Electronic Journal of Differential Equations, Vol. 2009(2009), No. 81, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF WEAK SOLUTIONS FOR NONLINEAR SYSTEMS INVOLVING SEVERAL P-LAPLACIAN OPERATORS

SALAH A. KHAFAGY, HASSAN M. SERAG

Abstract. In this article, we study nonlinear systems involving several p-Laplacian operators with variable coefficients. We consider the system

$$-\Delta_{p_i} u_i = a_{ii}(x)|u_i|^{p_i-2}u_i - \sum_{j\neq i}^{n} a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j + f_i(x),$$

where Δ_p denotes the p-Laplacian defined by $\Delta_p u \equiv \operatorname{div}[|\nabla u|^{p-2}\nabla u]$ with $p > 1$, $p \neq 2$; $\alpha_i \geq 0$; f_i are given functions; and the coefficients $a_{ij}(x)$ $(1 \leq i, j \leq n)$ are bounded smooth positive functions. We prove the existence of weak solutions defined on bounded and unbounded domains using the theory of nonlinear monotone operators.

1. Introduction

The generalized formulation of many boundary-value problems for partial differential equations leads to operator equations of the form

$$A(u) = f$$

on a Banach space V. For this operator equation, we have the so-called weak formulation:

Find $u \in V$ such that $(A(u), v) = (f, v)$ for all $v \in V$.

Then functional analysis has tools for proving existence of generalized (weak) solutions for a relatively wide class of differential equations that appear in mathematical physics and industry.

The existence of weak solutions for 2×2 nonlinear systems involving several p-Laplacian operators have been proved, using the method of sub and super solutions in [5], and using the theory of nonlinear monotone operators in [6].

Here, we use the theory of nonlinear monotone operators to prove the existence of weak solutions for the following nonlinear systems involving several p-Laplacian operators with variable coefficients defined on a bounded domain Ω of $\mathbb{R}^N$ with

---

2000 *Mathematics Subject Classification.* 74H20, 35J65.
Key words and phrases. Existence of weak solution; nonlinear system, p-Laplacian.
©2009 Texas State University - San Marcos.
Submitted December 9, 2008. Published July 10, 2009.

boundary $\partial\Omega$,

$$
\begin{gathered}
\begin{aligned}
-\Delta_{p_i} u_i &\equiv -\operatorname{div}[|\nabla u_i|^{p_i-2}\nabla u_i] \\
&= a_{ii}(x)|u_i|^{p_i-2}u_i - \sum_{j\neq i}^{n} a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j + f_i(x) \quad \text{in } \Omega,
\end{aligned} \\
u_i = 0, \quad i = 1,2,\dots,n, \quad \text{on } \partial\Omega.
\end{gathered}
$$

Then, we generalize our results to systems defined on the whole space $\mathbb{R}^N$.

This article is organized as follow: In section 2 we introduce some technical results and definitions concerning the theory of nonlinear monotone operators. We study the existence of weak solutions for $n \times n$ nonlinear systems defined on a bounded domain in section 3, and on unbounded domains in section 4.

2. Preliminary results

First, we introduce some results concerning the theory of nonlinear monotone operators [4].

Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is:
Bounded if it maps bounded sets into bounded; i.e., for each $r > 0$ there exists $M > 0$ (M depending on r) such that

$$
\|u\| \le r \text{ implies } \|A(u)\| \le M, \quad \forall u \in V;
$$

coercive if $\lim_{\|u\|\to\infty} \langle A(u), u\rangle / \|u\| = \infty$;
monotone if $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \ge 0$ for all $u_1, u_2 \in V$;
strictly monotone if $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0$ for all $u_1, u_2 \in V$, $u_1 \neq u_2$;
continuous if $u_k \to u$ implies $A(u_k) \to A(u)$, for all $u_k, u \in V$;
strongly continuous if $u_k \xrightarrow{w} u$ implies $A(u_k) \to A(u)$, for all $u_k, u \in V$;
continuous on finite-dimensional subspaces if $A : V_n \to V_n'$ is continuous for each subspace V_n of finite dimension.
demicontinuous if $u_k \to u$ implies $A(u_k) \xrightarrow{w} A(u)$, for all $u_k, u \in V$;
the operator A is said to be satisfy the M_0-condition if $u_k \xrightarrow{w} u$, $A(u_k) \xrightarrow{w} f$, and $[\langle A(u_k), u_k\rangle \to \langle f, u\rangle]$ imply $A(u) = f$.

