\begin{array}{cc} \mathbf{V}_1: & \mathbf{V}_2: \\ x' = 1 & x' = -1 \\ y' = z + \alpha x^2 & y' = 0 \\ z' = x & z' = 0 \end{array} \tag{4.1}$$

The stasis curve is the y-axis. Fixing $x_0 = 0$, trajectories under $\mathbf{V}_1$ are given by cubic curves

$$\begin{gathered} x = t \\ y = \frac{1}{6}(2\alpha + 1)t^3 + z_0 t + y_0 \\ z = \frac{1}{2}t^2 + z_0. \end{gathered} \tag{4.2}$$

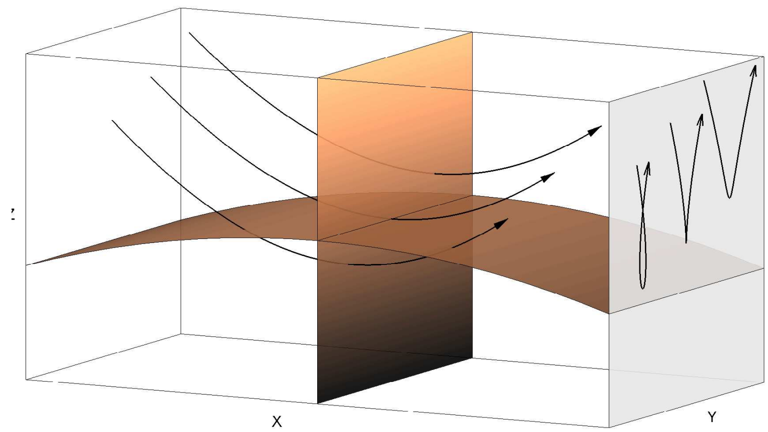

FIGURE 8. Generic structure for $\alpha < 0$, with $y' = 0$ and $z' = 0$ nullsurfaces

The trajectory under $\mathbf{V}_1$ at the origin has unit velocity $(1; 0; 0)$, unit curvature $(0; 1; 0)$, and torsion of magnitude $2\alpha + 1$. Figure 8 shows some sample trajectories along with the null surfaces $z' = 0$ and $y' = 0$. Two-cycles occur where these cubic trajectories self-intersect since the trajectory points at the intersection can be joined by a straight line trajectory in the other system.

To solve for the point of intersection note that $z(t) = z(-t)$ is trivially satisfied, set $y(t) = y(-t)$ and solve for $z_0 = -t^2(2\alpha + 1)/6$. Thus the sign of z_0 must be opposite that of torsion $2\alpha + 1$.

Parameterizing the stasis curve (the y-axis) with arclength σ, the switching surface $\Upsilon = \Sigma_{1,2} \cup S \cup \Sigma_{2,1}$ from Theorem 3.7 is computed by taking $t = \pm\delta/2$:

$$\Upsilon : \left\{ \begin{matrix} x(\delta, \sigma) = \delta \\ y(\delta, \sigma) = \sigma \\ z(\delta, \sigma) = \frac{1}{3}(1 - \alpha)\delta^2 \end{matrix} \right\} \tag{4.3}$$

The switching surface Υ partitions into $\Sigma_{1,2}$ for $\delta > 0$ and $\Sigma_{2,1}$ for $\delta < 0$, with the stasis curve characterized by $\delta = 0$.

This generically yields three different topologies characterized by $\alpha < -\frac{1}{2}$, $-\frac{1}{2} < \alpha < 1$, and $1 < \alpha$, and sample trajectories for each topology are shown as projections onto the $x = 0$ plane in figure 9.

The different topologies arise from an interplay between the curvatures of the switching manifold and the flow $\mathbf{V}_1$. If $\alpha < 1$ the switching manifold is curved in the same direction as the flow. If $\alpha < -\frac{1}{2}$, making torsion negative, the switching manifold is more sharply curved than the flow, and if $-\frac{1}{2} < \alpha < 1$, making torsion positive but less than 3, the switching manifold is not as sharply curved as the flow. If $1 < \alpha$, making torsion greater than 3, then the switching manifold has opposite curvature to the flow. Sample two-cycles and switching manifold for each case are shown in figure 10 as projections onto the $y = 0$ plane.

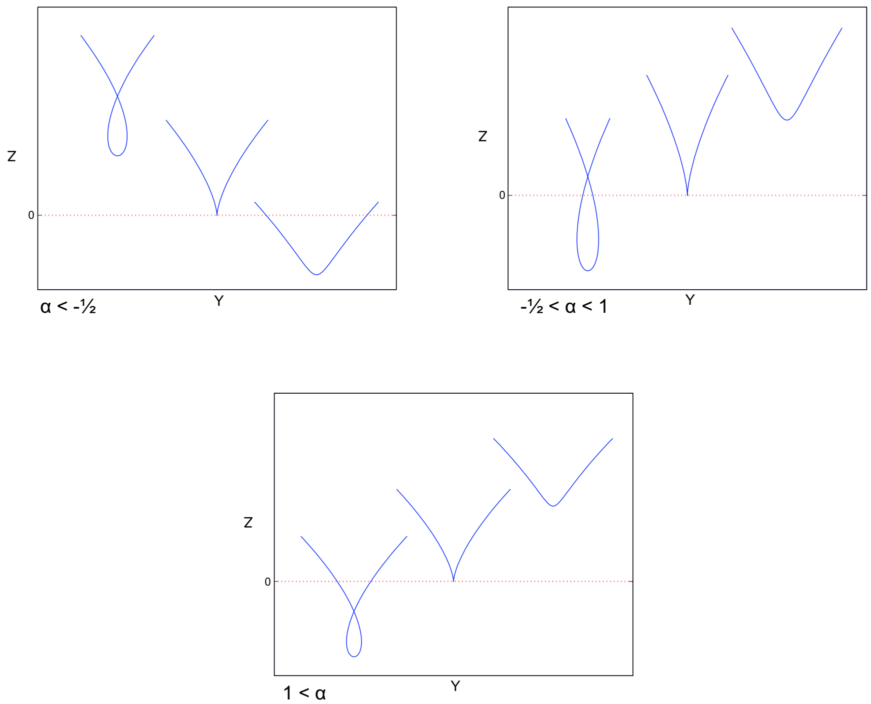

FIGURE 9. Cubic trajectories

For the $-\frac{1}{2} < \alpha < 1$ case, taking

$$\mathbf{H}(x,y,z) = \begin{cases} (-1;0;0) & \text{if } z > \frac{1}{3}(1-\alpha)x^2 \\ (1; z+\alpha x^2; x) & \text{if } z < \frac{1}{3}(1-\alpha)x^2 \end{cases}$$

creates a piecewise continuous system $(x'; y'; z') = \mathbf{H}(x,y,z)$ in which all trajectories are two-cycles except along the stasis curve.

4.2. **A Normal Form.** In the previous example the trajectory at the origin had unit velocity and curvature, and different structures arose for different values of the torsion. This suggests using Frenet frame neighborhoods of size ϵ in the direction of velocity, ϵ^2 in the direction of curvature, and ϵ^3 in the direction of torsion. In this section we show that any generic pair of systems near a pseudo-regular stasis point can be simultaneously diffeomorphed to a pair of form

$$\begin{aligned} x' &= 1 & \quad x' &= -1 + O(x,y,z) \\ y' &= z + \alpha x^2 + O(z^2, xz, yz) & \quad y' &= 0 \\ z' &= x + O(x^2, xy, xz) & \quad z' &= 0 \end{aligned} \tag{4.4}$$