Remark 2.1.
(i) Strongly continuous operators are continuous, and they are continuous on finite dimensional subspaces.
(ii) Strongly continuous operators are bounded and satisfy the M_0-condition.
(iii) Strictly monotone operators are monotone operators.
(iv) Monotone and continuous operators satisfy the M_0-condition.

Theorem 2.2. *Let V be a separable reflexive Banach space and $A : V \to V'$ an operator which is: coercive, bounded, continuous on finite-dimensional subspaces and satisfying the M_0-condition. Then the equation $A(u) = f$ admits a solution for each $f \in V'$.*

Next, we introduce the Sobolev space $W^{1,p}(\Omega), 1 < p < \infty$, defined as the completion of $C^\infty(\Omega)$ with respect to the norm (see [1])

$$
\|u\|_{W^{1,p}} = \Big[\int_\Omega |\nabla u|^p + |u|^p\Big]^{1/p} < \infty. \tag{2.1}
$$

Since we are studying a Dirichlet problem, we define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}} = \Big[\int_\Omega |\nabla u|^p\Big]^{1/p} < \infty, \tag{2.2}$$

which is equivalent to the norm given by (2.1). Both spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are well defined reflexive Banach Spaces. The space $W_0^{1,p}(\Omega)$ is compactly imbedded in the space $L^p(\Omega)$; i.e.,

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \tag{2.3}$$

which implies

$$\|u\|_{L^p(\Omega)} \le c\, \|u\|_{W_0^{1,p}(\Omega)}, \quad \text{i.e.,} \int_\Omega a(x)|u|^p \le c' \int_\Omega |\nabla u|^p \tag{2.4}$$

for every $u \in W_0^{1,p}(\Omega)$, where $a(x)$ is a smooth bounded positive function.

Now, we introduce some results [2] concerning the eigenvalue problem

$$\begin{gathered} -\Delta_p u \equiv -\operatorname{div}[|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

We will say that $\lambda \in \mathbb{R}$ is an eigenvalue of (2.5) if there exists $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \lambda \int_\Omega a(x)|u|^{p-2}u\varphi$$

hods for all $\varphi \in W_0^{1,p}(\Omega)$. Then u is called an eigenfunction corresponding to the eigenvalue λ.

Lemma 2.3. *The eigenvalue problem* (2.5) *admits a positive principal eigenvalue $\lambda = \lambda_a(\Omega) > 0$ which is associated with a positive eigenfunction $u \ge 0$ a.e. in Ω normalized by $\|u\|_p = 1$. Moreover, the first eigenvalue is characterized by*

$$\lambda_a(\Omega) = \inf\Big\{\int_\Omega |\nabla u|^p : \int_\Omega a(x)|u|^p = 1\Big\}. \tag{2.6}$$

Also, from the characterization of the first eigenvalue given by (2.6), we have

$$\lambda_a(\Omega)\int_\Omega a(x)|u|^p \le \int_\Omega |\nabla u|^p. \tag{2.7}$$

3. Nonlinear systems defined on bounded domains

Let us consider the nonlinear system

$$\begin{gathered} -\Delta_{p_i} u_i = a_{ii}(x)|u_i|^{p_i-2}u_i - \sum_{j\neq i}^n a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j + f_i(x) \quad \text{in } \Omega, \\ u_i = 0, \quad i = 1,2,\dots,n, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where $a_{ii}(x)$ is a smooth bounded positive function, Ω is a bounded domain of $\mathbb{R}^N$, and

$$\alpha_i \ge 0, \quad f_i \in L^{p_i^*}(\Omega), \tag{3.2}$$

$$\frac{1}{p_i} + \frac{1}{p_i^*} = 1, \quad \frac{\alpha_i + 1}{p_i} = \frac{1}{2}, \quad i = 1,2,\dots,n. \tag{3.3}$$

Theorem 3.1. *For $(f_i) \in \prod_{i=1}^n L^{p_i^*}(\Omega)$, there exists a weak solution (u_i) in the space $\prod_{i=1}^n W_0^{1,p_i}(\Omega)$ for the system (3.1), if*

$$\lambda_{a_{ii}}(\Omega) > 1, \quad i = 1, 2, \dots, n. \tag{3.4}$$

Proof. We transform the weak formulation of (3.1) to the operator form $(A-B)U = F$, where, A, B and F are operators defined on $\prod_{i=1}^n W_0^{1,p_i}(\Omega)$ by

$$(AU, \Phi) \equiv (A(u_1, u_2, \dots, u_n), (\phi_1, \phi_2, \dots, \phi_n)) = \sum_{i=1}^n \int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \nabla \phi_i, \tag{3.5}$$

$$\begin{aligned}(BU, \Phi) &\equiv (B(u_1, u_2, \dots, u_n), (\phi_1, \phi_2, \dots, \phi_n)) \\ &= \sum_{i=1}^n [\int_\Omega a_{ii}(x)|u_i|^{p_i-2} u_i \phi_i - \sum_{j\neq i}^n \int_\Omega a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j} u_j \phi_i],\end{aligned} \tag{3.6}$$

$$(F, \Phi) \equiv ((f_1, f_2, \dots, f_n), (\phi_1, \phi_2, \dots, \phi_n)) = \sum_{i=1}^n \int_\Omega f_i \phi_i. \tag{3.7}$$

Now, consider the operator J defined by

$$(J(u), \phi) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi. \tag{3.8}$$

This operator is bounded: Since

$$|(J(u), \phi)| \le \int_\Omega |\nabla u|^{p-1} |\nabla \phi|,$$

using Hölder's inequality, we obtain

$$|(J(u), \phi)| \le \Big[\int_\Omega |\nabla u|^p\Big]^{\frac{p-1}{p}} \Big[\int_\Omega |\nabla \phi|^p\Big]^{1/p} = \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \|\phi\|_{W_0^{1,p}(\Omega)}.$$

Also, we can prove that J is continuous, let us assume that $u_k \to u$ in $W_0^{1,p}(\Omega)$. Then $\|u_k - u\|_{W_0^{1,p}(\Omega)} \to 0$, so that $\|\nabla u_k - \nabla u\|_{L^p(\Omega)} \to 0$. Applying Dominated Convergence Theorem, we obtain

$$\|(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u)\|_{L^p(\Omega)} \to 0,$$

and hence

$$\|J(u_k) - J(u)\|_{L^p(\Omega)} \le \|(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u)\|_{L^p(\Omega)} \to 0.$$

Finally, J is strictly monotone:

$$\begin{aligned}(J(u_1) - J(u_2), u_1 - u_2) &= \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla u_1 + \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla u_2 \\ &\quad - \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 - \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1;\end{aligned}$$

using Hölder's inequality, we obtain

$$
\begin{aligned}
&(J(u_1) - J(u_2), u_1 - u_2) \\
&\geq \int_\Omega |\nabla u_1|^p + \int_\Omega |\nabla u_2|^p - \Big[\int_\Omega |\nabla u_1|^p\Big]^{\frac{p-1}{p}} \Big[\int_\Omega |\nabla u_2|^p\Big]^{\frac{1}{p}} \\
&\quad - \Big[\int_\Omega |\nabla u_2|^p\Big]^{\frac{p-1}{p}} \Big[\int_\Omega |\nabla u_1|^p\Big]^{1/p} \\
&= \|u_1\|^p_{W_0^{1,p}(\Omega)} + \|u_2\|^p_{W_0^{1,p}(\Omega)} - \|u_1\|^{p-1}_{W_0^{1,p}(\Omega)}\|u_2\|_{W_0^{1,p}(\Omega)} - \|u_2\|^{p-1}_{W_0^{1,p}(\Omega)}\|u_1\|_{W_0^{1,p}(\Omega)},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&(J(u_1) - J(u_2), u_1 - u_2) \\
&\geq (\|u_1\|^{p-1}_{W_0^{1,p}(\Omega)} - \|u_2\|^{p-1}_{W_0^{1,p}(\Omega)})(\|u_1\|_{W_0^{1,p}(\Omega)} - \|u_2\|_{W_0^{1,p}(\Omega)}) > 0.
\end{aligned}
$$

Now, AU can be written as the sum of $J_1(u_1), J_2(u_2), \dots, J_n(u_n)$ where

$$
(J_i(u_i), \phi_i) = \int_\Omega |\nabla u_i|^{p_i - 2}\nabla u_i \nabla \phi_i, \quad i = 1, 2, \dots, n,
$$

and as above, the operators J_1, $J_2, \dots$ and J_n are bounded, continuous and strictly monotone; so their sum, the operator A, will be the same.