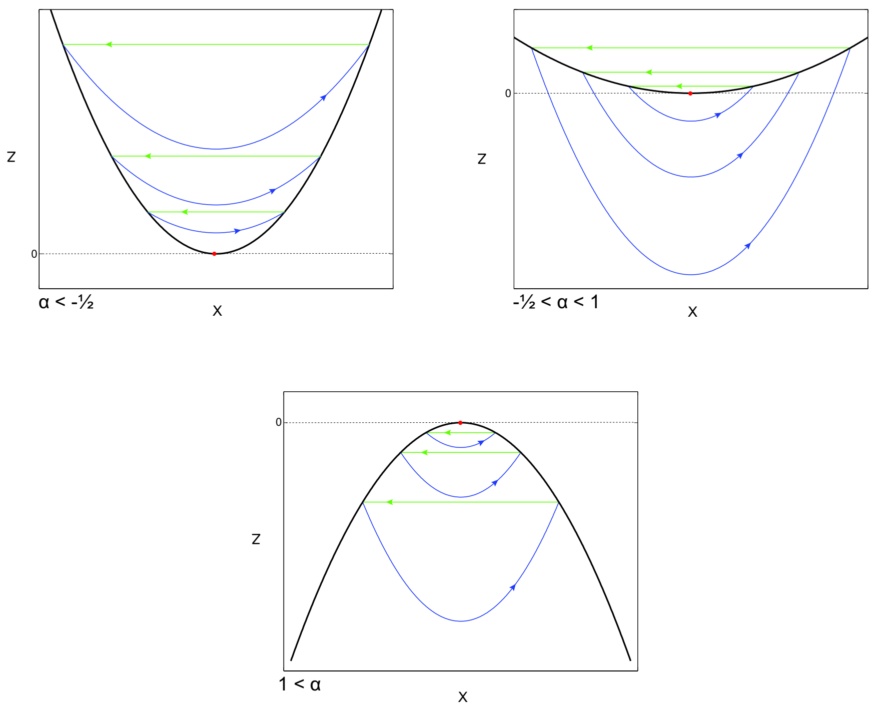

FIGURE 10. Two-Cycles and switching manifolds

which, by taking a Frenet frame neighborhood and renormalizing

$$
\begin{aligned}
t_{\text{new}} &= \epsilon t_{\text{old}}\\
x_{\text{new}} &= \epsilon x_{\text{old}}\\
y_{\text{new}} &= \epsilon^3 y_{\text{old}}\\
z_{\text{new}} &= \epsilon^2 z_{\text{old}}
\end{aligned}
$$

converges to the normal form (4.1) as $\epsilon \to 0$. Assuming sufficient differentiability, one could apply this to higher dimensional systems with an argument that the first three Frenet dimensions dominate the topology of the flow near the origin.

For a pair of C^1 vector fields $\mathbf{V}_1$, $\mathbf{V}_2$ near a regular stasis point, we begin by assuming the stasis point is at the origin and that we have rectified the second flow $\mathbf{V}_2 = (-1; 0; 0)$ near the origin.

Theorem 4.1. *A C^1 system*

$$
\begin{aligned}
x' &= f(x, y, z) \qquad & x' &= -1\\
y' &= g(x, y, z) \qquad & y' &= 0\\
z' &= h(x, y, z) \qquad & z' &= 0
\end{aligned}
\tag{4.5}
$$

with a stasis point at $\mathbf{0}$, and properties

(A1) *∇h and ∇g are independent.*
(A2) *$\partial_x g(\mathbf{0}) \neq 0$ or $\partial_x h(\mathbf{0}) \neq 0$*

is, in a neighborhood of $\mathbf{0}$*, diffeomorphic to*

$$\begin{gathered}
\begin{aligned}
x' &= 1 & \qquad x' &= \tilde{f}(x,y,z) \\
y' &= \tilde{g}(x,y,z) & y' &= 0 \\
z' &= \tilde{h}(x,y,z) & z' &= 0
\end{aligned} \\
\tilde{f}(0,0,0) < 0 \\
\tilde{g}(0,y,0) = 0 \\
\partial_x \tilde{g}(0,y,0) = 0 \\
\tilde{h}(0,y,z) = 0
\end{gathered} \tag{4.6}$$

Note that condition (A2) is sufficient (but not necessary) for pseudo-regularity. The system (4.6) is equivalent to the system (4.4). We prove Theorem 4.1 by defining a sequence of four diffeomorphisms that will preserve the direction of $\mathbf{V}_2$ and diffeomorph $\mathbf{V}_1$ to the required form.

Proof. Without loss of generality we can assume

(A1') $(\partial_x g\, \partial_z h - \partial_z g\, \partial_x h)(\mathbf{0}) \neq 0$

(A2') $\partial_x h(\mathbf{0}) \neq 0$.

The first diffeomorphism brings the surface $0 = h(x,y,z)$ to the $x = 0$ plane. By (A2') we can parameterize the surface $0 = h(x,y,z)$ by $x = p(y,z)$. Define

$$\begin{gathered}
u = x - p(y,z) \\
v = y \\
w = z
\end{gathered}$$

with derivative

$$\begin{pmatrix} 1 & \partial_y p & \partial_z p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which preserves $\mathbf{V}_2 = (-1;0;0)$. Applying this diffeomorphism and switching back to x, y, z coordinates, we now have flows of the form (4.5) with $h(0,y,z) = 0$. The quantities $\partial_x h(\mathbf{0})$ and $(\partial_x g\, \partial_z h - \partial_z\, g \partial_x h)(\mathbf{0})$ are invariant under this diffeomorphism, and hence conditions (A1') and (A2') hold in the new coordinates.