For the operator B,

$$
B\colon \prod_{i=1}^n W_0^{1,p_i}(\Omega) \to \prod_{i=1}^n L^{p_i}(\Omega),
$$

we can prove that it is a strongly continuous operator. To prove that, let us assume that $u_{ik} \xrightarrow{w} u_i$ in $W_0^{1,p_i}(\Omega)$, $i = 1, 2, \dots, n$. Then, using (2.3), $(u_{ik}) \to (u_i)$ in $\prod_{i=1}^n L^{p_i}(\Omega)$. By the Dominated Convergence Theorem,

$$
\begin{gathered}
a_{ii}(x)|u_{ik}|^{p_i-2}u_{ik} \to a_{ii}(x)|u_i|^{p_i-2}u_i \quad \text{in } L^{p_i}(\Omega), \\
-a_{ij}(x)|u_{ik}|^{\alpha_i}|u_{jk}|^{\alpha_j}u_{jk} \to -a_{ij}(x)|u_{_i}|^{\alpha_i}|u_j|^{\alpha_j}u_j \quad \text{in } L^{p_j}(\Omega),
\end{gathered}
$$

Since

$$
\begin{aligned}
(BU_k - BU, W) &= (B(u_{1k}, u_{2k}, \dots, u_{nk}) - B(u_1, u_2, \dots, u_n), (w_1, w_2, \dots, w_n)) \\
&= \sum_{i=1}^n \Big[\int_\Omega a_{ii}(x)(|u_{ik}|^{p_i-2}u_{ik} - |u_i|^{p_i-2}u_i)w_i \\
&\quad - \sum_{j\neq i}^n \int_\Omega a_{ij}(x)(|u_{ik}|^{\alpha_i}|u_{jk}|^{\alpha_j}u_{jk} - |u_{_i}|^{\alpha_i}|u_j|^{\alpha_j}u_j)w_i\Big],
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\|BU_k - BU\| \leq \sum_{i=1}^n &\Big[\|a_{ii}(x)(|u_{ik}|^{p_i-2}u_{ik} - |u_i|^{p_i-2}u_i)\|_{L^{p_i}(\Omega)} \\
&+ \sum_{j\neq i}^n \|a_{ij}(x)(|u_{ik}|^{\alpha_i}|u_{jk}|^{\alpha_j+1} - |u_{_i}|^{\alpha_i}|u_j|^{\alpha_j+1})\Big)\|_{L^{p_i}(\Omega)}] \to 0.
\end{aligned}
$$

This proves that B is a strongly continuous operators. According to Remark 2.1, the operator $A - B$ satisfies the M_0-condition. Now, to apply Theorem 2.2, it remains to prove that $A-B$ is a coercive operator

$$
\begin{aligned}
&((A-B)U,U)\\
&=\sum_{i=1}^{n}\int_\Omega|\nabla u_i|^{p_i}-\sum_{i=1}^{n}\Big[\int_\Omega a_{ii}(x)|u_i|^{p_i}-\sum_{j\neq i}^{n}\int_\Omega a_{ij}(x)|u_i|^{\alpha_i+1}|u_j|^{\alpha_j+1}\Big]\\
&\geq\sum_{i=1}^{n}\int_\Omega|\nabla u_i|^{p_i}-\sum_{i=1}^{n}\int_\Omega a_{ii}(x)|u_i|^{p_i}.
\end{aligned}
$$

Using (2.7), we obtain

$$
\begin{aligned}
((A-B)U,U)&\geq\sum_{i=1}^{n}\int_\Omega|\nabla u_i|^{p_i}-\sum_{i=1}^{n}\frac{1}{\lambda_{a_{ii}}(\Omega)}\int_\Omega|\bigtriangledown u_i|^{p_i}\\
&=\sum_{i=1}^{n}(1-\frac{1}{\lambda_{a_{ii}}(\Omega)})\int_\Omega|\bigtriangledown u_i|^{p_i},
\end{aligned}
$$

and hence,

$$
((A-B)U,U)\geq k\sum_{i=1}^{n}\|u_i\|^{p_i}_{W_0^{1,p_i}(\Omega)}=k\|(u_i)\|_{\prod_{i=1}^{n}W_0^{1,p_i}(\Omega)}.
$$

So that

$$
((A-B)U,U)\to\infty\quad\text{as }\|(u_i)\|_{\prod_{i=1}^{n}W_0^{1,p_i}(\Omega)}\to\infty.
$$

This proves the coercivity condition and so, the existence of a weak solution for systems (3.1). $\square$

4. Nonlinear systems defined on $\mathbb{R}^N$

We consider the nonlinear system

$$
\begin{gathered}
-\Delta_{p_i}u_i=a_{ii}(x)|u_i|^{p_i-2}u_i-\sum_{j\neq i}^{n}a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j+f_i(x),\quad x\in\mathbb{R}^N,\\
\lim_{|x|\to\infty}u_i(x)=0,\quad i=1,2,\dots,n,\quad x\in\mathbb{R}^N.
\end{gathered}\tag{4.1}
$$

We assume that $1<p_i<N$, $i=1,2,\dots,n$, and the coefficients $a_{ii}(x)$ and $a_{ij}(x)$ are smooth bounded positive functions such that

$$
0<a_{ii}(x)\in L^{\frac{N}{p_i}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\quad 0<a_{ij}(x)\in L^{\frac{N}{\alpha_i+\alpha_j+2}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N).\tag{4.2}
$$

To discuss this problem, we need the following results which are studied in [3] and that we recall briefly.

Let us introduce the Sobolev reflexive Banach space

$$
D^{1,p}(\mathbb{R}^N)=\{u\in L^{\frac{Np}{N-p}}(\mathbb{R}^N):\nabla u\in(L^p(\mathbb{R}^N))^n\},
$$

which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$
\|u\|_{D^{1,p}(\mathbb{R}^N)}=\Big[\int_{\mathbb{R}^N}|\nabla u|^p\Big]^{1/p}<\infty.\tag{4.3}
$$

Moreover $D^{1,p}(\mathbb{R}^N)$ is embedded continuously in the space $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$; that is, $D^{1,p}(\mathbb{R}^N)\hookrightarrow L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, which implies

$$
\|u\|_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)}\leq k\ \|u\|_{D^{1,p}(\mathbb{R}^N)}.\tag{4.4}
$$

Lemma 4.1. *The eigenvalue problem*

$$\begin{gathered} -\Delta_{p} u \equiv -\operatorname{div}[|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \quad u > 0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{4.5}$$

admits a positive principal eigenvalue $\lambda = \lambda_a(\Omega)$ which is associated with a positive eigenfunction $u \in D^{1,p}(\mathbb{R}^N)$. Moreover, the principal eigenvalue $\lambda_a(\Omega)$ is characterized by

$$\lambda_a(\Omega) \int_{\mathbb{R}^N} a(x)|u|^p \le \int_{\mathbb{R}^N} |\nabla u|^p, \quad \forall\, u \in D^{1,p}(\mathbb{R}^N) \tag{4.6}$$

where

$$0 < a(x) \in L^{\frac{N}{p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{4.7}$$

In this section, we assume that

$$\begin{gathered} \alpha_i \ge 0, \quad f_i \in L^{\frac{Np_i}{N(p_i-1)+p_i}}(\mathbb{R}^N), \quad \alpha_i + \alpha_j + 2 < N, 1 < p_i < n \\ \frac{1}{p_i} + \frac{1}{p_i^*} = 1, \quad \frac{\alpha_i + 1}{p_i} = \frac{1}{2}, \quad i = 1, 2, \dots, n. \end{gathered} \tag{4.8}$$

Theorem 4.2. *For $(f_i) \in \prod_{i=1}^n L^{\frac{Np_i}{N(p_i-1)+p_i}}(\mathbb{R}^N)$, there exists a weak solution (u_i) in $\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)$ for system (4.1), if*

$$\lambda_{a_{ii}}(\Omega) > 1, \quad i = 1, 2, \dots, n. \tag{4.9}$$

Proof. As in section 3, we transform the weak formulation of the system (4.1) to the operator form $(A - B)U = F$, where, A, B and F are operators defined on $\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)$ by

$$\begin{aligned} (AU, \Phi) &\equiv (A(u_1, u_2, \dots, u_n), (\phi_1, \phi_2, \dots, \phi_n)) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^N} |\nabla u_i|^{p_i-2}\nabla u_i \nabla \phi_i = \sum_{i=1}^n (J_i(u_i), \phi_i), \end{aligned} \tag{4.10}$$

$$\begin{aligned} (BU, \Phi) &\equiv (B(u_1, u_2, \dots, u_n), (\phi_1, \phi_2, \dots, \phi_n)) \\ &= \sum_{i=1}^n [\int_{\mathbb{R}^N} a_{ii}(x)|u_i|^{p_i-2}u_i\phi_i - \sum_{j\neq i}^n \int_{\mathbb{R}^N} a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j\phi_i], \end{aligned} \tag{4.11}$$

$$(F, \Phi) \equiv ((f_1, f_2, \dots, f_n), (\phi_1, \phi_2, \dots, \phi_n)) = \sum_{i=1}^n \int_{\mathbb{R}^N} f_i\phi_i. \tag{4.12}$$

First, we prove that A, B and F are bounded operators on $\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)$.