The null surface $h = 0$ is now the $x = 0$ plane, and the stasis curve is the intersection of $g = 0$ with this plane. By (A1'), $(\nabla g \times \nabla h)(\mathbf{0})$ has a y component and so we can parameterize the stasis curve as $(0, y, s(y))$ near the origin. The second diffeomorphism brings the stasis curve to the y-axis and is defined by

$$\begin{gathered}
u = x \\
v = y \\
w = z - s(y)
\end{gathered}$$

with derivative

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -s'(y) & 1 \end{pmatrix}$$

which preserves $\mathbf{V}_2 = (-1;0;0)$. Applying this diffeomorphism and switching back to x, y, z coordinates, we now have flows of the form (4.5) with $g(0,y,0) = 0$ and $h(0,y,z) = 0$. The quantity $(\partial_x g\, \partial_z h - \partial_z g \partial_x h)(\mathbf{0})$ is invariant under this diffeomorphism, and hence condition (A1') hold in the new coordinates.

Trajectories of the first system projected onto the $x = 0$ plane will have cusps at points $(0, y, 0)$, and the tangent at these cusps will have direction determined by

$$r(y) = \frac{dy}{dz} = \frac{\partial_x g}{\partial_x h}$$

which is finite under condition (A2'). The third diffeomorphism brings these cusps upright (see Figure 11) and is defined as

$$\begin{aligned} u &= x \\ v &= y - z\, r(y) \\ w &= z \end{aligned}$$

which preserves the y-axis, the $x = 0$ plane, and has derivative

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - z\, r'(y) & -r(y) \\ 0 & 0 & 1 \end{pmatrix}$$

which preserves $\mathbf{V}_2 = (-1; 0; 0)$. Applying this diffeomorphism and switching back to x, y, z coordinates, we can now assume our flows are of the form (4.5) with $g(0, y, 0) = 0$, $h(0, y, z) = 0$, and $\partial_x g(0, y, 0) = 0$.

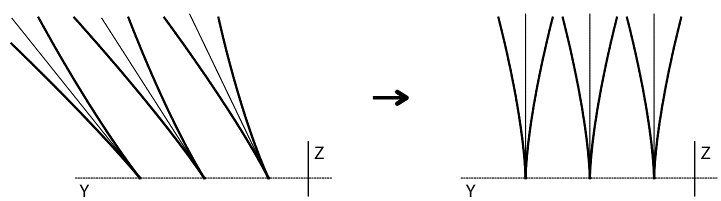

FIGURE 11. Cusp directions

The final diffeomorphism makes the second system flow at unit speed in the x-direction, which we achieve by rectifying the isotemporal surfaces of the second flow. That is, for trajectories $(x(t), y(t), z(t))$ with $x(0) = 0$, we define $T(x(t), y(t), z(t)) = t$ and diffeomorph with

$$\begin{aligned} u &= T(x, y, z) \\ v &= y \\ w &= z \end{aligned}$$

which is the identity on the $x = 0$ plane and has derivative

$$\begin{pmatrix} \partial_x T & \partial_y T & \partial_z T \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying this diffeomorphism and switching back to x, y, z coordinates makes the flows of the form (4.6). $\square$

5. Open Questions

To what extent does the switching system structure described in section 3 generalize to more than two flows?

For a computational challenge, given two $\mathbb{R}^3$ flows $\mathbf{F}_1$ and $\mathbf{F}_2$ near a stasis point, determine which of the three topological cases described in section 4 hold. In particular this would give a criterion to determine when there is a neighborhood of the stasis curve that is foliated with two-cycles.

Our construction focused on generic cases, which raises the question as to what types of behaviors occur in non-generic cases.

Appendix:Implicit Function Theorem

There are many statements and proofs of the implicit function theorem (see [6]), a succinct version for the current work is as follows:

Theorem 5.1 (Implicit Function Theorem). *If $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$ is defined near $\mathbf{x}_0$ with $\mathbf{F}(x_0) = \mathbf{0}$, and $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}_0)$ of rank n, then there exists an $m - n$ dimensional manifold of solutions to $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ containing $\mathbf{x}_0$. The tangent plane to this manifold at $\mathbf{x}_0$ is perpendicular to the rows of $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}_0)$.*

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