For the operator A, by using (4.10) and applying Holder inequality, we have

$$\begin{aligned} |(AU, \Phi)| &\le \sum_{i=1}^n \int_{\mathbb{R}^N} |\nabla u_i|^{p_i-1}|\nabla\phi_i| \\ &\le \sum_{i=1}^n \Big[\int_{\mathbb{R}^N} |\nabla u_i|^{p_i}\Big]^{(p_i-1)/p_i} \Big[\int_{\mathbb{R}^N} |\nabla\phi_i|^{p_i}\Big]^{1/p_i} \\ &= \sum_{i=1}^n \|u_i\|_{D^{1,p_i}(\mathbb{R}^N)}^{p_i-1} \|\phi_i\|_{D^{1,p_i}(\mathbb{R}^N)} \\ &= \Big(\sum_{i=1}^n \|u_i\|_{D^{1,p_i}(\mathbb{R}^N)}^{p_i-1}\Big)\Big(\|(\phi_i)\|_{\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)}\Big). \end{aligned}$$

This proves the boundedness of the operator A.

For the operator B, we have

$$
\begin{aligned}
|(BU,\Phi)| &\le \sum_{i=1}^{n}[\int_{\mathbb{R}^N} a_{ii}(x)|u_i|^{p_i-1}|\phi_i| + \sum_{j\neq i}^{n}\int_{\mathbb{R}^N} a_{ij}(x)|u_i|^{\alpha_i}|u_j|^{\alpha_j+1}|\phi_i|] \\
&\le \sum_{i=1}^{n}\Big[\Big(\int_{\mathbb{R}^N} a_{ii}(x)^{\frac{N}{p}}\Big)^{\frac{p}{N}}\Big(\int_{\mathbb{R}^N}|u_i|^{\frac{Np_i}{N-p_i}}\Big)^{\frac{(p_i-1)(N-p_i)}{Np_i}}\Big(\int_{\mathbb{R}^N}|\phi_i|^{\frac{Np_i}{N-p_i}}\Big)^{\frac{N-p_i}{Np_i}} \\
&\quad + \sum_{j\neq i}^{n}\Big[\int_{\mathbb{R}^N}(a_{ij}(x))^{\frac{N}{\alpha_i+\alpha_j+2}}\Big]^{\frac{\alpha_i+\alpha_j+2}{N}}\Big[\int_{\mathbb{R}^N}|u_i|^{\frac{Np_i}{N-p_i}}\Big]^{\frac{\alpha_i(N-p_i)}{Np_i}} \\
&\quad \times\Big[\int_{\mathbb{R}^N}|u_j|^{\frac{Np_j}{N-p_j}}\Big]^{\frac{(\alpha_j+1)(N-p_j)}{Np_j}}\Big[\int_{\mathbb{R}^N}|\phi_i|^{\frac{Np_i}{N-p_i}}\Big]^{\frac{N-p_i}{Np_i}}\Big] \\
&\le \sum_{i=1}^{n}\Big[k_i\|u_i\|^{p_i-1}_{D^{1,p_i}(\mathbb{R}^N)}\|\phi_i\|_{D^{1,p_i}(\mathbb{R}^N)} \\
&\quad + \sum_{j\neq i}^{n} l_i\|u_i\|^{\alpha_i}_{D^{1,p_i}(\mathbb{R}^N)}\|u_j\|^{\alpha_j+1}_{D^{1,p_j}(\mathbb{R}^N)}\|\phi_i\|_{D^{1,p_i}(\mathbb{R}^N)}\Big] \\
&= \Big[\sum_{i=1}^{n}\Big[k_i\|u_i\|^{p_i-1}_{D^{1,p_i}(\mathbb{R}^N)} + \sum_{j\neq i}^{n} l_i\|u_i\|^{\alpha_i}_{D^{1,p_i}(\mathbb{R}^N)}\|u_j\|^{\alpha_j+1}_{D^{1,pj}(\mathbb{R}^N)}\Big]\Big] \\
&\quad \times\|(\phi_i)\|_{\prod_{i=1}^{n}D^{1,p_i}(\mathbb{R}^N)}
\end{aligned}
$$

For the operator F, we have $(F,\Phi)=\sum_{i=1}^{N}\int_{\mathbb{R}^n} f_i\phi_i$ and so

$$
\begin{aligned}
|(F,\Phi)| &= |\sum_{i=1}^{N}\int_{\mathbb{R}^n} f_i\phi_i| \\
&\le \sum_{i=1}^{N}\Big[\int_{\mathbb{R}^n}|f_i|^{\frac{np_i}{n(p_i-1)+p_i}}\Big]^{\frac{n(p_i-1)+p_i}{np_i}}\Big[\int_{\mathbb{R}^n}|\phi_i|^{\frac{np_i}{n-p_i}}\Big]^{\frac{n-p_i}{np_i}} \\
&= \sum_{i=1}^{N}(\|f_i\|_{L^{\frac{np_i}{n(p_i-1)+p_i}}(\mathbb{R}^n)})\|(\phi_i)\|_{\prod_{i=1}^{N}D^{1,p_i}(\mathbb{R}^n)}.
\end{aligned}
$$

Now, as in section 3, the operator A defined by $(AU,\Phi)=\sum_{i=1}^{n}(J_i(u_i),\Phi)$ is continuous. Also it is strictly monotone on $\prod_{i=1}^{n}D^{1,p_i}(\mathbb{R}^N)$, since

$$
\begin{aligned}
&(J_i(u_1)-J_i(u_2),u_1-u_2) \\
&\ge (\|u_1\|^{p_i-1}_{D^{1,p_i}(\mathbb{R}^N)}-\|u_2\|^{p_i-1}_{D^{1,p_i}(\mathbb{R}^N)})(\|u_1\|_{D^{1,p_i}(\mathbb{R}^N)}-\|u_2\|_{D^{1,p_i}(\mathbb{R}^N)})>0.
\end{aligned}
$$

For the operator B, we can prove that it is a strongly continuous operator by using Dominated Convergence theorem and continuous imbedding property for the space $\prod_{i=1}^{n}D^{1,p_i}(\mathbb{R}^N)$ into $\prod_{i=1}^{n}L^{\frac{Np_i}{N-p_i}}(\mathbb{R}^N)$. To prove that, let us assume that $u_{ik}\xrightarrow{w}u_i$ in $D^{1,p_i}(\mathbb{R}^N)$, $i=1,2,\dots,n$. Then $(u_{ik})\to(u_i)$ in $\prod_{i=1}^{n}L^{\frac{Np_i}{N-p_i}}(\mathbb{R}^N)$. Now, the sequence (u_{ik}) is bounded in $D^{1,p_i}(\mathbb{R}^N)$, $i=1,2,\dots,n$, then it is containing a subsequence again denoted by (u_{ik}) converges strongly to u_i in $L^{\frac{Np_i}{N-p_i}}(B_{r_0})$, $i=1,2,\dots,n$, for any bounded ball $B_{r_0}=\{x\in\mathbb{R}^N:\|x\|\le r_0\}$. Since

$u_{ik}, u_i \in L^{\frac{Np_i}{N-p_i}}(B_{r_0})$, Then using the Dominated Convergence Theorem, we have

$$\|a_{ii}(x)(|u_{ik}|^{p_i-2}u_{ik} - |u_i|^{p_i-2}u_i)\|_{\frac{Np_i}{N(p_i-1)+p_i}} \to 0,$$

$$\|a_{ij}(x)(|u_{ik}|^{\alpha_i-1}|u_{jk}|^{\alpha_j+1}u_{jk} - |u_i|^{\alpha_i-1}|u_j|^{\alpha_j+1}u_j)\|_{\frac{Np_i}{N(p_i-1)+p_i}} \to 0,$$

for $i = 1, 2, \dots, n$. Since

$$\begin{aligned}
((BU_k - BU), W) &= (B(u_{1k}, u_{2k}, \dots, u_{nk}) - B(u_1, u_2, \dots, u_n), (w_1, w_2, \dots, w_n)) \\
&= \sum_{i=1}^{n} \Big[\int_{\mathbb{R}^N} a_{ii}(x)(|u_{ik}|^{p_i-2}u_{ik} - |u_i|^{p_i-2}u_i)w_i \\
&\quad - \sum_{j\neq i}^{n} \int_{\mathbb{R}^N} a_{ij}(x)(|u_{ik}|^{\alpha_i}|u_{jk}|^{\alpha_j}u_{jk} - |u_i|^{\alpha_i}|u_j|^{\alpha_j}u_j)w_i \Big],
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|BU_k - BU\|_{\prod_{i=1}^n D^{1,p_i}(B_{r_0})} \\
&\le \sum_{i=1}^{n} \Big[\|a_{ii}(x)(|u_{ik}|^{p_i-2}u_{ik} - |u_i|^{p_i-2}u_i)\|_{\frac{Np_i}{N(p_i-1)+p_i}} \\
&\quad + \sum_{j\neq i}^{n} \|a_{ij}(x)(|u_{ik}|^{\alpha_i}|u_{jk}|^{\alpha_j+1} - |u_i|^{\alpha_i}|u_j|^{\alpha_j+1})\|_{\frac{Np_i}{N(p_i-1)+p_i}} \Big] \to 0.
\end{aligned}$$

As in [6], we can prove that, the norm

$$\|BU_k - BU\|_{\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)}$$

tends strongly to zero and then the operator B is strongly continuous. According to Remark 2.1, the operator $A-B$ satisfies the M_0-condition. Now, to apply Theorem 2.2, it remains to prove that the operator $A-B$ is a coercive operator,

$$\begin{aligned}
&((A-B)U, U) \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^N} |\nabla u_i|^{p_i} - \sum_{i=1}^{n} \Big[\int_{\mathbb{R}^N} a_{ii}(x)|u_i|^{p_i} - \sum_{j\neq i}^{n} \int_{\mathbb{R}^N} a_{ij}(x)|u_i|^{\alpha_i+1}|u_j|^{\alpha_j+1} \Big] \\
&\ge \sum_{i=1}^{n} \int_{\mathbb{R}^N} |\nabla u_i|^{p_i} - \sum_{i=1}^{n} \int_{\mathbb{R}^N} a_{ii}(x)|u_i|^{p_i}.
\end{aligned}$$

Using (4.6), we obtain

$$\begin{aligned}
((A-B)U, U) &\ge \sum_{i=1}^{n} \int_{\mathbb{R}^N} |\nabla u_i|^{p_i} - \sum_{i=1}^{n} \frac{1}{\lambda_{a_{ii}}(\Omega)} \int_{\mathbb{R}^N} |\bigtriangledown u_i|^{p_i} \\
&= \sum_{i=1}^{n} (1 - \frac{1}{\lambda_{a_{ii}}(\Omega)}) \int_{\mathbb{R}^N} |\bigtriangledown u_i|^{p_i}.
\end{aligned}$$

From (4.9), we deduce

$$((A-B)U, U) \ge k \sum_{i=1}^{n} \|u_i\|^{p_i}_{D^{1,p_i}(\mathbb{R}^N)} = k\|(u_i)\|_{\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)}.$$

So that $((A-B)U, U) \to \infty$ as $\|(u_i)\|_{\prod_{i=1}^n D^{1,p_i}(\mathbb{R}^N)} \to \infty$. This proves the coercivity condition and so, the existence of a weak solution for systems (4.1). $\square$

Acknowledgments. The authors wish to thank the anonymous referees for their interesting remarks.

References

[1] Adams, R.; *Sobolev Spaces*, Academic Press, New York, 1975.
[2] Anane, A.,; *Simplicite et isolation de la premiere valeur propre du $p-$Laplacien avec poids*, Comptes Rendus Acad. Sc. Paris, 305, 725-728, 1987.
[3] Fleckinger, J.; Manasevich, R.; Stavrakakies, N.; De Thelin, F.; *Principal Eigenvalues for some Quasilinear Elliptic Equations on* $\mathbb{R}^N$, Advances in Diff. Eqns., Vol. 2, No. 6, 981-1003, 1997.
[4] Francu, J; *Solvability of Operator Equations*, Survey Directed to Differential Equations, Lecture Notes of IMAMM 94, Proc. of the Seminar "Industerial Mathematics and Mathematical Modelling" , Rybnik, Univ. West Bohemia in Pilsen, Faculty of Applied Scinces, Dept. of Math., July, 4-8, 1994.
[5] Serag, H. and El-Zahrani, E.; *Maximum Principle and Existence of Positive Solution for Nonlinear Systems on* $\mathbb{R}^N$, Electron. J. Diff. Eqns., Vol. 2005, No. 85, 1-12, 2005.
[6] Serag, H. and El-Zahrani, E.; *Existence of Weak Solution for Nonlinear Elliptic Systems on* $\mathbb{R}^N$, Electron. J. Diff. Eqns., Vol. 2006, No. 69, 1-10,2006.

Salah A. Khafagy
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
E-mail address: `el_gharieb@hotmail.com`

Hassan M. Serag
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
E-mail address: `serraghm@yahoo.com`